Projective Geometry and Symmetry in Physics

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Synopsis

This mini course aims at introducing participants to classical theorems due to E.P. Wigner and A.D. Alexandrov which concern how symmetries are described in the fields of quantum physics and special relativity respectively.

The main tool here is the fundamental theorem of projective geometry and we shall rely on the Faure's paper for its proof as well as that of the Wigner's theorem on quantum symmetry. The Alexandrov-Zeeman's theorem on special relativity is then derived following the steps organized by Vroegindewey.

All of these will convince us how simple linear algebraic arguments produce remarkable results.

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1 Affine Spaces

Vector spaces are assumed to be based on a common scalar field $\mathbb{K}$. An **affine space** is a set A on which a vector space V acts freely and transitively. The dimension of the vector space V is referred to as the dimension of the affine space A and denoted by $\dim A$.

It is customary to write the action by the symbol of addition: $a + v = v + a \in A$ if $a \in A$ and $v \in V$. Given two points a, b in A, the unique vector $\overrightarrow{ab}$ specified by $a + \overrightarrow{ab} = b$ is called the displacement vector from a to b. Displacement vectors satisfy the following properties:

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}, \quad \overrightarrow{ba} = -\overrightarrow{ab}.$$

Note that, once a reference point o (called the origin) is chosen in A, A is identified with V by the correspondence $v \mapsto o + v$, i.e., a vector space V itself is an affine space by translation.

By an **affine map** from an affine space (A, V) into another affine space (A', V') we shall mean a map $\phi : A \to A'$ for which the correspondence $\overrightarrow{ab} \mapsto \overrightarrow{\phi(a)\phi(b)}$ gives a well-defined linear map f of V into V'. If ϕ is bijective, it is called an isomorphism of affine spaces and two affine spaces are said to be **isomorphic**. When relevant affine spaces are furnished with reference points, an affine map takes the form

$$V \ni v \mapsto o + v \mapsto \phi(o + v) - o' = f(v) + \phi(o) - o' \in V',$$

which is a combination (i.e., composition) of the linear map f and the translation by $\phi(o) - o' \in V'$.

Exercise 1. Two affine spaces are isomorphic if and only if they have the same dimension. Consequently any finite-dimensional affine space A is isomorphic to $\mathbb{K}^n$ ($n = \dim A$).

A subset S of an affine space A is called an affine subspace if displacement vectors $\overrightarrow{pq}$ for various $p, q \in S$ constitute a linear subspace of V. Thus an affine subspace itself is an affine space with the displacement vector space being a linear subspace of V.

Exercise 2. Given a subset S of A, let W be the linear subspace spanned by $\{\overrightarrow{pq}; p, q \in S\}$ and set $\langle S \rangle = \{p + w; p \in S, w \in W\}$. Show that $\langle S \rangle$ is a minimal affine subspace containing S.

According to the dimensionality of affine subspaces, one-dimensional affine subspaces are referred to as **lines**, two-dimensional affine subspaces as **planes** and $(n-1)$-dimensional affine subspaces as **hyperplanes**.

Affine subspaces bear a lattice structure by taking $E \cap F$ as a meet and $E \vee F = \langle E \cup F \rangle$ as a join. Then the usual dimension formula

$$\dim(E \vee F) = \dim E + \dim F - \dim(E \cap F).$$

holds under the condition that $E \cap F \neq \emptyset$.

Note that the dimension formula becomes complicated for multiple operations due to the intersection condition. This kind of defects disappear if one passes into projective spaces.

2 Projective Spaces

The **projective space** $\mathbb{P}(V)$ associated to a $\mathbb{K}$-vector space V is the set of one-dimensional subspaces of V. There is an obvious surjection $V^\times = V \setminus \{0\} \ni v \mapsto [v] = \mathbb{K}v \in \mathbb{P}(V)$ and $\mathbb{P}(V)$ is identified with the quotient space $V^\times/\mathbb{K}^\times$. If $f : X \to V$ is a linear injection, then it induces an injective map $[f]$ on $\mathbb{P}(X)$ by $\mathbb{P}(X) \ni \mathbb{K}x \mapsto \mathbb{K}f(x) \in \mathbb{P}(V)$. In particular, if X is a subspace of V, $\mathbb{P}(X) \subset \mathbb{P}(V)$. A subset of the form $\mathbb{P}(X)$ with $X \subset V$ is called a **projective subspace** of $\mathbb{P}(V)$ with the **dimension** of $\mathbb{P}(X)$ defined to be $\dim X - 1$. Projective subspaces of dimension one and two are also referred to as **projective lines** and **projective** planes respectively.

