Yet Another Lebesgue Integration

YAMAGAMI Shigeru

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The set $\mathbb{N}$ of natural numbers is assumed to start with the number 1. Here are some of frequently used symbols.

$$\mathbb{N} = \{1, 2, \dots\}, \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\},$$
$$\mathbb{R} = \text{the set of real numbers}, \quad \mathbb{R}_+ = [0, +\infty), \quad \overline{\mathbb{R}} = [-\infty, \infty],$$
$$\mathbb{C} = \text{the set of complex numbers.}$$
$$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\}, \quad a_n \downarrow a, \quad a_n \uparrow a.$$

Given a real-valued function $f : X \to \mathbb{R}$, set

$$[a < f < b] = \{x \in X; a < f(x) < b\}.$$

If X is a topological space,

$$[f] = \overline{[f \neq 0]}, \quad [f \neq 0] = \{x \in X; f(x) \neq 0\}.$$

The items marked by asterisque can be skipped for the first reading.

1 From Real Numbers to Riemannian Integrals

If one wants to get a deep understanding of analysis including integral calculus, one can not avoid what the real numbers are. In modern mathematics, we notice three major characteristics of real numbers.

- Algebraic operations such as addition, subtraction, multiplication and division.

- Order structure associated with inequalities.
- The completeness known as the continuity of real numbers.

We shall not discuss about the algebraic structure because we only need manipulations of an elementary level. Related to the order structure, we introduce two symbols $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, which are examples of the so-called binary operations and satisfy the associativity and commutativity laws. In particular, the notation such as $a_1 \vee a_2 \vee \cdots \vee a_n$ has a definite meaning withought worrying about how parentheses should be inserted. In fact, we see

$$a_1 \vee \cdots \vee a_n = \max\{a_1, \ldots a_n\}, \quad a_1 \wedge \cdots \wedge a_n = \min\{a_1, \ldots a_n\}.$$

Consider a set A consisting of real numbers. If A is bounded, it admits the least upper bound and greatest lower bound, which are denoted by $\sup A$, $\inf A$ respectively. These symbols are used for unbounded A as well to denote virtual objects $\pm\infty$. In what follows, the real line $\mathbb{R}$ is often extended by adding these symbolical objects; the **extended real line** $\overline{\mathbb{R}}$ is defined to be $[-\infty, +\infty]$. The positive infinity $+\infty$ is often denoted by ∞.

Thus we can assign $\sup A$ and $\inf A$ to a set A of real numbers irrelevant of the boundedness of A. The following is clear from the definition.

$$A \subset B \Longrightarrow \sup(A) \leq \sup(B), \quad \inf(A) \geq \inf(B).$$

In accordance with these order relations, we set $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

From the obvious inequality

$$\sup\{a_n; n \geq 1\} \geq \sup\{a_n; n \geq 2\} \geq \ldots$$

for a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we can define the limit

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k; k \geq n\}$$

as an element in $\overline{\mathbb{R}}$, which is referred to as the **upper limit** of the sequence $\{a_n\}$. Likewise, the **lower limit** of $\{a_n\}$ is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k; k \geq n\}.$$

Proposition 1.1. Given a sequence $\{a_n\}$ of real numbers, we have

(i) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ and

(ii) $a = \lim_{n\to\infty} a_n \iff \liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty}$ with $a \in \overline{\mathbb{R}}$.

A sequence $\{a_n\}$ in $\mathbb{R}$ is said to be **increasing** if $a_j \leq a_k$ $(j \leq k)$ and **decreasing** if $a_j \geq a_k$ $(j \leq k)$.

The notation $a_n \uparrow a$ is used if an increasing sequence $\{a_n\}$ converges to an element $a \in \overline{\mathbb{R}}$ and similarly for $a_n \downarrow a$.

Next consider summations of real numbers. To keep the generality, we shall work with a family $\{a_i\}_{i\in I}$ of real numbers. Then the summation

$$\sum_{i\in I} |a_i|$$

has the meaning as an element in $[0, \infty]$, irrelevant of the cardinality of I.

If this value is finite, the family $\{a_i\}_{i \in I}$ is said to be **summable** and its **sum** is defined to be

$$\sum_{i \in I} a_i = \sum_{i \in I} a_i \vee 0 - \sum_{i \in I} (-a_i) \vee 0 \ \in \ \mathbb{R}.$$

Since the result of sum is characterized by

$$\forall \epsilon > 0, \exists \text{a finite subset } F \subset I, \forall \text{finite } F' \supset F, \ \left| \sum_{i \in I} a_i - \sum_{i \in F'} a_i \right| \leq \epsilon,$$

we see that the operation of summation is linear.

If the index set I is partitioned as

$$I = \bigsqcup_{j \in J} I_j,$$

then, for each $j \in J$, the family $\{a_i\}_{i \in I_j}$ is summable, the family $\{\sum_{i \in I_j} a_i\}_{j \in J}$ of sums is summable as well and the following partitioned sum formula holds

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

Exercise 1. If $\sum_{i \in I} |a_i| < +\infty$, $\{i \in I; a_i \neq 0\}$ is a countable set.

We shall here review the Riemannian integrals. Let f be a function defined on a finite interval $[a, b]$. Given a division $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$, define its **mesh** by $|\Delta| = \min\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$.

Consider

$$\overline{S}(f, \Delta) = \sum_{i=1}^{n} \overline{f}_i (x_i - x_{i-1}), \qquad \underline{S}(f, \Delta) = \sum_{i=1}^{n} \underline{f}_i (x_i - x_{i-1}),$$

where

$$\overline{f}_i = \sup\{f(x); x \in [x_{i-1}, x_i]\}, \quad \underline{f}_i = \inf\{f(x); x \in [x_{i-1}, x_i]\}$$

denotes local suprema and infima of f.

Lemma 1.2. Let Δ be a subdivision of Δ' and Δ''. Then

$$\underline{S}(f, \Delta') \leq \underline{S}(f, \Delta) \leq \overline{S}(f, \Delta) \leq \overline{S}(f, \Delta'').$$

According to Darboux, we now introduce **upper and lower integrals** by

$$\overline{S}(f) = \inf\{\overline{S}(f, \Delta); \Delta\}, \qquad \underline{S}(f) = \sup\{\underline{S}(f, \Delta); \Delta\},$$

which satisfy the inequality $\underline{S}(f) \leq \overline{S}(f)$.

Definition 1.3. A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann-integrable if $\overline{S}(f) = \underline{S}(f)$ with this common value denoted by

$$\int_a^b f(x)\, dx$$

and referred to as the **integral** of f.

Remark . The above definition is due to Darboux (1875), which is a rewriting of the Riemann's (1857).

Similar definition works for functions of multiple variables and we arrive at the Riemannian integral of functions defined on rectangular solids. For the later use, however, the case of continuous functions suffices and we shall next review some of basic results.

Theorem 1.4. Any continuous function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable and we have

$$\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} \sum_{i=1}^n f(x_j)(x_j - x_{j-1}).$$

Riemannian integral itself has the meaning only for bounded functions defined on bounded regions. Since this is too restrictive for applications, the notion of integration is usually modified so that it includes unbounded cases, known as **improper integrals**. Among these improper integrals, the absolutely convergent one has a legitimate meaning of integration; the truely improper integrals are absolutely divergent ones. Thus a continuous function f satisfying

$$\int_{\mathbb{R}^n} |f(x)|\,dx < +\infty,$$

for example, is considered to be in the scope of regular integrations, whereas a truely improper situation means that

$$\lim_{R\to\infty} \int_0^R f(x)\,dx$$

exists with the integration $\int_0^\infty |f(x)|\,dx$ divergent.

Example 1.5. Here are examples of (i) a good improper integration and (ii) a bad improper integration.

(i)
$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx.$$

(ii)
$$\int_0^\infty \frac{\sin x}{1+x}\,dx.$$

Exercise 2. Check the above statements. Hint: Use the integration-by-parts in the improper convergence of (ii).

2 Compact Sets and Continuous Functions

The following is the source of continuity of real numbers and can be proved by a squeezing argument.

Theorem 2.1 (Bolzano)**.** Any bounded sequence $\{a_n\}_{n\geq 1}$ of real numbers admits a convergent subsequence. Recall that a subsequence of $\{a_n\}$ is a sequence of the form $\{a_{n_k}\}_{k\geq 1}$ with $\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$ a strictly increasing function.

Corollary 2.2. A bounded subset K of a Euclidean space $\mathbb{R}^n$ has the following property: Any sequence in K admits a subsequence which coverges to a point in K.

Definition 2.3. A subset K in a metric space (X, d) is said to be **compact** if it has the property in the above corollary. A metric space is compact if X itself is compact.

A metric space (X, d) is said to be **locally compact** if for any $a \in X$ we can find a positive real $r > 0$ such that the closed ball $\overline{B}_r(a)$ is compact. The Euclidean space $\mathbb{R}^n$ is a typical example of a locally compact space.

We shall here review some terminologies on **metric space**. A **metric** in a set X is a function $d : X \times X \to [0, +\infty)$ satisfying the following properties.

(i) $d(x, y) = 0 \iff x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) [Triangle Inequality] $d(x, y) \leq d(x, z) + d(z, y)$.

Example 2.4.

(i) The Euclidean metric $d(x, y) = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}$ in the set $\mathbb{R}^n$.

(ii) * A metric
$$d(f, g) = \sup\{|f(x) - g(x)|; x \in X\}$$
in the set $B(X)$ of bounded functions on a set X.

(iii) * A metric
$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$
in a set X. This is the metric induced from that in $B(X)$ by an embedding $X \ni x \mapsto \delta_x \in B(X)$. Here the function δ_x is defined by $\delta_x(y) = \delta_{x,y}$.

Example 2.5. * A metric space (X, d) is said to be bounded if $\sup\{d(x, y); x, y \in X\} < +\infty$. Given a positive real number $M > 0$, $M \wedge d$ is a bounded metric on X. For a bounded metric space (K, d_K), the product set $X = K^{\mathbb{N}}$ is a metric space with the metric given by
$$d_X(x, y) = \sum_{n \geq 0} \frac{1}{2^n} d_K(x_n, y_n).$$

Exercise 3. * Show that, if K is compact, so is the sequential product X by a diagonal argument.

Exercise 4. * For the choice $K = \{0, 1, \ldots, p - 1\}$, relate the product space $K^{\mathbb{N}}$ with the p-adic expansion of real numbers in $[0, 1]$.

Proposition 2.6. * In a metric space X, show that

(i) if A, B are compact subsets of X, so is $A \cup B$ and

(ii) any closed subset of a compact set $K \subset X$ is compact.

Exercise 5. * Check the above statements.

In a metric space (X, d),

$$B_r(a) = \{x \in X; d(x, a) < r\}, \quad \overline{B}_r(a) = \{x \in X; d(x, a) \le r\}$$

represent **open and closed balls** with center $a \in X$ and radius $r > 0$ respectively.

Remark . By definition, $B_r(a)$ is an open set. Since $\overline{B}_r(a)$ is a closed set, it includes the closure of $B_r(a)$ but these are not necessarily identical as can be seen from Example 2.4 (iii).

Exercise 6. * The condition of being locally compact is equivalent to requiring the following: For any $a \in X$, we can find a $r > 0$ such that $\overline{B_r(a)}$ is compact.

Given a point x and a non-empty subset A in X, the extended real number

$$d(x, A) = \inf\{d(x, a); a \in A\}$$

is called the **distance** between $x \in X$ and $A \subset X$. We see that $d(x, A) = 0 \iff x \in \overline{A}$ and the $d(x, A)$ is continuous as a function of $x \in X$ because of

$$|d(x, A) - d(y, A)| \le d(x, y), \quad x, y \in X,$$

which is a consequence of the triangle inequality.

Exercise 7. Check the above inequality.

Lemma 2.7. * Write $K_r = \{x \in X; d(x, K) \le r\}$ for a positive real $r > 0$ and $K \subset X$.

If K is a compact subset of a locally compact metric space X, we can find $r > 0$ such that K_r is compact.

Proof. We first claim that

$$\exists r > 0, \forall x \in K, \overline{B}_r(x) \text{ is compact.}$$

Otherwise, we can find a sequence $\{x_n\}_{n \ge 1}$ in K so that $\overline{B}_{1/n}(x_n)$ is not compact. Passing to a subsequence, we may assume that $x_n \to x \in K$. By the local compactness, we can find $r > 0$ such that $\overline{B}_r(x)$ is compact.

Now we choose n so that it satisfies $d(x_n, x) \le r/2$ and $1/n \le r/2$. Then

$$\overline{B}_{1/n}(x_n) \subset \overline{B}_{r/2}(x_n) \subset \overline{B}_r(x)$$

reveals that the non-compactness of the closed ball $\overline{B}_{1/n}(x_n)$ contradicts with the compactness of $\overline{B}_r(x)$.

Returning to the problem in question, we show that $K_{r/2}$ is compact for the positive number $r > 0$ described in the first stage.

Given a sequence $y_n \in K_{r/2}$, choose a sequence $x_n \in K$ so that $d(x_n, y_n) \le 2r/3$ for $n \ge 1$ and then, passing to a subsequence, assume that $x_n \to x \in K$. Now, if n satisfies $d(x_n, x) \le r/3$, the inequality $d(y_n, x) \le d(x_n, y_n) + d(x_n, x) \le r$ shows that we can find a convergent subsequence $\{y_{n'}\}$ by the compactness of $\overline{B}_r(x)$. Finally, we remark that $K_{r/2}$ is a closed subset. □

Next, we review on the notion of completeness of metric spaces. Recall that a sequence $\{a_n\}_{n\geq 1} \subset X$ converges to a point $a \in X$ if

$$\lim_{n\to\infty} d(a_n, a) = 0.$$

The element a is called the **limit point** of $\{a_n\}$ and written by $a = \lim_{n\to\infty} a_n$. A convergent sequence satisfies the Cauchy's condition: $\lim_{m,n\to\infty} d(a_m, a_n) = 0$, i.e.,

$$\forall\epsilon > 0, \exists N, \forall m, n \geq N,\ d(a_m, a_n) \leq \epsilon.$$

Exercise 8. * A convergent sequence has a unique limit point, i.e., $\lim_n a_n = a$ and $\lim_n a_n = a'$ imply $a = a'$.

A metric space is **complete** if any Cauchy sequence is convergent. Euclidean spaces are typical examples of complete metric spaces.

When a metric space is not complete, we can extend it to a complete one by adding possible limit points. The resultant metric space is called the **completion** of X, which is uniquely determined by X. The real line $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Here we shall review the definition of continuous functions.

(i) A local definition: $\forall x \in X$, $\forall\epsilon > 0$, $\exists\delta > 0$, $y \in B_\delta(x) \Longrightarrow |f(x) - f(y)| \leq \epsilon$.