Given a subset S of $\mathbb{P}(V)$, $\mathbb{P}(X)$ with $X = \sum_{[x]\in S} \mathbb{K}x$ is a minimal projective subspace containing S. Projective subspaces are therefore identified with the Grassmannian $\mathbb{G}(V)$ of V. As in the case of affine subspaces, it bears the lattice structure by $\mathbb{P}(X) \wedge \mathbb{P}(Y) = \mathbb{P}(X \cap Y)$ and $\mathbb{P}(X) \vee \mathbb{P}(Y) = \mathbb{P}(X + Y)$ for which the dimension formula holds without qualifications:

$$\dim \mathbb{P}(X + Y) = \dim \mathbb{P}(X) + \dim \mathbb{P}(Y) - \dim \mathbb{P}(X \cap Y).$$

Here is a simple relation to affine spaces: In the obvious set-theoretical decomposition

$$\mathbb{P}(V \oplus \mathbb{K}) = [V \oplus 1] \sqcup \mathbb{P}(V),$$

the subset $[V \oplus 1]$ is identified with the affine space V by $V \ni\mapsto [v \oplus 1]$ so that any affine line $a + \mathbb{K}v$ in V is extended to the unique projective line $\ell = [a + \mathbb{K}v \oplus 1] \vee [v \oplus 0]$ in $\mathbb{P}(V \oplus \mathbb{K})$ which is not contained in $\mathbb{P}(V)$. Conversely any projective line in $\mathbb{P}(V \oplus \mathbb{K}) \setminus \mathbb{P}(V)$ which intersects with $[V \oplus 1]$ is of this form. Elements in $\mathbb{P}(V)$ can be regarded as the limit points at infinity of affine lines in V.

Returning to projective spaces, a subset S of $\mathbb{P}(V)$ is said to be **collinear** if S is contained in a projective line (i.e., a one-dimensional projective subspace) of $\mathbb{P}(V)$. Consider a map $\varphi : \mathbb{P}(V) \to \mathbb{P}(V')$, which is **collinear** in the sense that it maps collinear subsets into collinear subsets, i.e., it satisfies $\varphi(p\vee q) \subset \varphi(p)\vee\varphi(q)$ for $p, q \in \mathbb{P}(V)$. Repeated use of the collinearity shows $\varphi(p\vee q\vee r) \subset \varphi(p)\vee\varphi(q)\vee\varphi(r)$ and so on. If the collinearity on φ is strengthened to the condition $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$ for $p, q \in \mathbb{P}(V)$, it is said to be **strongly collinear**.

Clearly the map $[f]$ induced from an injective linear map $f : V \to V'$ is strongly collinear. To get a collinear map, the linearity on f can be weakened to the following one: An additive map $f : V \to V'$ is said to be **semilinear** if we can find a map $\sigma : \mathbb{K} \to \mathbb{K}$ so that $f(\lambda v) = \sigma(\lambda)f(v)$ for $\lambda \in \mathbb{K}$ and $v \in V$. Note that, for a non-trivial semilinear map $f \neq 0$, σ is uniquely determined by f and called a twist, which turns out to be a unit-preserving ring homomorphism. Since $\mathbb{K}$ is a field, any such σ is injective with the image $\sigma(\mathbb{K})$ a subfield of $\mathbb{K}$.

Given an injective semilinear map $f : V \to V'$, it induces a map $[f] : \mathbb{P}(V) \to \mathbb{P}(V')$ by $[f]([v]) = [f(v)]$ ($v \in V \setminus \{0\}$), which is obviously collinear. If the associated twist σ is an automorphism of $\mathbb{K}$, then $[f]$ is injective and strongly collinear.