(ii) A global definition: $\forall a, b \in \mathbb{R}$, $[a < f < b] \equiv \{x \in X; a < f(x) < b\}$ is an open subset.

Exercise 9. Check the equivalence of (i) and (ii).

Given continuous functions $f, g : X \to \mathbb{R}$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}$, the composite function $\Phi(f, g) : x \mapsto \Phi(f(x), g(x))$ is continuous. Particularly,

$$f + g, \quad fg, \quad f \vee g, \quad f \wedge g$$

are continuous.

Exercise 10. Check the continuity of functions $(a, b) \mapsto a \vee b, a \wedge b$ on $\mathbb{R}^2$.

For a function f on a metric space (or a topological space), its **support** is defined to be $[f] \equiv \overline{[f \neq 0]}$. By definition, the support is a closed subset satisfying $f(x) = 0$ $(x \notin [f])$ and we see

$$[f + g] \cup [f \vee g] \cup [f \wedge g] \subset [f] \cup [g], \quad [fg] \subset [f] \cap [g].$$

Exercise 11. (i) Check the above inclusions for supports.

(ii) Give an example of continuous functions f, g on $\mathbb{R}$ satisfying $[fg] \neq [f] \cap [g]$.

Exercise 12. Show that the support of f is minimal among closed subsets F satisfying $f(x) = 0$ $(x \notin F)$.

For a locally compact metric space X, we denote by $C_c(X)$ the set of real-valued continuous functions of compact support, which is a vector space and satisfies

$$f, g \in C_c(X) \Longrightarrow f \vee g, f \wedge g, fg \in C_c(X).$$

Exercise 13 (F. Riesz)**.** * Let F be a compact subset of a metric space (X, d) and $h : F \to [0, +\infty)$ be a continuous function. Then h is extended to a continuous function f on X by

$$f(x) = \begin{cases} h(x) & \text{if } x \in F, \\ d(x, F) \sup\left\{ \frac{h(y)}{d(x,y)}; y \in F \right\} & \text{if } x \notin F. \end{cases}$$

Definition 2.8. Given a function $f : X \to \mathbb{R}$ on a metric space (X, d) and a positive real $\delta > 0$, we introduce the degree of uniform continuity of f by

$$C_f(\delta) = \sup\{|f(x) - f(y)|; d(x, y) \leq \delta\} \in \overline{\mathbb{R}}.$$

Exercise 14. For a differentiable function $f : \mathbb{R} \to \mathbb{R}$, we have $C_f(\delta) \leq M\delta$ with $M = \sup\{|f'(x)|; x \in \mathbb{R}\}$.

Theorem 2.9 (Uniform Continuity)**.** For a continuous function $f : X \to \mathbb{R}$ on a compact metric space, we have

$$\lim_{\delta \to 0} C_f(\delta) = 0.$$

Proof. If f is not uniformly continuous,

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in X, d(x, y) \leq \delta, |f(x) - f(y)| > \epsilon.$$

In particular, for the choice $\delta = 1/n$,

$$\exists x_n, y_n \in X, d(x_n, y_n) \leq \frac{1}{n}, |f(x_n) - f(y_n)| \geq \epsilon.$$

Furthermore if we choose a subsequence $\{x_{n'}\}_{n \geq 1}$ so that $x_{n'} \to a$, then $y_{n'} \to a$. Now by the continuity of f,

$$\lim_{n \to \infty} f(x_{n'}) = f(a) = \lim_{n \to \infty} f(y_{n'}),$$

which, however, contradicts with $|f(x_{n'}) - f(y_{n'})| \geq \epsilon$. $\square$

Exercise 15. Give an example of a bounded continuous function on $\mathbb{R}$ which is not uniformly continuous.

Consider a continuous function $f : [a, b] \to \mathbb{R}$ on a rectangular solid $[a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. If we minimize the degree of uniform continuity by passing to a subvidision of $[a, b]$, we see that the Riemannian integral

$$\int_{[a,b]} f(x)\, dx$$

exists. In fact, for a division Δ of $[a, b]$, we have the inequality

$$\overline{S}(f, \Delta) - \underline{S}(f, \Delta) \leq C_f(|\Delta|)\,(b_1 - a_1) \ldots (b_n - a_n).$$

Exercise 16. * Give the definition of division Δ and its magnitude $|\Delta|$ of mesh.

For a function $f \in C_c(\mathbb{R}^n)$, we choose a rectangular solid $[a, b]$ large enough so that it contains the support $[f]$ and set

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{[a,b]} f(x)\,dx.$$

This integral does not depend on the choice of $[a, b]$ and we see the following.

(i) $C_c(\mathbb{R}^n) \ni f \mapsto \int_{\mathbb{R}^n} f(x)\,dx$ is linear.

(ii) If $f \geq 0$, $\int_{\mathbb{R}^n} f(x)\,dx \geq 0$.

(iii) For $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(x+y)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$

(iv) For a non-singular linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(Tx)\,dx = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f(x)\,dx.$$

Exercise 17. * Let $\Phi : [a, b] \to \mathbb{R}$ be an increasing function defined on a bounded closed interval $[a, b] \subset \mathbb{R}$. For a continuous function $f : [a, b] \to \mathbb{R}$, show that the limit

$$\lim_{|\Delta|\to 0} \sum_{j=1}^{n} f(x_j)(\Phi(x_j) - \Phi(x_{j-1})) = \int_a^b f(t)d\Phi(t)$$

exists, which is written as in the right hand side and referred to as the **Stieltjes integral** of f with respect to Φ.

Given a subset A of a set X, the function defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

is called the **indicator function** of A.

If $A \subset \mathbb{R}^n$ is a bounded subset in a Eucledian space, we may expect its n-dimensional volume $|A|$ as an integral of the form

$$|A| = \int 1_A(x)\,dx.$$

Example 2.10. * Let $\{x_i\}_{i\geq 1}$ be a sequence in a metric space X and $\{r_i\}_{i\geq 1}$ be a sequence of positive reals converging to 0. We can construct various strange sets in the form of open sets

$$U = \bigcup_{i\geq 1} B_{r_i}(x_i)$$

or their complementary closed sets. For example, let $\{x_i\}$ be a sequential arrangement of all rational numbers and $r_i = r/2^i$. Then

$$|U| \leq \sum_i |(x_i - r_i, x_i + r_i)| = \sum_{i=1}^{\infty} \frac{r}{2^{i-1}} = 2r.$$

3 Uniform Convergence

A sequence $\{f_n : X \to \mathbb{R}\}_{n\geq 1}$ of functions defined on a set X converges to a function $f : X \to \mathbb{R}$ if

$$\forall x \in X, \lim_{n\to\infty} f_n(x) = f(x).$$

The function f is uniquely determined by the sequence $\{f_n\}$ and referred to as the limit function. A sequence $\{f_n\}$ of functions is said to be **increasing** (**decreasing**) if the sequence $\{f_n(x)\}$ is increasing (decreasing) for all x. A sequence which is increasing or decreasing is said to be **monotone**. When an increasing sequence $\{f_n\}$ converges to a function f, we write $f_n \uparrow f$. Similarly $f_n \downarrow f$ means that f is the limit function of a decreasing sequence $\{f_n\}$.

For a function $f : X \to \mathbb{R}$, set

$$\|f\|_\infty = \sup\{|f(x)|; x \in X\} \in [0, +\infty].$$

The boundedness of f is then equivalent to $\|f\|_\infty < +\infty$. A sequence f_n of functions **converges uniformly** to a function f if

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

By the inequality $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty$, uniform convergence implies (point-wise) convergence. Riemannian integrals behave well with respect to uniform convergence.

Remark . The symbol ∞ in the notation $\| \ \|_\infty$ comes from a formula such as

$$\lim_{p\to+\infty} (|a_1|^p + \cdots + |a_n|^p)^{1/p} = |a_1| \vee \cdots \vee |a_n|.$$

Proposition 3.1. For a Riemannian integrable function $f : [a, b] \to \mathbb{R}$, we have

$$\left| \int_{[a,b]} f(x)\, dx \right| \leq (b_1 - a_1) \cdots (b_n - a_n) \|f\|_\infty.$$

Corollary 3.2. If $f_n \to f$ (uniformly),

$$\lim_{n\to\infty} \int_{[a,b]} f_n(x)\, dx = \int_{[a,b]} f(x)\, dx.$$

Example 3.3. Pushing limits of continuous functions and their integrals.

Theorem 3.4 (Dini)**.** If a sequence $\{f_n\}_{n\geq 1}$ of continuous functions defined on a compact set K satisfies $\forall x \in K$, $f_n(x) \downarrow 0$, then we have $\lim_{n\to\infty} \|f_n\|_\infty = 0$.

Proof. The denial of $\|f_n\|_\infty \to 0$ is

$$\exists r > 0, \forall N \geq 1, \exists n \geq N, \|f_n\|_\infty > r,$$

which means that we can find $n_1 < n_2 < \ldots$ satisfying $\|f_{n_j}\|_\infty > r$. Thus, from the condition $\|f_{n_j}\|_\infty > r$, $\exists x_j \in X$, $f_{n_j}(x_j) > r$. We shall now derive a contradiction by choosing a subsequence $\{x_{j'}\}_{j\geq 1}$ so that $x_{j'} \to x \in X$.

For each $m \geq 1$, if we restrict $j \geq 1$ so that it satisfies $n_{j'} \geq m$, then

$$f_m(x) = f_m(x) - f_m(x_{n'}) + f_m(x_{n'}) \geq f_m(x) - f_m(x_{j'}) + f_{n_{j'}}(x_{j'}) > f_m(x) - f_m(x_{j'}) + r.$$

Since f_m is continuous and $x_{j'} \to x$, the last inequality implies $f_m(x) \geq r$, which contradicts with the assumption $f_m(x) \to 0$ as $m \to \infty$. $\square$

Corollary 3.5. If a sequence $f_n : \mathbb{R}^m \to \mathbb{R}$ of continuous functions of compact support satisfies $f_n \downarrow 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} f_n(x)\, dx = 0.$$

Definition 3.6. * A function $f : X \to \mathbb{R}$ on a metric space is said to be **lower semicontinuous** if it satisfies the following equivalent conditions.

(i) $\lim_{n\to\infty} x_n = x$ implies $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

(ii) $\forall x \in X, \forall \epsilon > 0, \exists \delta > 0, d(x,y) \leq \delta \Longrightarrow f(y) \geq f(x) - \epsilon$.

(iii) For any $a \in \mathbb{R}$, $[f > a]$ is an open subset.

A function f is **upper semicontinous** if $-f$ is lower semicontinuous.

Exercise 18. * Check the equivalence of the above three conditions and write down similar conditions for upper semicontinuity.

Proposition 3.7. * The limit of an increasing sequence of lower semicontinuous functions is lower semicontinous and similarly for decreasing sequences of upper semicontinuous functions.

Proof. If lower semicontinuous functions f_n converge to f, then, for any $\alpha \in \mathbb{R}$, $[f > \alpha] = \bigcup_{n\geq 1}[f_n > \alpha]$ is an open subset. $\square$

Exercise 19. Express indicator functions $1_{[a,b]}, 1_{(a,b]}, 1_{[a,b)}, 1_{(a,b)}$ of intervals as limits of continuous functions.

Theorem 3.8 (Baire)**.** * Given a function $f : X \to (-\infty, \infty]$ on a metric space X which has a lower bound, define a sequence $\{f_n : X \to \mathbb{R}\}_{n\geq 1}$ of functions by

$$f_n(x) = \inf\{f(x') + nd(x,x'); x' \in X\}.$$

(i) We have $|f_n(x) - f_n(y)| \leq nd(x,y)$ $(x, y \in X)$. In particular, f_n is Lipschitz-continuous.

(ii) If f is assumed to be lower semicontinuous in addition, $f_n \uparrow f$ $(n \to \infty)$.

Proof. (i) Let $x, y \in X$. We have

$$\forall \epsilon > 0, \exists x' \in X, \quad f(x') + nd(x,x') \leq f_n(x) + \epsilon$$

and then

$$f_n(x)-f_n(y) \geq f_n(x)-(f(x')+nd(y,x')) \geq -nd(y,x')+nd(x',x)-\epsilon \geq -nd(x,y)-\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain $f_n(x) - f_n(y) \geq -nd(x,y)$. By the symmetry in x, y, the claimed inequality follows.

(ii) By definition, $f_n \uparrow$ and $f_n \leq f$. If $f(x)$ is lower semicontinuous at $x = a$,

$$\forall \epsilon > 0, \exists \delta > 0, \quad d(x,a) \leq \delta \Longrightarrow f(x) \geq f(a) - \epsilon,$$

which implies

$$\inf\{f(x) + nd(x,a); d(x,a) \leq \delta\}.$$

If we take n large enough so that $n\delta + \inf_{z\in X} f(z) \geq f(a) - \epsilon$, then

$$d(x,a) \geq \delta \Longrightarrow f(x) + nd(a,x) \geq \inf_{z\in X} f(z) + n\delta \geq f(a) - \epsilon.$$

Combining these, we have

$$f_n(a) \geq f(a) - \epsilon \quad \text{if } n \geq \frac{f(a) - \epsilon - \inf f(z)}{\delta},$$

which shows $\lim_{n\to\infty} f_n \geq f$. □

Exercise 20. * In the proof of (ii), we have implicitly assumed that $f(a) < +\infty$. Supply the arguments for the case $f(a) = +\infty$.

4 Vector Lattices and Integrals

A real vector space L consisting of real-valued functions on a set X is called a **vector lattice** on X if

$$f, g \in L \implies f \vee g, f \wedge g \in L,$$

where

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

For a vector lattice L, we set

$$L^+ = \{f \in L; f \geq 0\}.$$

Example 4.1.

(i) The set $C_c(X)$ of continuous functions of compact support on a locally compact metric space X.

(ii) * Consider a function on a product set $X = \{1, 2, \ldots, N\}^{\mathbb{N}}$ (N being a preassigned natural number) whose values determined by finitely many components of variables. Then the set L of all such functions is a vector lattice.

(iii) * The set L of functions, say f, on a set X, with the property that $[f \neq 0]$ is a finite set.

Exercise 21. Check the following identity:

$$|f| = f \vee 0 - f \wedge 0 \in L.$$

Especially, we have $L = L^+ - L^+$.

Exercise 22. The following three conditions are equivalent for a real vector space L of real-valued functions on a set X.

(i) L is a vector lattice. (ii) $f \in L$ implies $f \vee 0 \in L$. (iii) $f \in L$ implies $|f| \in L$.

Definition 4.2. A functional $I : L \to \mathbb{R}$ on a vector lattice L is called a **Daniell integral** (or simply integral) on L if the following conditions are satisfied.

(i) [Linearity] $I(\alpha f+\beta g)=\alpha I(f)+\beta I(g)$, $\alpha,\beta\in\mathbb{R}$, $f,g\in L$.

(ii) [Positivity] $f\geq 0\Longrightarrow I(f)\geq 0$.

(iii) [Continuity] $f_n\downarrow 0\Longrightarrow I(f_n)\downarrow 0$.

An **integration system** is, by definition, a couple (L,I) of a vector lattice L and an integral I on L.

Example 4.3. In the following examples, the continuity of integral is a consequence of the Dini's theorem.

(i) The Riemannian integral

$$I(f)=\int_{\mathbb{R}^n} f(x)\,dx$$

for $f\in C_c(\mathbb{R}^n)$.

(ii) * Given a finite probability distribution $\{p_1,\dots,p_N\}$,

$$I(f)=\sum_{k_1,\dots,k_n} f(k_1,\dots,k_n,*)p_{k_1}\dots p_{k_n}.$$

(iii) * For the choice $L=C_c(X)$ with X a discrete space,

$$I(f)=\sum_{x\in X} f(x).$$

(iv) * An integral on $L=C(S^n)$ is defined by a Riemannian integral of the form

$$I(f)=\int_{0<|x|\leq 1} f\left(\frac{x}{|x|}\right)\,dx.$$

Exercise 23. * Given an increasing function $\Phi:\mathbb{R}\to\mathbb{R}$, show that the Stieltjes integral

$$I(f)=\int_{-\infty}^{\infty} f(t)d\Phi(t)$$

gives an integral on $L=C_c(\mathbb{R})$.

Exercise 24. Show that the positivity of integral implies

$$f\geq g\Longrightarrow I(f)\geq I(g).$$

Exercise 25. Show that the continuity of integral implies

$$f_n\uparrow f\Longrightarrow I(f_n)\uparrow I(f).$$

Exercise 26. * Show that the continuity of integral is equivalent to the following condition:

Given a function f and a sequence $\{h_n\}_{n\geq 1}$ in L^+, we have

$$f\leq\sum_{n=1}^{\infty} h_n \implies I(f)\leq\sum_{n=1}^{\infty} I(h_n).$$

(The sum $\sum_n h_n$ is NOT assumed to belong to L.)

Exercise 27. * Let X be a locally compact metric space. Show that positive linear functionals on $C_c(X)$ are automatically continuous. Hint: Let $f_n \downarrow 0$ and $K = [f_1]$. If we set $g(x) = 1 - 1 \wedge Nd(x, K)$, g is continuous and $g|_K = 1$. By choosing N large enough, we see that $g \in C_c(X)$ by Lemma 2.7 and then use the ineqaulity $0 \leq f_n \leq \|f_n\| \, g$.

Definition 4.4. Given a vector lattice L on a set X, we set

$$L_\uparrow = \{f : X \to (-\infty, +\infty]; \exists \text{ a sequence} f_n \in L, f_n \uparrow f\},$$
$$L_\downarrow = \{f : X \to [-\infty, +\infty); \exists \text{ a sequence} f_n \in L, f_n \downarrow f\}.$$

The following notation is also used.

$$L_\uparrow^+ = \{f \in L_\uparrow; f \geq 0\}.$$

Proposition 4.5.

(i) $L_\downarrow = -L_\uparrow$ and $L \subset L_\uparrow \cap L_\downarrow$.

(ii) $\alpha, \beta \in \mathbb{R}_+,\ f, g \in L_\uparrow \Longrightarrow \alpha f + \beta g, f \vee g, f \wedge g \in L_\uparrow$.

(iii) $\alpha, \beta \in \mathbb{R}_+,\ f, g \in L_\downarrow \Longrightarrow \alpha f + \beta g, f \vee g, f \wedge g \in L_\downarrow$.

Remark . By definition, $0f(x) = 0$ even if $f(x) = \pm\infty$.

Exercise 28. Check the above properties.

Example 4.6. Consider the vector lattice $L = C_c(\mathbb{R}^n)$. For a subset $A \subset \mathbb{R}^n$, (i) $1_A \in L_\uparrow \iff A$ is open and (ii) $1_A \in L_\downarrow \iff A$ is closed.

Exercise 29. The function $f(x) = x/(x^2 + 1)$ belongs to neither $C_c(\mathbb{R})_\uparrow$ nor $C_c(\mathbb{R})_\downarrow$.

Exercise 30. * For the vector lattive $L = C_c(\mathbb{R}^n)$, show that $L_\uparrow = \{f : \mathbb{R}^n \to (-\infty, \infty]; f \text{ is lower semicontinuous}\}$ according to the following steps:

(i) Choose a sequence $h_n \in C_c(\mathbb{R}^n)$ of positive-valued functions which increasingly converges to the constant function 1.

(ii) Apply the Baire's theorem to a function fh_n with $f \in L_\uparrow$ to find an increasing sequence in m $f_{n,m} \in C_c(\mathbb{R}^n)$ such that $f_{n,m} \uparrow fh_n$.

(iii) If we set $f_m = \vee_{1 \leq n \leq m} f_{n,m}$, then $f_m \uparrow f$.

Exercise 31. * If X is a metric space and L consists of continuous functions, we have $L_\uparrow \cap L_\downarrow \subset C(X)$. Particularly, if X is locally compact and $L = C_c(X)$, $L_\uparrow \cap L_\downarrow = L$.

Lemma 4.7. If increasing sequences f_n, g_n in a vector lattice L satisfies the inequality

$$\lim_n f_n \leq \lim_n g$$

for limit functions (we do not assume that $\lim_n f_n$, $\lim_n g_n$ belong to L), then the following holds.

$$\lim_n I(f_n) \leq \lim_n I(g_n).$$

Proof. From the assumption, $f_m \leq \lim_{n\to\infty} g_n$ and hence $f_m = \lim_{n\to\infty} f_m \wedge g_n$. By applying the continuity of integral to $(f_m - f_m \wedge g_n) \downarrow 0$, we have

$$I(f_m) = \lim_{n\to\infty} I(f_m \wedge g_n) \leq \lim_{n\to\infty} I(g_n).$$

Finally take the limit on m. $\square$

Definition 4.8. We can define a functional $I_\uparrow : L_\uparrow \to (-\infty, +\infty]$ by

$$I_\uparrow(f) = \lim_{n\to\infty} I(f_n), \quad f_n \uparrow f, \; f_n \in L.$$

Likewise, $I_\downarrow : L_\downarrow \to [-\infty, +\infty)$ is defined by

$$I_\downarrow(f) = \lim_{n\to\infty} I(f_n), \quad f_n \downarrow f, \; f_n \in L.$$

Exercise 32. With help of the lemma, show that the functionals $I_\uparrow$, $I_\downarrow$ are well-defined.

Example 4.9. For the ordinary integral I on $L = C_c(\mathbb{R}^n)$,

$$I_\uparrow(1_{(a,b)}) = (b_1 - a_1)\dots(b_n - a_n) = I_\downarrow(1_{[a,b]}).$$

Exercise 33. * For the Stieltjes integral $I : C_c(\mathbb{R}) \to \mathbb{R}$ associated to an increasing function $\Phi : \mathbb{R} \to \mathbb{R}$,

$$I_\uparrow(1_{(a,b)}) = \Phi(b-0) - \Phi(a+0), \qquad I_\downarrow(1_{[a,b]}) = \Phi(b+0) - \Phi(a-0).$$

Proposition 4.10.

(i) $I_\downarrow(-f) = -I_\uparrow(f)$ for $f \in L_\uparrow$ (recall that $-L_\uparrow = L_\downarrow$).

(ii) Functionals $I_\uparrow$, $I_\downarrow$ are extensions of I, i.e., for $f \in L$, $L_\uparrow(f) = I(f) = I_\downarrow(f)$. Especially, $I_\uparrow(0) = I_\downarrow(0) = 0$.

(iii) Functionals $I_\uparrow$, $I_\downarrow$ are semilinear, i.e., for $\alpha, \beta \in \mathbb{R}_+$ and $f, g \in L_\uparrow$ (or $f, g \in L_\downarrow$),
$$I_\uparrow(\alpha f + \beta g) = \alpha I_\uparrow(f) + \beta I_\uparrow(g)$$
(or the identity holds with $I_\uparrow$ replaced by $I_\downarrow$).

(iv) If $f, g \in L_\uparrow$, $f \leq g$, $I_\uparrow(f) \leq I_\uparrow(g)$.

Proof. To see (iv), notice that $f_n \uparrow f$, $g_n \uparrow g$ imply $f_n \vee g_n \uparrow f \vee g = g$. $\square$

Lemma 4.11. (i) $f_n \uparrow f$ with $f_n \in L_\uparrow$ implies $f \in L_\uparrow$ and $I_\uparrow(f_n) \uparrow I_\uparrow(f)$.

(ii) $f_n \downarrow f$ with $f_n \in L_\downarrow$ implies $f \in L_\downarrow$ and $I_\downarrow(f_n) \downarrow I_\downarrow(f)$.

Proof. By symmetry it suffices to prove (i). For each $f_n \in L_\downarrow$, choose a sequence $\{f_{n,m}\}_{m\geq 1}$ so that $f_{n,m} \uparrow f_n$. To get the monotonivcity for $\{f_{n,m}\}_{n\geq 1}$, we introduce their pushing-ups by

$$g_{n,m} = f_{1,m} \vee f_{2,m} \vee \dots \vee f_{n,m}.$$

Here $g_{1,m} = f_{1,m}$ by definition. Clearly $g_{n,m}$ is increasing in n. Since $f_{n,m}$ is increasing in m, so is $g_{n,m}$ in m. Moreover

$$f_{n,m} \le g_{n,m} \le f_1 \vee f_2 \vee \cdots \vee f_n = f_n$$

shows that $g_{n,m} \uparrow f_n$ for each n.

With this perparation, we pick up the diagonal $\{g_{n,n}\}_{n\ge 1}$, which is an increasing sequence in L. Taking the limit $m \to \infty$ in the obvious inequality

$$f_{n,m} \le g_{n,m} \le g_{m,m} \le f_m, \quad m \ge n,$$

we obtain

$$f_n \le \lim_{m\to\infty} g_{m,m} \le f$$

and then

$$f = \lim_{m\to\infty} g_{m,m} \in L_\uparrow$$

as a consequence of the limit $n \to \infty$.

If the integration is applied in the above inequality,

$$I(f_{n,m}) \le I(g_{m,m}) \le I_\uparrow(f_m). \quad m \ge n$$

Taking the limit $m \to \infty$,

$$I_\uparrow(f_n) \le I_\uparrow(f) \le \lim_{m\to\infty} I_\uparrow(f_m)$$

and then

$$\lim_{n\to\infty} I_\uparrow(f_n) = I_\uparrow(f)$$

if we let $n \to \infty$ furthermore. □

Corollary 4.12. For a sequence $f_n \in L_\uparrow^+$, $\sum_n f_n \in L_\uparrow$ and

$$I_\uparrow\left(\sum_n f_n\right) = \sum_n I_\uparrow(f_n).$$

Example 4.13. Let $L = C_c(\mathbb{R})$ and $\mathbb{Q} = \{q_n\}_{n\ge 1}$. Then, for any $\epsilon > 0$,

$$A = \bigcup_{n\ge 1} (q_n - \epsilon/2^n, q_n + \epsilon/2^n)$$

is an open subset of $\mathbb{R}$ and hence $1_A \in L_\uparrow$. Moreover,

$$1_A \le \sum_{n\ge 1} 1_{(q_n-\epsilon/2^n, q_n+\epsilon/2^n)}$$

shows that

$$I_\uparrow(1_A) \le \sum_{n=1}^{\infty} \frac{2\epsilon}{2^n} = 2\epsilon.$$

Think of the meaning of this inequality carefully.

5 Extension of Integrals

Definition 5.1. Given a function $f : X \to \overline{\mathbb{R}}$, its **upper** and **lower integrals** are defined by

$$\overline{I}(f) = \inf\{I_\uparrow(g); g \in L_\uparrow, f \leq g\}, \qquad \underline{I}(f) = \sup\{I_\downarrow(g); g \in L_\downarrow, g \leq f\},$$

which are elements in the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$. Recall that $\inf(\emptyset) = +\infty$ and $\sup(\emptyset) = -\infty$.

Example 5.2. The indicator function of the set $\mathbb{Q} \subset \mathbb{R}$ (known as the Dirichlet function) satisfies $\overline{I}(1_{\mathbb{Q}}) = 0$.

Exercise 34.

(i) Prove $1_{\mathbb{Q}}(x) = \lim_{m\to\infty} \lim_{n\to\infty} (\cos(\pi m! x))^{2n}$.

(ii) Concerning Darboux's upper and lower integrals, show $\underline{S}(1_{\mathbb{Q}\cap[a,b]}) = 0$, $\overline{S}(1_{\mathbb{Q}\cap[a,b]}) = b - a$.

Proposition 5.3.

(i) $\underline{I}(f) = -\overline{I}(-f)$ for any f.

(ii) $\overline{I}(\lambda f) = \lambda \overline{I}(f)$ for $0 \leq \lambda < +\infty$. In particular, $\overline{I}(0) = 0$.

(iii) If $f \leq g$, $\overline{I}(f) \leq \overline{I}(g)$.

(iv) Suppose that the sum $f+g$ is well-defined, i.e., there is no $x \in X$ satisfying $f(x) = \pm\infty$ and $g(x) = \mp\infty$. Then we have $\overline{I}(f+g) \leq \overline{I}(f) + \overline{I}(g)$.

(v) $\underline{I}(f) \leq \overline{I}(f)$.

(vi) For $f \in L_\uparrow \cup L_\downarrow$, we have $\underline{I}(f) = \overline{I}(f)$. Moreover this value is equal to $I_\uparrow(f)$ or $I_\downarrow(f)$ according to $f \in L_\uparrow$ or $f \in L_\downarrow$.

Proof. The assertions (i)–(iv) are immediate from the definition.

For (v), let $g = -f$ in (iv) and use $\overline{I}(0) = 0$.

To see (vi), first notice that $\overline{I}(f) = I_\uparrow(f)$ ($f \in L_\uparrow$) and $\underline{I}(f) = I_\downarrow(f)$ ($f \in L_\downarrow$). Especially, $\underline{I}(f) = \overline{I}(f) = I(f)$ for $f \in L$.

Now let $f \in L_\uparrow$ and choose $f_n \in L$ so that $f_n \uparrow f$. Then

$$I_\uparrow(f) = \lim_n I(f_n) = \lim_n \underline{I}(f_n) \leq \underline{I}(f).$$

On the other hand, for $f \in L_\uparrow$, we have $\overline{I}(f) = I_\uparrow(f)$ as already checked. Thus $\overline{I}(f) = \underline{I}(f)$. □

Exercise 35. Supply the details for (i)–(iv).