Lemma 2.1. If $\{\varphi(p), \varphi(q), \varphi(r)\}$ with $p, q, r \in \mathbb{P}(V)$ is not collinear, φ is injective on the projective plane $p \vee q \vee r$. When φ is strongly collinear, φ gives a bijection between $p \vee q \vee r$ and $\varphi(p) \vee \varphi(q) \vee \varphi(r)$.

Proof. In fact, given two points $a \neq b$ in $p \vee q \vee r$, choose a point $c \in p \vee q \vee r$ so that $p \vee q \vee r = a \vee b \vee c$. Any point $x \in p \vee q \vee r$ satisfies $x \in c \vee d$ for some $d \in a \vee b$. Thus $\varphi(a) = \varphi(b)$ implies $\varphi(x) \in \varphi(c) \vee \varphi(d)$ with $\varphi(d) \in \varphi(a) \vee \varphi(b) = \varphi(a)$. Consequently $\varphi(p), \varphi(q), \varphi(r) \in \varphi(c) \vee \varphi(a)$ are collinear, a contradiction.

If φ is strongly collinear, a realization of a point $x' \in p' \vee q' \vee r'$ ($p' = \varphi(p)$, $q' = \varphi(q)$, $r' = \varphi(r)$) as $x' \in p' \vee s'$ with $s' \in q' \vee r'$ is traced back to find $s' = \varphi(s)$ ($s \in q \vee r$) and $x' = \varphi(x)$ ($x \in p \vee s$); the restriction $\varphi : p \vee q \vee r \to \varphi(p) \vee \varphi(q) \vee \varphi(r)$ is surjective. □

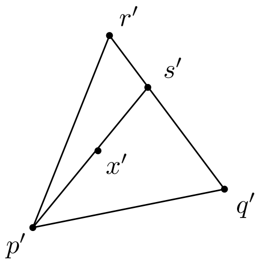

Figure1

Corollary 2.2. Assume that $\varphi(\mathbb{P}(V))$ is not collinear. If two points p, q of $\mathbb{P}(V)$ satisfy $\varphi(p) \neq \varphi(q)$, the restriction $\varphi : p \vee q \to \varphi(p) \vee \varphi(q)$ is injective. When φ is strongly collinear, it is bijective.

Proof. By assumption, we can find a point $r \in \mathbb{P}(V)$ so that $\{\varphi(p), \varphi(q), \varphi(r)\}$ is not collinear. □

In what follows, the image $\varphi(\mathbb{P}(V))$ (of a collinear map φ) is assumed to be not collinear to avoid singular situations.* Here is our main theorem.

Theorem 2.3. Any collinear map $\varphi : \mathbb{P}(V) \to \mathbb{P}(V')$ with a non-collinear image is of the form $\varphi = [f]$ with $f : V \to V'$ an injective semilinear map. Moreover, f is uniquely determined by φ up to scalar multiplication.

Corollary 2.4 (Fundamental Theorem of Projective Geometry)**.** Assume that $\dim V \geq 3$ and let $\varphi : \mathbb{P}(V) \to \mathbb{P}(V)$ be a collinear bijection with a collinear inverse φ^{-1}. Then it is of the form $[f]$, where $f : V \to V$ a semilinear bijection with the twist σ given by an automorphism of $\mathbb{K}$.

Proof. Since φ^{-1} is collinear, we can find injective semilinear maps g with a twist τ so that $\varphi^{-1} = [g]$. Clearly $[fg] = [f][g]$ and $[gf] = [g][f]$ are the identity on $\mathbb{P}(V)$ and the uniqueness of liftings shows that fg and gf are scalar multiples of the identity on V. Let $fg = \mu 1_V$ with $0 \neq \mu \in \mathbb{K}$. Then

$$\mu\lambda v = (fg)(\lambda v) = f(\tau(\lambda)g(v)) = \sigma(\tau(\lambda))(fg)(v) = \sigma(\tau(\lambda))\mu v$$

for $\lambda \in \mathbb{K}$ and $v \in V$ shows that $\sigma\tau$ is the identity and similarly for $\tau\sigma$. Thus σ is an automorphism of $\mathbb{K}$. □

Remark 1. Both of φ and φ^{-1} in the above corollary are strongly collinear.

* If $\varphi(\mathbb{P}(V))$ is collinear, collinearity loses its control over φ and there appear lots of strange collinear maps.