Definition 5.4. A function $f : X \to \mathbb{R}$ is **Lebesgue-integrable** or simply integrable if $\underline{I}(f) = \overline{I}(f) \in \mathbb{R}$ (the upper and the lower integrals are finite and equal). The totality of integrable functions is denoted by $\mathbf{L^1}$. For $f \in L^1$, the value $\underline{I}(f) = \overline{I}(f) \in \mathbb{R}$ is denoted by $I(f)$. The notation is guaranteed by Proposition4.10 (ii) and Proposition5.3 (vi). For the case $L = C_c(\mathbb{R}^n)$ with the ordinary integral, L^1 is denoted by $L^1(\mathbb{R}^n)$.

Exercise 36. If a function $f : [a,b] \to \mathbb{R}$ on an interval $[a,b]$ is Riemann-integrable, so is Lebesgue-integrable and we have

$$I(f) = \int_a^b f(t)\,dt.$$

Lemma 5.5. A function $f : X \to \mathbb{R}$ is integrable if and only if

$$\forall \epsilon > 0,\ \exists f_+ \in L_\uparrow,\ \exists f_- \in L_\downarrow, \quad f_- \le f \le f_+,\ I_\uparrow(f_+) - I_\downarrow(f_-) \le \epsilon.$$

Moreover, if f_- increases (f_+ decreases) in such a way that $f_- \le f \le f_+$ and $I_\uparrow(f_+) - I_\downarrow(f_-) = I_\uparrow(f_+ - f_-) \ge 0$ goes to 0, then

$$I_\downarrow(f_-) \uparrow I(f), \quad I_\uparrow(f_+) \downarrow I(f).$$

Proof. Use the inequality $I_\downarrow(f_-) \le \underline{I}(f) \le \overline{I}(f) \le I_\uparrow(f_+)$. $\square$

Theorem 5.6.

(i) The set L^1 is a vector lattice on X and includes $L_\uparrow \cap L_\downarrow$.

(ii) $I : L^1 \to \mathbb{R}$ is a positive linear functional satisfying $I(f) = I_\uparrow(f) = I_\downarrow(f)$ for $f \in L_\uparrow \cap L_\downarrow$. In particular, $I : L^1 \to \mathbb{R}$ is an extension of the initial integral $I : L \to \mathbb{R}$.

Proof. Let $f, g \in L^1$. Assume that $f_+, g_+ \in L_\uparrow$ and $f_-, g_- \in L_\downarrow$ satisfy $f_- \le f \le f_+$, $g_- \le g \le g_+$. Then $f_- + g_- \le f + g \le f_+ + g_+$ and we see that

$$I_\uparrow(f_+ + g_+) - I_\downarrow(f_- + g_-) = (I_\uparrow(f_+) - I_\downarrow(f_-)) + (I_\uparrow(g_+) - I_\downarrow(g_-))$$

can be chosen arbitrarily small, i.e., $f + g \in L^1$ and $I(f+g) = I(f) + I(g)$.

Next, let $\lambda > 0$. Since $\lambda f_- \le \lambda f \le \lambda f_+$, we see that

$$I_\uparrow(\lambda f_+) - I_\downarrow(\lambda f_-) = \lambda(I_\uparrow(f_+) - I_\downarrow(f_-))$$

can be arbitrarily small, i.e., $\lambda f \in L^1$ and $I(\lambda f) = \lambda I(f)$.

If we notice $-f_+ \le -f \le -f_-$ ($-f_+ \in L_\downarrow$, $-f_- \in L_\uparrow$),

$$I_\uparrow(-f_-) - I_\downarrow(-f_+) = I_\uparrow(f_+) - I_\downarrow(f_-)$$

can be chosen small as well, i.e., $-f \in L^1$ and $I(-f) = -I(f)$.

Here we have checked that L^1 is a vector space and I is a linear functional on L^1.

To show that L^1 is closed under the lattice operation, it suffices to check $f \in L^1 \Longrightarrow f \vee 0 \in L^1$, which can be seen as follows. From $f_- \vee 0 \le f \vee 0 \le f_+ \vee 0$, we have the inequality

$$0 \le f_+ \vee 0 - f_- \vee 0 \le f_+ - f_-,$$

which is used to see that

$$0 \le I_\uparrow(f_+ \vee 0) - I_\downarrow(f_- \vee 0) = I_\uparrow(f_+ \vee 0 - f_- \vee 0) \le I_\uparrow(f_+ - f_-)$$

can be chosen arbitrarily small. In particular, for $f \ge 0$, $I(f) = I(f \vee 0) \ge 0$ as a limit of $I_\uparrow(f_+ \vee 0) \ge 0$.

Finally, if $f \in L_\uparrow \cap L_\downarrow$, we can find $f_\pm \in L$ such that $f_- \le f \le f_+$, which, together with Proposition5.3 (vi), shows that $\underline{I}(f) = \overline{I}(f) \in [I(f_-), I(f_+)]$ is finite. $\square$

Example 5.7. If a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{R\to+\infty} \int_{-R}^{R} |f(t)|\, dt < +\infty,$$

then $f \in L^1$ and

$$I(f) = \lim_{R\to+\infty} \int_{-R}^{R} f(t)\, dt.$$

This is an example of good improper integrals.

The integration

$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt \equiv \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin t}{t}\, dt = \pi$$

is an example of improper integrals which are not Lebesgue-integrable.

Exercise 37. Determine the integrability of

$$\int_0^{\infty} \sin\left(\frac{1}{t^{\alpha}}\right) dt$$

for $\alpha > 0$.

Exercise 38. * Prove the following property of integrable functions.

$$\forall \epsilon > 0, \forall f \in L^1, \exists g \in L, I(|f-g|) \leq \epsilon.$$

Theorem 5.8. Let (L, I), (M, J) be integration systems on sets X, Y respectively and $\phi : X \to Y$ be a map satisfying $M \circ \phi \subset L$ and $I(f \circ \phi) = J(f)$ $(f \in M)$. Then we have $M^1 \circ \phi \subset L^1$ and $I(f \circ \phi) = J(f)$ $(f \in M^1)$.

Proof. Check $M_{\uparrow} \circ \phi \subset L_{\uparrow}$, $I_{\uparrow}(f \circ \phi) = J_{\uparrow}$ and so on step by step. □

Corollary 5.9.

(i) If $\phi : X \to Y$ is bijective and $L = M \circ \phi$, we have $L^1 = M^1 \circ \phi$ and $I(f) = J(f \circ \phi)$ $(f \in M^1)$.

(ii) If integration systems (L, I), (M, J) on a set X satisfies $L \subset M$, $J|_L = I$, i.e., (M, J) is an extension of (L, I), we have $L^1 \subset M^1$ and the associated integration on M^1 coincides with I on L^1.

Example 5.10. For an open set $A \subset \mathbb{R}^n$, an integration system is obtained through the inclusion $C_c(A) \subset C_c(\mathbb{R}^n)$ with the associated space of integrable functions on A denoted by $L^1(A)$, for which $L^1(A) \subset L^1(\mathbb{R}^n)$ and the integration on $L^1(A)$ is an extension of that for $L^1(\mathbb{R}^n)$.

In particular, for the one-dimensional case, if we set $A = (a, b)$, then

$$\int_a^b f(x)\, dx = \int_{\mathbb{R}} f(x) 1_A(x)\, dx.$$

Remark . In §8, we shall introduce

$$\int_A f(x)\, dx = \int f(x) 1_A(x)\, dx$$

for a more general set A.

Example 5.11. For $f \in L^1(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, $f(x+y)$ is integrable as a function of $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} f(x+y)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$

Exercise 39. For a function $f \in L^1(\mathbb{R}^n)$ and a positive real $\lambda > 0$, check the identity

$$\int_{\mathbb{R}^n} f(\lambda x)\,dx = \lambda^{-n} \int_{\mathbb{R}^n} f(x)\,dx.$$

6 Convergence Theorems in Integration

Lemma 6.1 (subadditivity of upper integrals)**.** If a function $f : X \to [0, +\infty]$ has an expression $f = \sum_{n=1}^{\infty} f_n$ $(f_n \geq 0)$,

$$\overline{I}(f) \leq \sum_{n=1}^{\infty} \overline{I}(f_n).$$

Proof. If there is an n such that $\overline{I}(f_n) = +\infty$, the inequality is trivial. So assume that $\overline{I}(f_n) < +\infty$ $(n \geq 1)$. Given any $\epsilon > 0$, if we choose $g_n \in L_\uparrow^+$ so that

$$f_n \leq g_n, \quad I(g_n) = I_\uparrow(g_n) \leq \overline{I}(f_n) + \frac{\epsilon}{2^n},$$

then $f \leq \sum_n g_n$. Now Corollary 4.12 (i) gives $\sum_n g_n \in L_\uparrow^+$ and $I_\uparrow(\sum_n g_n) = \sum_n I_\uparrow(g_n)$, whence

$$\overline{I}(f) \leq I_\uparrow\left(\sum_n g_n\right) = \sum_n I_\uparrow(g_n) \leq \sum_n \overline{I}(f_n) + \sum_n \frac{\epsilon}{2^n} = \sum_n \overline{I}(f_n) + \epsilon.$$

□

Theorem 6.2 (Monotone Convergence Theorem)**.** If a function $f : X \to \mathbb{R}$ is the limit of an increasing sequence $f_n \in L^1$ of integrable functions, then f is integrable if and only if $\lim\limits_{n\to\infty} I(f_n) < +\infty$. If this is the case, the following holds.

$$I(f) = \lim_{n\to\infty} I(f_n).$$

Proof. Since $I(f_n) = \overline{I}(f_n) \leq \overline{I}(f)$, $\lim_{n\to\infty} I(f_n) = +\infty$ implies $\overline{I}(f) = +\infty$ and hence $f \notin L^1$. Let $\lim\limits_{n\to\infty} I(f_n) < +\infty$. Since $f - f_0 = \sum_{n=1}^{\infty}(f_n - f_{n-1})$, the above lemma implies

$$\overline{I}(f-f_0) \leq \sum_{n=1}^{\infty} \overline{I}(f_n-f_{n-1}) = \sum_{n=1}^{\infty} I(f_n-f_{n-1}) = \sum_{n=1}^{\infty}(I(f_n)-I(f_{n-1})) = \lim_{n\to\infty} I(f_n)-I(f_0).$$

and therefore

$$\overline{I}(f) \leq \overline{I}(f_0) + \overline{I}(f - f_0) = I(f_0) + \overline{I}(f - f_0) \leq \lim_n I(f_n).$$

On the other hand, $f_n \leq f$ and $f_n \in L^1$ imply

$$I(f_n) = \underline{I}(f_n) \leq \underline{I}(f).$$

By taking limit, we obtain

$$\lim_n I(f_n) \le \underline{I}(f) \le \overline{I}(f) \le \lim_n I(f_n).$$

$\square$

Corollary 6.3. The positive linear functional I is continuous on L^1, i.e., if a decreasing sequence $f_n \in L^1$ satisfies $f_n \downarrow 0$, then $I(f_n) \downarrow 0$. As a conclusion, we obtain an integration system (L^1, I) which extends (L, I).

Theorem 6.4 (Dominated Convergence Theorem)**.** If a sequence $f_n \in L^1$ and a function $g \in L^1$ satisfy $|f_n| \le g$ $(n \ge 1)$, then $\inf_{n\ge 1} f_n$, $\sup_{n\ge 1} f_n$, $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are all integrable and

$$I(\liminf f_n) \le \liminf I(f_n) \le \limsup I(f_n) \le I(\limsup f_n).$$

In particular, if the limit function $f = \lim_{n\to\infty} f_n$ exists, $f \in L^1$ and the following holds.

$$I(f) = \lim_{n\to\infty} I(f_n).$$

Proof. For a natural number m, we see

$$-g \le \inf_{n\ge m} f_n \le f_m \wedge \cdots \wedge f_n \le f_m \vee \cdots \vee f_n \le \sup_{n\ge m} f_n \le g$$

and

$$f_m \wedge \cdots \wedge f_n \downarrow \inf_{n\ge m} f_n, \quad f_m \vee \cdots \vee f_n \uparrow \sup_{n\ge m} f_n.$$

whence, by the monotone convergence theorem and the positivity of I, we have $\inf_{n\ge m} f_n, \sup_{n\ge m} f_n \in L^1$ and

$$I(\inf_{n\ge m} f_n) = \lim_n I(f_m \wedge \cdots \wedge f_n) \le \lim_n I(f_m) \wedge \cdots \wedge I(f_n) = \inf_{n\ge m} I(f_n)$$
$$I(\sup_{n\ge m} f_n) = \lim_n I(f_m \vee \cdots \vee f_n) \ge \lim_n I(f_m) \vee \cdots \vee I(f_n) = \sup_{n\ge m} I(f_n).$$

In other words, we have

$$-I(g) \le I(\inf_{n\ge m} f_n) \le \inf_{n\ge m} I(f_n) \le \sup_{n\ge m} I(f_n) \le I(\sup_{n\ge m} f_n) \le I(g)$$

and then, again by the monotone convergence theorem, $\liminf_n f_n, \limsup_n f_n \in L^1$ and

$$-I(g) \le I(\liminf_n f_n) \le \liminf_n I(f_n) \le \limsup_n I(f_n) \le I(\limsup_n f_n) \le I(g).$$

$\square$

Corollary 6.5. Let $f(x,t)$ be a real-valued function on $\mathbb{R}^n \times (a,b)$ satisfying (i) for each $t \in (a,b)$, $f(x,t)$ is integrable as a function of $x \in \mathbb{R}^n$, (ii) for each $x \in \mathbb{R}^n$, $f(x,t)$ is differentiable as a function of $t \in (a,b)$ and (iii) we can find an integrable function g on $\mathbb{R}^n$ so that $|f(x,t) \le g(x)$ for $x \in \mathbb{R}^n$, $t \in (a,b)$. Then $\dfrac{\partial f}{\partial t}(x,t)$ is integrable as a function of x and we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} f(x,t)\,dx = \int_{\mathbb{R}^n} \frac{\partial f}{\partial t}(x,t)\,dx$$

for $a < t < b$.

Proof. By a mean value theorem, we have

$$\left| \frac{f(x, t+h) - f(x,t)}{h} \right| \leq g(x)$$

for h in a neighborhood of $0 \in \mathbb{R}$. Then apply the dominated convergence theorem to the limit function

$$\frac{\partial f}{\partial t}(x,t) = \lim_{h \to 0} \frac{f(x,t+h) - f(x,t)}{h}.$$

. ◻

Example 6.6. (i) For a real number t,

$$\int_0^\infty e^{-x^2+tx}\, dx = \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^\infty x^n e^{-x^2}\, dx.$$

(ii) For a positive number $t > 0$,

$$\frac{d^n}{dt^n} \int_0^\infty e^{-tx^2}\, dx = (-1)^n \int_0^\infty x^{2n} e^{-tx^2}\, dx.$$

Exercise 40. For a positive real $t > 0$, show that

$$\sum_{n=1}^\infty \frac{1}{n^t} = \frac{1}{\Gamma(t)} \int_0^\infty \frac{x^{t-1}}{e^x - 1}\, dx.$$

Exercise 41. For a continuous function $f(x)$ $(0 \leq x \leq 1)$, compute the limit

$$\lim_{n \to \infty} \int_0^n f\left(\frac{x}{n}\right) e^{-x}\, dx.$$

Exercise 42. For $f \in L^1(\mathbb{R}^n)$, show that $\int |f(x+y) - f(x)|\, dx$ is a continuous function of $y \in \mathbb{R}^n$.