For a later use, we give here an affine version of the fundamental theorem. Consider a bijection $\phi : V \to V$ such that both of ϕ and its inverse ϕ^{-1} map affine lines onto affine lines; for an affine line ℓ, both of $\phi(\ell)$ and $\phi^{-1}(\ell)$ are affine lines.

Lemma 2.5. Assume that $\mathbb{K} \neq \mathbb{Z}_2$. Let ℓ_1, ℓ_2 be two affine lines in V. Then ℓ_1 and ℓ_2 are parallel if and only if so are $\phi(\ell_1)$ and $\phi(\ell_2)$.

Proof. If not, we have two parallel lines in V which are mapped into skew lines. With suitable choices of affine coordinates for initial and final spaces, we may assume that two parallel lines are

$$\{(\lambda, 0, 0, \dots, 0); \lambda \in \mathbb{K}\}, \quad \{(\lambda, 1, 0, \dots, 0); \lambda \in \mathbb{K}\},$$

which are mapped under ϕ into a pair of skewlines

$$\{(\lambda, 0, 0, 0, \dots, 0); \lambda \in \mathbb{K}\}, \quad \{(0, \lambda, 1, 0, \dots, 0); \lambda \in \mathbb{K}\}$$

respectively.

Choose two points in each of skewlines with four points in total, couple them into two pairs of points $\{\alpha_1, \beta_1\}$, $\{\alpha_2, \beta_2\}$ and try to find a crossing point $\ell_{\alpha_1,\beta_1} \cap \ell_{\alpha_2,\beta_2}$. In the above coordinates, $\alpha_j = (a_j, 0, \dots, 0)$ and $\beta_j = (0, b_j, 1, 0, \dots, 0)$ with $a_1 \neq a_2$ and $b_1 \neq b_2$. Then lines ℓ_{α_j,β_j} are parametrized by

$$s(a_1, 0, \dots, 0) + (1 - s)(0, b_1, 1, 0, \dots, 0), \quad t(a_2, 0, \dots, 0) + (1 - t)(0, b_2, 1, 0, \dots, 0)$$

respectively and, by equating them, we see that there are no such solutions.

On the other hand, in the case of parallel lines, a similar computation shows that there is a solution if and only if $a_1 - a_2 \neq b_1 - b_2$, which is the case if $\mathbb{K}$ contains more than two elements.

Thus we can find four points $\{\alpha_j, \beta_j; j = 1, 2\}$ so that ℓ_{α_1,α_2} and ℓ_{β_1,β_2} are parallel and $\ell_{\alpha_1,\beta_1} \cap \ell_{\alpha_2,\beta_2} \neq \emptyset$. Then $\phi(\ell_{\alpha_1,\beta_1}) \cap \phi(\ell_{\alpha_2,\beta_2}) \neq \emptyset$ whereas $\ell_{\phi(\alpha_1),\phi(\beta_1)} \cap \ell_{\phi(\alpha_2),\phi(\beta_2)} = \emptyset$, which contradicts with $\phi(\ell_{\alpha_j,\beta_j}) = \ell_{\phi(\alpha_j),\phi(\beta_j)}$ $(j = 1, 2)$. □

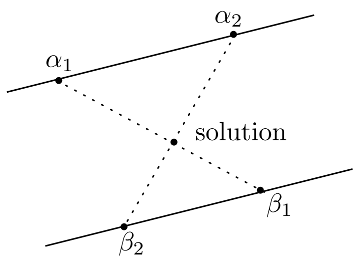

Figure2 Crossing

Corollary 2.6. A collinear bijection $\varphi : \mathbb{P}(V) \to \mathbb{P}(V)$ with a collinear inverse φ^{-1} is well-defined by $\varphi([v]) = [v']$, where v and v' are related by $\phi(a + \mathbb{K}v) = \phi(a) + \mathbb{K}v'$ for any $a \in V$.

Proof. The bijection φ is well-defined because ϕ induces a bijection on directional vectors of affine lines.