Exercise 43. For an integrable function $f \in L^1(\mathbb{R}^n)$ and a bounded continuous function g on $\mathbb{R}^n$, show that fg is integrable.

Exercise 44. Given an integrable function $f \in L^1(\mathbb{R})$ and a positive real $a > 0$, show that

$$f_a(x) = \sqrt{\frac{a}{\pi}} \int_{-\infty}^\infty f(x-t) e^{-at^2}\, dt = \sqrt{\frac{a}{\pi}} \int_{-\infty}^\infty f(t) e^{-a(t-x)^2}\, dt$$

is a continuous and integrable function satisfying

$$\lim_{x \to \pm\infty} f_a(x) = 0, \qquad \int_{-\infty}^\infty |f_a(x)|\, dx \leq \int_{-\infty}^\infty |f(x)|\, dx.$$

7 Monotone Completion

The vector lattice L^1 is an ultimate extension of L from the view point of integration, which is , however, too far when one commits oneself to sequential extensions. In fact, we have used the operations of uncountable limits in the definition of upper and lower integrals.

In this section, we shall introduce the notion of monotone operation, which turns out to be a useful tool for proving properties preserving sequential limits.

Definition 7.1. A set $M \subset \mathbb{R}^X$ of real-valued functions on a set X is called a **monotone class** if a monotone sequence $\{f_n\}$ in M admits a limit function f such that $f(x) \in \mathbb{R}$ for $x \in X$, then $f \in M$.

Proposition 7.2. For any subset $S \subset \mathbb{R}^X$, there exists a monotone class $M \supset S$ such that

$$\text{if } M' \text{ is a monotone class containing } S\text{, then } M' \text{ contains } M.$$

The monotone class M of this property is unique, denoted by $M(S)$ and referred to as **the monotone class generated by S**.

Proof. Let M be the intersection of all the monotone classes containing S. □

Theorem 7.3. Given a vector lattice L on a set X, the monotone class $M(L)$ generated by L is a vector lattice and stable under taking sequential limits.

Proof. We shall check $f, g \in M(L) \Longrightarrow f + g \in M(L)$, for example. The proof will be done in two steps by applying the above proposition.

(i) $[L + M(L) \subset M(L)]$ For $f \in L$,

$$\{g \in M(L); f + g \in M(L)\}$$

is a monotone class containing L and therefore coincides with $M(L)$.

(ii) $[M(L) + M(L) \subset M(L)]$ For $g \in M(L)$,

$$\{f \in M(L); f + g \in M(L)\}$$

includes L by the first step, which is a monotone class in an obvious way. Thus it is equal to $M(L)$.

Similarly for other properties.

The last assertion follows if one notices that, in a vector lattice, the limit function of a convergent sequence can be written as repetitions of monotone convergence (lim sup or lim inf). □

Exercise 45. Supply the proof for scalar multiplication and lattice operations.

Corollary 7.4. The intersection $L^1 \cap M(L)$ is a vector lattice and we obtain an integration system by the restriction of $I : L^1 \to \mathbb{R}$ to $L^1 \cap M(L)$.

Exercise 46. Let L (resp. L') be a vector lattice on a set X (resp. X'). If a map $\phi : X \to X'$ satisfies $L' \circ \phi \subset L$, then $M(L') \circ \phi \subset M(L)$.

Definition 7.5. An integration system (L, I) is said to be **monotone-complete** if the monotone convergence theorem holds: Given a function $f : X \to \mathbb{R}$, $f_n \in L$, $f_n \uparrow f$ and $\lim_n I(f_n) < +\infty$ imply $f \in L$ and $I(f_n) \uparrow I(f)$.

The integration system $L^1 \cap M(L)$ is monotone-complete, which is called the **monotone-completion** of L with respect to $I : L \to \mathbb{R}$.

Lemma 7.6. For a function $f \in L_\downarrow$, $g \in L_\uparrow$, set $[f, g] = \{h : X \to \overline{\mathbb{R}}; f \le h \le g\}$.

(i) We have $M(L) \cap [f, g] = M(L \cap [f, g])$.

(ii) If the integration system (L, I) is monotone-complete, $M(L) \cap [f, g] = L \cap [f, g]$ for $f, g \in L$.

(iii) We have $M(L) = \displaystyle\bigcup_{f \in L_\downarrow, g \in L_\uparrow} [f, g] \cap M(L)$, i.e., given $f \in M(L)$, we can find $f_- \in L_\downarrow$ and $f_+ \in L_\uparrow$ such that $f_- \le f \le f_+$. Particularly, $M(L)^+ = M(L^+)$.

Proof. We assume $f \le g$ to avoid the trivial case.

(i) Choose $f_n, g_n \in L$ so that $f_n \downarrow f$ and $g_n \uparrow g$. Consider

$$M = \{h \in M(L); (f \vee h) \wedge g \in M([f, g] \cap L)\}.$$

Notice here that we have the identity $(f \vee h) \wedge g = f \vee (h \wedge g)$.

Now, it is immediate to see that M is a monotone class. Moreover, for $h \in L$, we have

$$(f \vee h) \wedge g = \lim_{n \to \infty} \lim_{m \to \infty} (f_m \vee h) \wedge g_n \in ([f, g] \cap L)_{\downarrow\uparrow}$$

and hence $L \subset M$. Thus $M = M(L)$ and, if $f \le h \le g$ $(h \in M)$ furthermore, $h = (f \vee h) \wedge g \in M([f, g] \cap L)$. In other words, $M(L) \cap [f, g] \subset M(L \cap [f, g])$. The reverse inclusion is obvious.

(ii) From the monotone completeness, $L \cap [f, g]$ is a monotone class and then (i) shows that

$$M(L) \cap [f, g] = M(L \cap [f, g]) = L \cap [f, g].$$

(iii) The set of functions h satisfying

$$h \in M(L), \quad \exists f \in L_\downarrow, g \in L_\uparrow, f \le h \le g$$

constitutes a monotone clas (Corollary4.12) and contains L, whence it coincides with $M(L)$. In particular, for $h \in M(L)^+$, we can find $g \in L_\uparrow$ such that $h \le g$. By applying (i), we see

$$h \in M(L) \cap [0, g] = M(L \cap [0, g]) \subset M(L^+).$$

The reverse inclusion $M(L^+) \subset M(L)^+$ follows because $M(L)^+$ is a monotone class containing L^+. $\square$

Exercise 47. If $1_X \in L_\uparrow^+$, $M(L)^+ = M(L^+)$ is a special case of (i). In particular, this is the case for $L = C_c(\mathbb{R}^n)$.

Theorem 7.7. In a monotone-complete integration system (L, I), we have $M(L)_\uparrow^+ = L_\uparrow^+$ and $L^1 \cap M(L) = L$. In particular, the monotone completion of a monotone-complete integration system coincides with itself.

Proof. For $f \in M(L)^+_\uparrow$, we can write $f_n \uparrow f$ ($f_n \in M(L)^+$). By (iii) in the above lemma, there exists $g_n \in L_\uparrow$ such that $f_n \le g_n$. Since $L_\uparrow$ is closed under taking limits of increasing sequences,

$$h \equiv \sup_{n\ge 1} g_n = \lim_{n\to\infty} g_1 \vee \cdots \vee g_n$$

belongs to $L_\uparrow$ and satisfies $f \le h$. Now take $h_n \in L^+$ so that $h_n \uparrow h$ and put $f_n = f \wedge h_n$. Then by (ii) in the above lemma, we see $f_n \in M(L) \cap [0, h_n] = L \cap [0, h_n] \subset L^+$ and $f_n \uparrow f$, i.e., $f \in L^+_\uparrow$.

Next consider $f \in L^1 \cap M(L)$. By the fact already checked, $(\pm f) \vee 0$ is in $L^1 \cap M(L)^+ \subset L^+_\uparrow$ and therefore we can find $g_n, h_n \in L^+$ so that

$$g_n \uparrow (f \vee 0), \quad h_n \uparrow (-f) \vee 0.$$

On the other hand,

$$I(g_n) \le I(f \vee 0) < +\infty, \quad I(h_n) \le I((-f) \vee 0) < +\infty,$$

together with monotone completeness, shows that

$$f \vee 0 = \lim_{n\to\infty} g_n, \quad (-f) \vee 0 = \lim_{n\to\infty} h_n$$

is in L^+ and hence $f = f \vee 0 - (-f) \vee 0 \in L$. $\square$

Corollary 7.8. In a (not necessarily monotone-complete) integration system (L, I), $M(L)^+_\uparrow = (L^1 \cap M(L))^+_\uparrow$. Especially, for $f \in M(L)^+_\uparrow$, $\overline{I}(f) = \underline{I}(f)$ and there exists $f_n \in L^1 \cap M(L)^+$ fulfilling $f_n \uparrow f$. Furthermore, for any such increasing sequence f_n, we have $I(f_n) \uparrow \overline{I}(f) = \underline{I}(f)$.

Exercise 48. Construct a counter-example to the equality $M(L)_\uparrow = (L^1 \cap M(L))_\uparrow$.

Definition 7.9. We extend the integration I to $M(L)^+_\uparrow$ by the following formula:

$$I(f) = \lim_{n\to\infty} I(f_n), \quad f_n \uparrow f, \ f_n \in (L^1 \cap M(L))^+,$$

which coincides with the integration value on $L^1 \cap M(L)^+_\uparrow$ and is equal to $I_\uparrow$ on $L^+_\uparrow$.

Remark . Without further conditios, the equality $\underline{I}(f) = \overline{I}(f)$ is not sufficient to assign the value of integration for the case $\underline{I}(f) = \overline{I}(f) = \pm\infty$.

Example 7.10. Let $f_{\alpha,\beta}$ ($\alpha > 0$, $\beta > 0$) be a function on $\mathbb{R}$ defined by

$$f_{\alpha,\beta}(x) = \begin{cases} |x|^{-\alpha} e^{-|x|^\beta} & \text{if } 0 < |x|, \\ +\infty & \text{if } x = 0. \end{cases}$$

Then $f \in M(C_c(\mathbb{R}))^+_\uparrow$ and

$$I(f) = \begin{cases} \frac{2}{\beta}\Gamma\left(\frac{1-\alpha}{\beta}\right) & \text{if } \alpha < 1, \\ +\infty & \text{if } \alpha \ge 1. \end{cases}$$

8 Measurable Sets and Measurable Functions

Definition 8.1.

(i) A subset $A \subset X$ is said to be **L-measurable** if $1_A \in M(L)$. The set of L-measurable sets is denoted by $\mathcal{M}(L)$.

(ii) A vector lattice L is **σ-finite** if $X \in \mathcal{M}(L)$, i.e., $1_X \in M(L)$.

An integration system (L, I) is σ-finite if L is σ-finite.

Proposition 8.2. Let X be a locally compact metric space.

(i) Any compact subset K of X is $C_c(X)$-measurable.

(ii) The vector lattice $C_c(X)$ is σ-finite if and only if $X = \bigcup_{n\geq 1} X_n$ with $\{X_n\}$ a sequence of compact subsets of X. A locally compact metric space with this property is said to be **σ–compact**.

Proof. (i) For a sufficiently large $n \geq 1$, $K_{1/n}$ is compact (Lemma 2.7) and contains the support of a continuous function $1 - 1 \wedge (nd(\cdot, K))$. Since

$$1_K = \lim_{n\to\infty} (1 - 1 \wedge nd(\cdot, K)),$$

this implies the $C_c(X)$-measurability of K.

(ii) The condition is sufficient by the part (i). So we need to check the necessity. Let

$$M = \{f \in M(L)^+; [f \neq 0] \subset \bigcup_{n\geq 1} K_n\}.$$

Here $\{K_n\}$ is a sequence of compact subsets, which can be chosen depending on f. Then M is a monotone class containing $C_c(X)^+$ and hence $M = M(L^+) = M(L)^+$. Consequently, $1_X \in M(L)^+$ implies $[1_X] = X = \bigcup_{n\geq 1} K_n$. $\square$

Corollary 8.3. If $X = \mathbb{R}^n$ or X is a countable discrete space, $C_c(X)$ is σ-finite.

In what follows, vector lattices are assumed to be σ-finite unless otherwise stated.

Proposition 8.4. If L is a σ-finite vector lattice, $\mathcal{M}(L)$ is a **σ-Boolean algebra**: The following holds.

(i) $\emptyset, X \in \mathcal{M}(L)$.

(ii) $\{A_n\}_{n\geq 1} \subset \mathcal{M}(L) \implies \bigcup_{n\geq 1} A_n, \bigcap_{n\geq 1} A_n \in \mathcal{M}(L)$.

(iii) $A \in \mathcal{M}(L) \implies X \setminus A \in \mathcal{M}(L)$.

Exercise 49. Supply the proof of the above statements.

Example 8.5. For $L = C_c(\mathbb{R}^n)$, $\mathcal{M}(L)$ contains all open sets and closed sets.

Definition 8.6. For a measurable set $A \in \mathcal{M}(L)$, its **I-measure** $|A|_I \in [0, +\infty]$ is defined by $|A|_I = I(1_A)$, i.e.,

$$|A|_I = \begin{cases} I(1_A) & \text{if } 1_A \text{ is integrable,} \\ +\infty & \text{otherwise.} \end{cases}$$

The **Lebesgue measure** is by definition the I-measure with I given by the Riemannian integral on $L = C_c(\mathbb{R}^n)$. The Lebesgue measure of a set A is simply denoted by $|A|$ with the suffix I omitted.

Example 8.7. The Lebesgue measure of a rectangular solid $[a, b] \subset \mathbb{R}^n$ is given by

$$|[a, b]| = (b_1 - a_1) \dots (b_n - a_n).$$

Exercise 50. Lebesgue measures possess (i) the invariance under translations and (ii) the covariance $|T(A)| = |\det(T)|\,|A|$ for an invertible linear transformation T.

In particular, Lebesgue measures have the meaning irrelevant of the choice of orthogonal coordinates.

Proposition 8.8. The I-measure has the following properties.

(i) $|\emptyset|_I = 0$.

(ii) $A = \bigsqcup_{n \geq 1} A_n \quad \Longrightarrow \quad |A|_I = \sum_{n=1}^{\infty} |A_n|_I.$

Definition 8.9. A **measure** is a function μ defined on a σ-Boolean algebra $\mathcal{B} \subset 2^X$ with values in $[0, +\infty]$ such that (i) $\mu(\emptyset) = 0$, (ii) $\{A_n\}_{n \geq 1} \subset \mathcal{B}$ and $A_m \cap A_n = \emptyset$ $(m \neq n)$ imply

$$\mu\left(\bigsqcup_{n \geq 1} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

(this property being referred to as the σ-additivity).

A measure μ is **σ-finite** if $X = \bigcup_{n \geq 1} X_n$ with $\mu(X_n) < +\infty$. If $\mu(X) < +\infty$, μ is a **finite measure** and a finite measure is a **probability measure** if $\mu(X) = 1$.

A **measure space** is by definition atriplet $(X, \mathcal{B}, \mu)$ of a set X, a σ-Boolean algebra $\mathcal{B} \subset 2^X$ and a measure μ on $\mathcal{B}$.

Remark . The power set of X is denoted by 2^X, i.e., 2^X is the set of subsets of X.

Exercise 51. If a decreasing sequence of sets $A_n \in \mathcal{B}$ satisfies $A_n \downarrow \emptyset$ and $\mu(A_1) < +\infty$, then $\mu(A_n) \downarrow 0$ (the continuity of measure).

Exercise 52. For an integration I on a σ-finite vector lattice L, the associated measure on $\mathcal{M}(L)$ is σ-finite. [Hint: Use (iii) in Lemma7.6.]

Lemma 8.10. Given a σ-Boolean algebra $\mathcal{B}$ on a set X and a function $f : X \to \overline{\mathbb{R}}$, the following four conditions are equivalent. (i) $\forall a \in \mathbb{R}$, $[f > a] \in \mathcal{B}$. (ii) $\forall a \in \mathbb{R}$, $[f \geq a] \in \mathcal{B}$. (iii) $\forall a \in \mathbb{R}$, $[f < a] \in \mathcal{B}$. (iv) $\forall a \in \mathbb{R}$, $[f \leq a] \in \mathcal{B}$.

Here remark that $\{x \in X; f(x) > a\} = [f > a]$.

Exercise 53. Prove the above statements.

Definition 8.11. Given a σ-Boolean algebra $\mathcal{B} \subset 2^X$, a function $f : X \to \overline{\mathbb{R}}$ is **$\mathcal{B}$-measurable** if it satisfies the equivalent conditions in the above lemma.

In a probability measure space, measurable functions are also referred to as **random variables**.

A function $f : X \to \mathbb{R}$ is called a **simple function** if its image is a finite set.

Proposition 8.12. Let S be the set of simple measurable functions. Then S is a vector lattice, which is closed under taking pointwise product, with the generated monotone class $M(S)$ being the set of $\mathbb{R}$-valud $\mathcal{B}$-measurable functions.

Proof. For a measurable simple function f, if we set $f(X) = \{a_1 < \cdots < a_m\}$,

$$A_i \equiv [f = a_i] = [f \geq a_i] \cap [f \leq a_i] \in \mathcal{B}$$

and

$$f = \sum_{i=1}^{m} a_i \, 1_{A_i}, \quad \bigsqcup_i A_i.$$

Conversely, any function of this form is a simple measurable function.

Let $g \in S$ and express

$$g = \sum_{j=1}^{n} b_j \, 1_{B_j}, \quad \bigsqcup_j B_j.$$

If we notice $\bigsqcup_{i,j} A_i \cap B_j$, the following expressions are obtained,

$$f + g = \sum_{i,j} (a_i + b_j) \, 1_{A_i \cap B_j}, \quad fg = \sum_{i,j} a_i b_j \, 1_{A_i \cap B_j},$$
$$f \vee g = \sum_{i,j} (a_i \vee b_j) \, 1_{A_i \cap B_j}, \quad f \wedge g = \sum_{i,j} (a_i \wedge b_j) \, 1_{A_i \cap B_j},$$

which show that these functions are simple and measurable.

Next, let M be the set of $\mathbb{R}$-valued measurable functions. Then M is a monotone class. In fact, given a sequence $\{f_n\}$ of $\mathbb{R}$-valued measurable functions, we have

$$f_n \downarrow f \Longrightarrow [f \geq a] = \bigcap_{n \geq 1} [f_n \geq a] \in \mathcal{B},$$
$$f_n \uparrow f \Longrightarrow [f \leq a] = \bigcap_{n \geq 1} [f_n \leq a] \in \mathcal{B}.$$

Combining these facts, $M(S) \subset M$. To see the reverse inclusion, for $f \in M$, if we set

$$f_{m,n}(x) = \begin{cases} m & \text{if } f(x) > m, \\ -m + 2mj/n & \text{if } -m + 2m(j-1)/n < f(x) \leq -m + 2mj/n \text{ for } 1 \leq j \leq n, \\ -m & \text{if } f(x) \leq -m, \end{cases}$$

then

$$|f_{m,n}(x) - f(x)| \leq \frac{1}{n} \quad \text{if } -m < f(x) \leq m$$

and hence, by taking limits,

$$\lim_{n \to \infty} f_{m,n}(x) = f(x) \quad \text{if } -m < f(x) \leq m.$$

Finally take the limit

$$\lim_{m\to\infty}\lim_{n\to\infty} f_{m,n}(x) = f(x) \quad \text{for any } x \in X,$$

which shows that $M \subset M(S)$. $\square$

Exercise 54. Given a sequence $\{f_n : X \to \overline{\mathbb{R}}\}_{n\geq 1}$ of $\mathcal{B}$-measurable functions, we have $\{x \in X; \lim_{n\to\infty} f_n(x) \text{ exists}\} \in \mathcal{B}$.

With a measure $\mu : \mathcal{B} \to [0, +\infty]$ specified, a simple measurable function f is said to be μ-finite if $\mu([f = a]) < +\infty$ $(0 \neq \forall a \in \mathbb{R})$. The set S_μ of μ-finite simple measurable functions is a vector lattice and, if μ is σ-finite, $M(S) = M(S_\mu)$. For $f \in S_\mu$, write $f = \sum_{i=1}^n a_i\, 1_{A_i}$ $(\mu(A_i) < +\infty)$ and set

$$I_\mu(f) = \sum_{i=1}^n a_i\, \mu(A_i).$$

Then the value does not depend on the expression in the right hand side and therefore it defines a positive linear functional $I_\mu : S_\mu \to \mathbb{R}$. Since the continuity of I_μ follows from the σ-additivity of measure (cf. the exercise below), an integration(S_μ, I_μ) is obtained. The integration constructed in this way is often denoted by

$$\int f(x)\, \mu(dx).$$

The notation

$$\int_A f(x)\, \mu(dx) = \int 1_A(x) f(x)\, \mu(dx)$$

is also used for a measurable set $A \in \mathcal{B}$.

Exercise 55. * If a decreasing sequence $f_n \in S_\mu$ satisfies $f_n \downarrow 0$, then, given, $\forall \epsilon > 0$,

$$I(f_n) = I(1_{[f_n\leq\epsilon]} f_n) + I(1_{[f_n>\epsilon]} f_n) \leq \epsilon I(f_1) + \|f_1\|_\infty\, \mu([f_n > \epsilon])$$

and therefore $[f_n > \epsilon] \downarrow \emptyset$, together with the continuity of μ gives $\lim_{n\to\infty} I(f_n) \leq \epsilon I(f_1)$ $(\forall \epsilon > 0)$.

Theorem 8.13. Given a σ-finite integration system (L, I) on a set X, we have

$$f : X \to \overline{\mathbb{R}} \text{ is } \mathcal{M}(L)\text{-measurable} \iff f_\pm = (\pm f) \vee 0 \in M(L)^+_\uparrow.$$

For $f \in M(L)^+_\uparrow$, the following holds.

$$I(f) = \lim_{r\to 1+0} \sum_{n=-\infty}^{\infty} r^n \left| [r^n < f \leq r^{n+1}] \right|_I.$$

Proof. Let $f_\pm = 0 \vee (\pm f) \in M(L)^+_\uparrow$. Then, for $r \geq 0$,

$$1_X \wedge (n(f_\pm - f_\pm \wedge r)) \uparrow 1_{[f_\pm > r]}$$

and then (recall the σ-finiteness $1_X \in M(L)$) $[f_\pm > r] \in \mathcal{M}(L)$. Thus we have $[f > a] \in \mathcal{M}(L)$ $(a \in \mathbb{R})$.

Conversely, assume that $[f > a] \in \mathcal{M}(L)$ $(\forall a \in \mathbb{R})$. Then, for $h = f_\pm$, $[h > r] \in \mathcal{M}(L)$ $(0 \leq r < +\infty)$. Now, given $r > 1$ and $n \in \mathbb{Z}$, we have $[r^n < h \leq r^{n+1}] \in \mathcal{M}(L)$ and

$$h_r \equiv \sum_{n=-\infty}^{\infty} r^n \, 1_{[r^n < h \leq r^{n+1}]} \in M(L)^+.$$

On the other hand, $h_{r^{1/n}} \uparrow h$ $(n \to \infty)$ implies $h \in M(L)^+_\uparrow$ $(h = f_\pm)$ and the desired expression is obtained by monotone convergence theorem. □

Exercise 56. By observing $h_r(x) = r^n \iff r^n < h(x) \leq r^{n+1}$, show the following.

(i) $\lim_{r \to 1} h_r(x) = h(x)$ $(x \in X)$.

(ii) $m \leq n$ implies $h_{r^{1/m}} \leq h_{r^{1/n}}$.

Corollary 8.14. Starting with an integration system (L, I), construct the measure space $(X, \mathcal{M}(L), \mu)$ and then the integration system (S_μ, I_μ). Then the monotone completions of (L, I) and (S_μ, I_μ) coincide. In particular, $M(S_\mu) = M(L)$.

Corollary 8.15.

(i) For a $\mathcal{M}(L)$-measurable function f, $|f|^\alpha \in M(L)^+_\uparrow$ $(\alpha > 0)$.

(ii) The vector lattice $M(L)$ is closed under taking pointwise multiplication, i.e., $f, g \in M(L)$ implies $4fg = (f+g)^2 - (f-g)^2 \in M(L)$.

Exercise 57. Check the assertion (i) in Corollary.

By the above discussions, we see that an integration I and a measure μ give equivalent information. With this in mind, we use the notation $L^1(X, \mu)$ to indicate the integration in use.

As an application of results obtained so far, we shall give a characterization of Lebesgue measures.

Theorem 8.16. * The Riemannian integration on $C_c(\mathbb{R}^n)$ is characterized as the one which is invariant under translations up to scalar multiplication.

Proof. Let μ be the measure associated with an integration, which is invariant under translations, and set $C = \mu((0,1] \times \cdots \times (0,1])$. Since μ is translationally invariant as well, we have, for $m = 1, 2, \ldots$,

$$\mu((0, 1/2^m] \times \cdots \times (0, 1/2^m]) = \frac{C}{2^{mn}} = |(0, 1/2^m] \times \cdots \times (0, 1/2^m]|.$$

Thus, for a simple function f which is adapted to a disjoint union of translations of these intervals,

$$\int f(x)\, \mu(dx) = C \int f(x)\, dx.$$

Since any $g \in C_c(\mathbb{R}^n)$ is a uniform limit of such simple functions with their supports contained in a common bounded subset, the above relation remains valid for $f \in C_c(\mathbb{R}^n)$. □

Exercise 58 (Chebyshev's inequality)**.** For $\alpha > 0$, show the following inequality.

$$\int_X |f(x)|^\alpha \, dx \geq r^\alpha \mu([|f| \geq r]).$$

Example 8.17 (Construction of non-measurable sets)**.** * Consider $\mathbb{R}$ as an additive group and T be a countable dense subgroup including $\mathbb{Z}$. For example, we can set $T = \mathbb{Q}$ or $T = \mathbb{Z} + \theta\mathbb{Z}$ ($\theta \notin \mathbb{Q}$). Next choose a representative set $W \subset [0, 1)$ (here the axiom of choice is needed), which gives a non-measurable subset of $\mathbb{R}$.

If W were Lebesgue-integrable, we have

$$|(W + t + \mathbb{Z}) \cap [0, 1)| = |W| \quad (t \in T).$$

In fact, if we choose $n \in \mathbb{Z}$ so that $n \leq t < n + 1$, then

$$(W + t + \mathbb{Z}) \cap [0, 1) = (W + t - n - 1) \cap [0, t - n) \bigsqcup (W + t - n) \cap [t - n, 1)$$

and therefore

$$|(W+t+\mathbb{Z})\cap[0, 1)| = |(W+t)\cap[n+1, t+1)|+|(W+t)\cap[t, n+1)| = |W+t| = |W|.$$

Since $\mathbb{R} = \bigsqcup_{t\in T/\mathbb{Z}}(W + t + \mathbb{Z})$,

$$[0, 1) = \bigsqcup_{t\in T/\mathbb{Z}} (W + t + \mathbb{Z}) \cap [0, 1)$$

gives

$$1 = \sum_{t\in T/\mathbb{Z}} |W|.$$

Since the summand in the right hand side takes the value 0 or $+\infty$, this is a contradiction.

9 Null Sets and Null Functions

Definition 9.1. If a measurable function $f : X \to \overline{\mathbb{R}}$ satisfies $I(|f|) < \infty \iff I((\pm f) \vee 0) < \infty$, f is said to be **integrable**. For an integrable function in this sense, we define its integral by

$$I(f) = I(f \vee 0) - I((-f) \vee 0) \in \mathbb{R}.$$

In the case of probability space, the integral of an integrable random variable ξ is called the **expectation** of ξ and denoted by $\langle \xi \rangle$.

A complex-valued function $f : X \to \mathbb{C}$ is measurable (integrable) if its real and imaginary parts $\Re f$, $\Im f$ are both measurable (integrable).

For an integrable complex-valued function f, its integration is defined by $I(f) = I(\Re f) + iI(\Im f)$.

Exercise 59. Integrable complex-valued functions constitute a complex vector space and the integration is a complex-linear functional.

Proposition 9.2. For a measurable function $f : X \to \mathbb{C}$, $|f|$ is mesurable as well and f is integrable if and only if $|f|$ is integrable. Moreover we have

$$|I(f)| \leq I(|f|).$$

Proof. The inequality is checked, for example, if we use the polar expression $I(f) = |I(f)|e^{i\theta}$ to compute as

$$|I(f)| = I(\Re f)\cos\theta + I(\Im f)\sin\theta = I\big((\Re f)\cos\theta + (\Im f)\sin\theta\big) \leq I(|f|).$$

□

Example 9.3. We can perform the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2+itx}\, dx = \sqrt{\pi} e^{-t^2/4}$$

via power series expansion.

Exercise 60. Given an integrable function $f : \mathbb{R} \to \mathbb{C}$ and a positive real $a > 0$, the following function

$$f_a(x) = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} f(x-t)e^{-at^2}\, dt$$

($x \in \mathbb{R}$) is extended to the complex plane $\mathbb{C}$ as an entire analytic function.

Definition 9.4. A measurable function $f : X \to \overline{\mathbb{R}}$ is called a **null function** if $I(|f|) = 0$. A measurable set A is a **null set** if its indicator 1_A is a null function, i.e., if $1_A \in M(L)$ and $I(1_A) = 0$. The set of null sets is denoted by $\mathcal{N}(I)$.

Example 9.5. The following are null sets with respect to the Lebesgue measure.

(i) Countable subsets of $\mathbb{R}^n$.

(ii) The Cantor set in $[0, 1]$.

(iii) Hypersurfaces in a Euclidean space.

Exercise 61. Show that hypersurfaces in $\mathbb{R}^n$ are null sets. (Hint: Use the implicit function theorem or you may postpone this problem after the Fubini's theorem).

Proposition 9.6. Here are properties of null sets.

(i) $\emptyset \in \mathcal{N}(I)$.

(ii) Any measurable subset of a null set $N \in \mathcal{N}(I)$ is a null set.

(iii) For a sequence $\{N_n\}_{n\geq 1}$ of null sets, $\bigcup_{n=1}^{\infty} N_n$ is a null set.

Exercise 62. Check the above properties.

Proposition 9.7.

(i) For a measurable function $f \in M(L)$, $[f \neq 0] \in \mathcal{M}(L)$ is a null set if and only if $I(|f|) = 0$.

(ii) For $f \in M(L)^+_\uparrow$, if $I(f) < +\infty$, $[f = +\infty] \in \mathcal{M}(L)$ is a null set.

Proof. (i) Use the monotone convergence theorem. From $|f| \wedge (n1_{[f\neq 0]}) \uparrow |f|$,

$$I(|f|) = \lim_{n\to\infty} I(|f| \wedge (n1_{[f\neq 0]}) \leq \lim_{n\to\infty} nI(1_{[f\neq 0]})$$

and $(n|f|) \wedge 1 \uparrow = 1_{[f\neq 0]}$ implies

$$I(1_{[f\neq 0]}) = \lim_{n\to\infty} I((n|f|) \wedge 1) \leq \lim_{n\to\infty} nI(|f|).$$

(ii) Set $A = [f = +\infty]$, Then $n|A|_I \leq I(f)$ follows from $n1_A \leq f$ ($n = 1, 2, \dots$). □

Corollary 9.8.

(i) The set $N(I)$ of $\mathbb{R}$-valued null functions is an ideal of the function algebra $M(L)$:

$$f \in N(I),\ g \in M(L) \quad \Longrightarrow \quad fg \in N(I).$$

(ii) For an integrable function $f, g \in L^1$, $I(|f - g|) = 0$ if and only if there exists a null set N such that $f(x) = g(x)$ ($x \notin N$).

In view of discussions so far, we see that the difference of null functions is irrelevant to integration values. This observation, in fact, is in match with our intuition on integrations. No one would worry about the difference between two integrals

$$\int_{(0,1]} f(x)\, dx, \qquad \int_{[0,1]} f(x)\, dx,$$

for example.

Definition 9.9. two functions $f, g : X \to \overline{\mathbb{R}}$ is said to be **equal almost everywhere** if $[f \neq g]$ is a subset of a null set. If this is the case, we write $f = g$ (a.e.).

The relation $f = g$ (a.e.) is an equivalence relation. We shall think of an equivalence class as representing a single function virtually. Various operations such as addition, multiplication and lattice operations can be regarded as defining for these virtual functions.

For example, if we are given two functions $f_j : X \setminus N_j \to \mathbb{R}$ which are defined except for null sets N_j respectively ($j = 1, 2$), then their sum $f_1(x) + f_2(x)$ has the meaning on $X \setminus (N_1 \cup N_2)$, whence the virtual function $f_1 + f_2$ is well-defined.

Exercise 63. Check the other operations such as multiplication or lattice operations.

Exercise 64. Show that, given an integrable function f, we can find a function $g \in L^1 \cap M(L)$ such that $f = g$ (a.e).

We can formulate various convergence theorems in the form of almost everywhere versions. For example, we have the following.

Theorem 9.10. If an integrable function $\{f_n\}_{n\geq 1}$, which is defined almost everywhere, satisfies

$$\sum_{n\geq 1} I(|f_n|) < +\infty,$$

then there exists a null set N such that (i) for any n, $X \setminus N$ is contained in the domain of f_n and (ii) for $x \in X \setminus N$, we have $\sum_n |f(x)| < +\infty$.

Consequently, if we set $f(x) = \sum_n f_n(x)$ $(x \notin N)$, f is an integrable function defined almost everywhere and satisfies

$$I(f) = \sum_n I(f_n).$$

Proof. Suppose that each f_j is defined outside of a null set N_j. If we set $\widetilde{f}_j(x) = f_j(x)$ $(x \notin \cup_j N_j)$ and 0 otherwise, then we may assume that $\widehat{f}_j$ is measurable (cf. the above exercise) and

$$\sum_{n\geq 1} |\widehat{f}_n| \in M_{\uparrow}^{+}.$$

Moreover,

$$I\left(\sum_{n\geq 1} |\widehat{f}_n|\right) = \sum_{n\geq 1} I(|\widehat{f}_n|) = \sum_{n\geq 1} I(|f_n|) < +\infty,$$

whence the set N_0 of x satisfying

$$\sum_{n\geq 1} |\widetilde{f}_n(x)| = +\infty$$

constitutes a null set . Now, if we set $N = \cup_{n\geq 0} N_j$, for $x \notin N$,

$$f(x) = \sum_{n\geq 1} \widetilde{f}_n(x) = \sum_{n\geq 1} f_n(x)$$

is absolutely convergent and therefore defines an integrable function almost everywhere, proving the convergence theorem. $\square$

Exercise 65. Check the other assertions.

Example 9.11. Given a sequential listing of rational numbers $\{q_n; n \geq 1\}$, the measurable function

$$f(x) = \sum_{n\geq 1} e^{-n^3(x-q_n)^2}$$

is convergent for almost every $x \in \mathbb{R}$ and we have

$$\int_{-\infty}^{\infty} f(x)\, dx = \sum_{n\geq 1} \sqrt{\frac{\pi}{n^3}} < +\infty.$$

As a conceptual application of the above convergence theorem, we present here the following.

Proposition 9.12.

(i) The vector space L^1 is the completion of L with respect to the norm $\|f\|_1 = I(|f|)$.

(ii) We have the expression $L^1 = L^1 \cap L_\uparrow - L^1 \cap L_\uparrow$ which holds up to null functions, i.e., given an integrable function $f \in L^1$, there exist a function $f_\pm \in L^1 \cap L_\uparrow$ and a null set N such that $f(x) = f_+(x) - f_-(x)$ $(x \notin N)$.

Proof. We first prove (ii). Take a decreasing sequence $f_n \geq f$ in $L_\uparrow$ so that $I(f - f_n) \leq 1/2^n$ and then choose $f_{n,m} \in L$ such that

$$f_{n,m} \uparrow f_n \ (m \to \infty), \quad I(f_n - f_{n,m}) \leq \frac{1}{2^{m+n}}.$$

Then we have

$$0 \leq I(f_{n,m} - f_{n,m-1}) = I(f_n - f_{n,m-1}) - I(f_n - f_{n,m}) \leq \frac{1}{2^{m+n-1}} + \frac{1}{2^{m+n}} = \frac{3}{2^{m+n}}$$

and therefore $\sum_{m,n\geq 1}(f_{n-1,m} - f_{n-1,m-1})$ is absolutely convergent almost everywhere. In the expression

$$\begin{aligned} \lim_{n\to\infty} f_n &= f_0 + \sum_{n\geq 1}(f_n - f_{n-1}) \\ &= f_{0,0} + \sum_{m\geq 1}(f_{0,m} - f_{0,m-1}) + \sum_{n\geq 1}(f_{n,0} - f_{n-1,0}) \\ &\quad + \sum_{m,n\geq 1}(f_{n,m} - f_{n,m-1}) - \sum_{m,n\geq 1}(f_{n-1,m} - f_{n-1,m-1}), \end{aligned}$$

the part

$$I(|f_{n,0} - f_{n-1,0}|) \leq I(|f_{n,0} - f_n|) + I(|f_n - f_{n-1}|) + I(|f_{n-1} - f_{n-1,0}|) \leq \frac{6}{2^n}$$

is aboslutely convergent as well and hence all these are absolutely convergent almost everywhere.

In view of the decomposition

$$\begin{aligned} f_{0,0} &= 0 \vee f_{0,0} - 0 \vee (-f_{0,0}), \\ f_{n,0} - f_{n-1,0} &= 0 \vee (f_{n,0} - f_{n-1,0}) - 0 \vee (f_{n-1,0} - f_{n,0}), \end{aligned}$$

the relation $f(x) = \lim_{n\to\infty} f_n(x)$ (a.e. $x \in X$) shows (ii) by choosing $f_\pm \in L^1 \cap L_\uparrow$ so that

$$f_+ = 0 \vee f_{0,0} + \sum_{m\geq 1}(f_{0,m} - f_{0,m-1}) + \sum_{n\geq 1} 0 \vee (f_{n,0} - f_{n-1,0}) + \sum_{m,n\geq 1}(f_{n,m} - f_{n,m-1})$$

$$f_- = 0 \vee (-f_{0,0}) + \sum_{n\geq 1} 0 \vee (f_{n-1,0} - f_{n,0}) + \sum_{m,n\geq 1}(f_{n-1,m} - f_{n-1,m-1}).$$

(i) Let $f_n \in L^1$ be a Cauchy sequence, i.e.,

$$\lim_{m\to\infty} \limsup_{n\to\infty} I(|f_m - f_n|) = 0.$$

By passing to a subsequence if necessary, Then we can find a subsequence $\{n_k\}_{k\geq 1}$ so that $I(|f_{n_k} - f_{n_{k+1}}|) \leq 1/2^k$ and and hence the limit function

$$f = \lim_{k\to\infty} f_{n_k} = f_{n_0} - \sum_{k=0}^{\infty}(f_{n_k} - f_{n_{k+1}})$$

is well-defined almost everywhere. Since we have

$$|f_n - f_{n_k}| \leq |f_n - f_{n_0}| + \sum_{k=0}^{\infty} |f_{n_k} - f_{n_{k+1}}| \in L^1,$$

the dominated convergence theorem, together with the property of Cauchy sequence gives

$$0 = \lim_{n\to\infty} I(|f_n - f_{n_k}|) = \lim_{n\to\infty} I(|f_n - f|).$$

□

Remark . As can be seen by the above argument, we have

$$L^1 \cap L_{\uparrow} = \{f; f_n \in L, f_n \uparrow f, \sup_{n\geq 1} I(f_n) < +\infty\}, \quad I(f_n) \uparrow I(f),$$

which suggests another approach to Lebesgue integration: If we adopt (ii) in the proposition as the definition of L^1, then the integration of $f = f_+ - f_-$ can be defined by $I(f) = I(f_+) - I(f_-)$. This is known as the F. Riesz' approach. By introducing the identitification of almost everwhere in an early stage of construction, we can reduce the limiting process to the minimum, .e., just once.

Theorem 9.13 (the law of large numbers)**.** * In a probability measure space (X, μ), suppose that a sequence $\{\xi_n\}_{n\geq 1}$ of integrable random variables satisfy the following conditions: (i) The sequence $\{\langle\xi_n\rangle\}_{n\geq 1}$ of expectations converges in the Cesaro's sense. (ii) Each variance $\sigma_n^2 = \langle(\xi_n - \langle\xi_n\rangle)^2\rangle$ is finite and the Cesaro mean of the variance sequence $\{\sigma_n^2\}_{n\geq 1}$ is bounded. (iii) Except for a null set, the sequence $\{\xi_n(x) - \langle\xi_n\rangle\}_{n\geq 1}$ is bounded.

Then the Cesaro mean of the sequence $\{\xi_n\}$ of random variables converges to a constant function almost everwhere.

$$\lim_{n\to\infty} \frac{\xi_1(x) + \cdots + \xi_n(x)}{n} = \lim_{n\to\infty} \frac{\langle\xi_1\rangle + \cdots + \langle\xi_n\rangle}{n} \quad (\mu\text{-a.e. } x \in X).$$

Proof. Set $\eta_n = \xi_n - \langle\xi_n\rangle$. We need to prove the following.

$$\lim_{n\to\infty} \frac{\eta_1(x) + \cdots + \eta_n(x)}{n} = 0 \quad (\mu\text{-a.e. } x \in X).$$

By the estimate

$$\begin{aligned}\frac{1}{n}\int(\eta_1(x) + \cdots + \eta_n(x))^2 \mu(dx) &= \frac{1}{n}\sum_{1\leq j,k\leq n}\int \eta_j(x)\eta_k(x)\,\mu(dx) \\ &= \frac{\sigma_1^2 + \cdots + \sigma_n^2}{n} \leq \sup_{n\geq 1}\frac{\sigma_1^2 + \cdots + \sigma_n^2}{n} < +\infty,\end{aligned}$$

we see

$$\sum_{k=1}^{\infty} \int \left(\frac{\eta_1(x) + \cdots + \eta_{k^2}(x)}{k^2} \right)^2 \mu(dx) < +\infty$$

and then

$$\sum_{k=1}^{\infty} \frac{1}{k^4} (\eta_1(x) + \cdots + \eta_{k^2}(x))^2 < +\infty \quad (\mu\text{-a.e. } x \in X).$$

Particularly,

$$\lim_{m\to\infty} \frac{\eta_1(x) + \cdots + \eta_{n^2}(x)}{m^2} = 0 \quad (\mu\text{-a.e. } x \in X).$$

For a general $n \geq 1$, choose $m \geq 1$ so that $m^2 \leq n < (m+1)^2$ and estimate as

$$\begin{aligned}
\left| \frac{\eta_1(x) + \cdots + \eta_n(x)}{n} \right| &\leq \left| \frac{\eta_1(x) + \cdots + \eta_n(x)}{m^2} \right| \\
&\leq \left| \frac{\eta_1(x) + \cdots + \eta_{m^2}(x)}{m^2} \right| + \frac{|\eta_{m^2+1}(x)| + \cdots + |\eta_n(x)|}{m^2} \\
&\leq \left| \frac{\eta_1(x) + \cdots + \eta_{m^2}(x)}{m^2} \right| + \frac{|\eta_{m^2+1}(x)| + \cdots + |\eta_{(m+1)^2}(x)|}{m^2} \\
&\leq \left| \frac{\eta_1(x) + \cdots + \eta_{m^2}(x)}{m^2} \right| + \frac{(m+1)^2 - m^2}{m^2} \sup_{k\geq 1} |\eta_k(x)|
\end{aligned}$$

and take the limit $n \to \infty$ $(m \to \infty)$. □

Example 9.14 (Borel's normal number theorem)**.** * Let $X = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ be a product-type compact metric space with $N \geq 2$ a natural number and μ be the probability measure on X associated to a probability distribution $\{p_j = 1/N\}_{0\leq j\leq N-1}$.

Let

$$X = \{0, 1, \ldots, N-1\}^{\mathbb{N}} \ni x = (x_k)_{k\geq 1} \mapsto \overline{x} = \sum_{k=1}^{\infty} N^{-k} x_k \in [0, 1]$$

be the measurable function associated to the N-digit expansion of real numbers: Given a sequence $d_1, d_2, \ldots, d_{m-1} \in \{0, 1, \ldots, N-1\}$ and $d_m \in \{0, 1, \ldots, N-2\}$, $x_k = d_k$ $(1 \leq k \leq m)$ if and only if

$$N^{-1}d_1 + \cdots + N^{-m}d_m \leq \overline{x} \leq N^{-1}d_1 + \cdots + N^{-m+1}d_{m-1} + N^{-m}(d_m + 1).$$

Cosider the random variables $\xi_k^{(j)} : X \to \{0, 1\}$ $(0 \leq j \leq N-1,\ k \geq 1)$ defined by

$$\xi_k^{(j)}(x) = \begin{cases} 1 & \text{if } x_k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle \xi_k^{(j)} \rangle = 1/N$

$$\langle f_k^{(j)} f_{k'}^{(j')} \rangle = \begin{cases} \frac{1}{N^2} & \text{if } k \neq k', \\ \frac{1}{N} & \text{if } k = k',\ j = j', \\ 0 & \text{if } k = k',\ j \neq j'. \end{cases}$$

$$\langle(\xi_k^{(j)} - 1/N)(\xi_{k'}^{(j')} - 1/N)\rangle = \begin{cases} 0 & \text{if } k = k', \\ \frac{N-1}{N^2} & \text{if } k = k',\ j = j', \\ -\frac{1}{N^2} & \text{if } k = k',\ j \neq j'. \end{cases}$$

Thus, for each $j \in \{0, 1, \dots, N-1\}$, the sequence $\{\xi_k^{(j)}\}_{k\geq 1}$ of random variables satisfies the assumption in the law of large numbers and therefore we have

$$\lim_{n\to\infty} \frac{\xi_1^{(j)}(x) + \cdots + \xi_n^{(j)}(x)}{n} = \frac{1}{N} \quad (\mu\text{-a.e. } x \in X).$$

Exercise 66. * Consider the meaning for $N = 10$.

10 Repeated Integration Formula

Integration systems are assumed to be σ-finite in this section.

Consider a surjection $\pi : \Omega \to X$. Suppose that we are given a σ-finie vector lattice F on Ω, an integration system (L, μ) on X and a family $\{(L_x, \mu_x)\}$ of integration systems indexed by $x \in X$ with (L_x, μ_x) based on $\pi^{-1}(x)$, which satisfy the following property: For a function $f \in F$,

(i) $\forall x \in X,\ f|_{\pi^{-1}(x)} \in L_x$ and

(ii) $\displaystyle\int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega)$ belongs to L as a function of x.

The triplet $(F, (L, \mu), \{(L_x, \mu_x)\})$ is then called a **fibered integration system**. By repeated use of monotone convergence theorem, we see that

$$I(f) = \int_X \mu(dx) \int_{\pi^{-1}(x)} \mu_x(d\omega)\, f(\omega)$$

defines an integral on F.

Let (L_X, I_X), (L_Y, I_Y) be (σ-finite) integration systems with the associated measures denoted by μ_X, μ_Y. Let S_X, S_Y be the vector lattices consisting of measurable simple functions which are integrable.

Also set

$$S_X \otimes S_Y = \left\{ \sum_{i=1}^n f_i \otimes g_i; f_i \in S_X, g_i \in S_Y \right\},$$

where $f \otimes g : X \times Y \to \mathbb{R}$ is the function defined by $(f \otimes g)(x, y) = f(x)g(y)$.

Proposition 10.1.

(i) The set $S_X \otimes S_Y$ is a vector lattice on the product set $X \times Y$.

(ii) If $f \in S_X \otimes S_Y$, $Y \mapsto f(x, y)$ is a function in S_Y for any $x \in X$ and its integration

$$\int_Y f(x, y)\, \mu_Y(dy)$$

with respect to I_Y belong to S_X as a function of x. Similarly with the role of X and Y intechanged.

(iii) For $f \in S_X \otimes S_Y$, we have

$$\int_X \left(\int_Y f(x,y)\, \mu_Y(dy) \right) \mu_X(dx) = \int_Y \left(\int_X f(x,y)\, \mu_X(dx) \right) \mu_Y(dy).$$

(iv) If we denote the value of the repteated integration in (iii) by $I(f)$, $I : S_X \otimes S_Y \to \mathbb{R}$ is an integration on $S_X \otimes S_Y$.

Exercise 67. Check the above assertions. Also supply an example for which $L_X \otimes L_Y$ is not closed under the lattice operation.

The fibered integration system obtained this way is referred to as a **repeated integration system**. Note here that we have two choices of repeated integration systems depending on the order of integrations.

Proposition 10.2. * For $L_X = C_c(X)$, $L_Y = C_c(Y)$ with X, Y locally compact metric spaces, we have the following inclusions.

$$C_c(X \times Y) \subset M(S_X \otimes S_Y).$$

Proof. As a metric on the product space $X \times Y$, we adopt

$$d(x, y; x', y') = \max\{d_X(x, x'), d_Y(y, y')\}.$$

Note that $B_r(x, y) = B_r(x) \times B_r(y)$.

Let $f \in C_c(X \times Y)$ and $K = [f]$ be the supprt of f. We first show the following:

$$\forall r > 0, \exists x_1, x_2, \dots, x_n \in K, \quad K \subset \bigcup_{j=1}^n B_r(x_j).$$

If this does not hold,

$$\exists r > 0, \forall \text{finite } F \subset K, \quad K \not\subset \bigcup_{y \in F} B_r(y),$$

whence we can find a sequence $\{x_j\}_{j \geq 1}$ in K so that

$$x_1 \in K, x_2 \notin B_r(x_1), x_3 \notin B_r(x_1) \cup B_r(x_2), \dots.$$

By the condition imposed, we have $d(x_i, x_j) \geq r$ $(1 \leq i < j)$ and therefore no subsequence of $\{x_j\}$ converges, which contradicts with the compactness of K.

By the fact just proved and the uniform continuity of the function f,

$$\forall \epsilon > 0, \exists \delta > 0, \exists x_1, \dots, x_n \in K, \forall x \in K, \exists i \geq 1, d(x, x_i) < \delta, |f(x) - f(x_i)| \leq \epsilon.$$

Now put $B_i = B_\delta(x_j)$ and expand $X = (B_1 \sqcup B_1^c) \cap \dots \cap (B_n \sqcup B_n^c)$, which yields the decomposition

$$X = \bigsqcup_{j=2}^{2^n} A_j \sqcup (B_1 \cup \dots \cup B_n)^c.$$

For each $2 \leq j \leq 2^n$, choose so that $a_j = x_i$ $(x_i \in A_j)$ and let

$$f_\epsilon(x) = \begin{cases} f(a_j) & \text{if } x \in A_j \ (2 \leq j \leq 2^n), \\ 0 & \text{if } x \in (B_1 \cup \dots \cup B_n)^c. \end{cases}$$

Then $f_\epsilon \in S_X \otimes S_Y$ and we obtain $\|f - f_\epsilon\|_\infty \leq \epsilon$. □

Theorem 10.3. Consider a fibered integration system $(\pi : \Omega \to X, F, (L, \mu), \{(L_x, \mu_x)\})$.

(i) Let $f \in M(F)^+_\uparrow$. Then $f|_{\pi^{-1}(x)} \in M(L_x)^+_\uparrow$ for any $x \in X$, the fibered integral

$$\int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega)$$

belongs to $M(L)^+_\uparrow$ as a function of x and the following holds.

$$I(f) = \int_X \left(\int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega) \right) \mu(dx).$$

(ii) For an integrable function $f \in M(F) \cap F^1$, there exists a null set $N \subset X$ such that $x \in X \setminus N$ implies $f|_{\pi^{-1}(x)} \in M(L_x) \cap L^1_x$, the function

$$X \setminus N \ni x \mapsto \int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega)$$

belongs to $M(L) \cap L^1$ and the following holds.

$$I(f) = \int_X \left(\int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega) \right) \mu(dx).$$

Proof. (i) By the σ-finiteness $1_\Omega \in M(F)^+$ of F, there exists a function $\phi \in F^+_\uparrow$ satisfying $1_\Omega \leq \phi$ (Lemma7.6 (iii)). If we choose a sequence $\phi_n \in F^+$ so that $\phi_n \uparrow \phi$ and set $\varphi_n = 1_\Omega \wedge \phi_n$, then $\varphi_n \in M(F) \cap F^1$ and $\varphi_n \uparrow 1_\Omega$.

Let M_n be the set of functions in $[0, \varphi_n]$ which meets the conclusion of the statement in (i), then M_n is a monotone class containing $F \cap [0, \varphi_n]$. In fact, it is closed under the limit of increasing sequences clearly. For the limit of decreasing sequences, we can reduce it to the case of increasing sequences by taking the difference of $\varphi_n \leq \phi_n$ with $\phi_n \in F^+$. Consequently, by Lemma7.6, we have $M_n \supset M(F) \cap [0, \varphi_n]$.

Next, for $f \in M(F)^+_\uparrow$, put $f_n = f \wedge \varphi_n$. Then $f_n \in M(F) \cap [0, \varphi_n]$ satisfies the conclusion in (i) and so does f by the monotone convergence theorem applied to $f_n \uparrow f$.

(ii) From the decomposition $f = f \vee 0 - (-f) \vee 0$, the problem is reduced to the case $f \in M(L)^+$, which follows from Proposition 9.7 (ii) and the fact

$$\left(\int_{\pi^{-1}(x)} f(\omega)\, \mu_x(d\omega) \right) \mu(dx) = I(f) < +\infty$$

obtained from (i). □

Example 10.4. Here is an example of fibered integration system: Let $\Omega = \mathbb{R}^2$, $X = \mathbb{R}$, $X = \mathbb{R}$, $\pi : \Omega \ni (t, x) \mapsto t \in\to X$, $F = C_c(\Omega)$, $L = C_c(X)$, and $L_t = C_c(\{t\} \times \mathbb{R}) \cong C_c(\mathbb{R})$ be an integration system on $\pi^{-1}(t) = \{t\} \times \mathbb{R} \cong \mathbb{R}$ specified by

$$\mu_t(dx) = \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx & \text{if } t > 0, \\ \delta(x) & \text{otherwise,} \end{cases} \qquad I(f) = \int_{-\infty}^{\infty} dt \int_{\mathbb{R}} f(t, x)\, \mu_t(dx).$$

Theorem 10.5 (Lebesgue-Fubini-Tonelli)**.** Consider the repeated integral I on $F = S_X \otimes S_Y$ in a repeated integration system.

(i) We have $M(L_X) \otimes M(L_Y) = M(S_X) \otimes M(S_Y) \subset M(F)$.

(ii) Let $f \in M(F)^+_\uparrow$. Then $f(x, \cdot) \in M(L_Y)^+_\uparrow$ for $x \in X$, $\int_Y f(x, y)\,\mu_Y(dy)$ belongs to $M(L_X)^+_\uparrow$ as a function of x, and the following holds.

$$I(f) = \int_X \mu_X(dx) \int_Y \mu_Y(dy) f(x, y).$$

Similar for the repeated inegration system for which the role of X and Y is interchanged.

(iii) Let $f \in M(F) \cap F^1$. Then there exists a null set $N_X \subset X$ such that $f(x, \cdot) \in M(L_Y) \cap L^1_Y$ for $x \in X \setminus N_X$, $\int_Y f(x, y)\,\mu_Y(dy)$ belong to $M(L_X) \cap L^1_X$ as a function of x and the following holds.

$$I(f) = \int_X \mu_X(dx) \int_Y \mu_Y(dy) f(x, y).$$

Similar for the repeated inegration system for which the role of X and Y is interchanged.

Corollary 10.6 (Reproducing Property of Lebesgue Measures)**.** Let $f \in L^1(\mathbb{R}^{m+n})$ be an integrable function. Then there exists a null set $N \subset \mathbb{R}^m$ such that $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for $x \in \mathbb{R}^m \setminus N$, the function

$$\mathbb{R}^m \setminus N \ni x \mapsto \int_{\mathbb{R}^n} f(x, y)\,dy$$

belongs to $L^1(\mathbb{R}^m)$ up to null functions and the following holds.

$$\int_{\mathbb{R}^{m+n}} f(x, y)\,dxdy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x, y)\,dy \right) dx.$$

Example 10.7.

$$\int_0^\infty \int_0^\infty e^{-t} e^{-tx^2}\,dtdx = \int_0^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2},$$

$$\int_0^\infty \int_0^\infty e^{-t} e^{-tx^2}\,dxdt = C \int_0^\infty e^{-t} \frac{1}{\sqrt{t}}\,dt = 2C^2.$$

Here

$$C = \int_0^\infty e^{-y^2}\,dy = \frac{1}{2} \int_0^\infty e^{-t} \frac{1}{\sqrt{t}}\,dt.$$

Exercise 68. Check that the repeated integration formula can be applied to the double integral ($t > 0$ being a parameter)

$$\int_t^\infty \int_0^\infty e^{-xy} \sin x\,dxdy$$

and derive the following.

$$\int_0^\infty e^{-tx} \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \arctan t.$$

Exercise 69. * For $\alpha \in \mathbb{R}$ and $\beta > 0$,

$$\int_{\mathbb{R}^n} \frac{e^{-|x|^\beta}}{|x|^\alpha}\, dx = \begin{cases} \frac{2\pi^{n/2}}{\beta\Gamma(n/2)}\Gamma\left(\frac{n-\alpha}{\beta}\right) & \text{if } \alpha < n, \\ +\infty & \text{if } \alpha \geq n. \end{cases}$$

Exercise 70. Determine the range of real α which satisfies

$$\int_0^1 \int_0^1 \frac{1}{|x-y|^\alpha}\, dxdy < +\infty.$$

Exercise 71. We have checked that, for an integrable function $f : \mathbb{R} \to \mathbb{C}$ and a positive real $a > 0$, the function

$$f_a(x) = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} f(x-t) e^{-at^2}\, dt$$

is integrable and analytic satisfying $\lim_{x\to\pm\infty} f_a(x) = 0$ (Exercise 43, Exercise 59). Show the following convergence here.

$$\lim_{a\to+\infty} \int_{-\infty}^{\infty} |f(x) - f_a(x)|\, dx = 0.$$