To see the collinearity of φ, let $[u]$ be a point in a projective line $[v] \vee [w]$ in $\mathbb{P}(V)$. Then the affine line $\mathbb{K}u$ is contained in the union of affine lines $\mathbb{K}v + \mathbb{K}w = \cup_{a \in \mathbb{K}v, b \in \mathbb{K}w} \ell_{a,b}$, whence $\phi(0) + \mathbb{K}u' = \phi(\mathbb{K}u)$ is

a subset of
$$\bigcup_{a\in\mathbb{K}v,b\in\mathbb{K}w}\ell_{\phi(a),\phi(b)}\subset\phi(0)+\mathbb{K}v'+\mathbb{K}w'$$
in view of $\phi(a)\in\phi(\mathbb{K}v)=\phi(0)+\mathbb{K}v'$ and $\phi(b)\in\phi(\mathbb{K}w)=\phi(0)+\mathbb{K}w'$. Consequently $u'\in\mathbb{K}v'+\mathbb{K}w'$, i.e., $\varphi([u])=[u']\in[v']\vee[w']=\varphi([v])\vee\varphi([w])$. □

The collinear bijection $\varphi:\mathbb{P}(V)\to\mathbb{P}(V)$ is now extended to a collinear bijection ψ of $\mathbb{P}(V\oplus\mathbb{K})\supset\mathbb{P}(V)$ by
$$\mathbb{P}(V\oplus\mathbb{K})\setminus\mathbb{P}(V)=[V\oplus 1]\ni[v\oplus 1]\mapsto[\phi(v)\oplus 1]\in[V\oplus 1]=\mathbb{P}(V\oplus\mathbb{K})\setminus\mathbb{P}(V).$$
Since affine lines in V correspond exactly to projective lines not contained in $\mathbb{P}(V)$, the assumption on ϕ together with the collinearity of φ and φ^{-1} on $\mathbb{P}(V)$ shows that ψ and ψ^{-1} are collinear on $\mathbb{P}(V\oplus\mathbb{K})$.

The fundamental theorem of projective geometry is now applied to ψ to find a semilinear bijection $f:V\oplus\mathbb{K}\to V\oplus\mathbb{K}$ such that $\psi=[f]$ with an automorphic twist σ. Choose a linear base $\{e_i\}$ of V and let $h:V\oplus\mathbb{K}\to V\oplus\mathbb{K}$ be a σ^{-1}-semilinear bijection defined by $h(\sum_i\lambda_ie_i\oplus\lambda)=\sum_i\sigma^{-1}(\lambda_i)e_i\oplus\sigma^{-1}(\lambda)$. Clearly h preserves $V\oplus 0$ and the composition hf is a $\mathbb{K}$-linear isomorphism which makes $V\oplus 0$ invariant. Consequently, hf is of the form
$$hf=\begin{pmatrix}A & b\\ 0 & \lambda\end{pmatrix}\quad\text{with}\quad A\in\mathrm{GL}(V),\ a\in V,\ 0\neq\lambda\in\mathbb{K}.$$
Thus ϕ as the restriction of f to $V\oplus 1\cong V$ is of the form
$$\phi(v)=g(v)+a,$$
where $a\in V$ and $g:V\to V$ is a σ-semilinear bijective transformation of V.

Theorem 2.7. Assume that $\dim V\geq 2$ with $\mathbb{K}\neq\mathbb{Z}_2$. Then any bijection $\phi:V\to V$ which maps affine lines onto affine lines is of the form described above.

Exercise 3. Investigate what happens when $\dim V=1$ or $\mathbb{K}=\mathbb{Z}_2$.

3 Proof of the Main Theorem

The following proof is taken from [Faure] but you can find similar arguments in [Artin] and [Baer].

The idea of lifting φ to f is simple. Choose any point $[a]\in\mathbb{P}(V)$ as a reference and let $a'\in\varphi([a])$ be a representative vector. If $[v]\in\mathbb{P}(V)$ satisfies $\varphi([v])\neq[a']$, then φ is injective on the line $[a]\vee[v]$ and the collinear relation $[a']\in\varphi([a+v])\vee\varphi([v])$ with $[a']\notin\{\varphi([a+v]),\varphi([v])\}$ enables us to find a unique $0\neq v'\in\varphi([v])$ satisfying $\varphi([a+v])=[a'+v']$ (the midpoint positioning), which we shall set $v'=f(v)$.

To deal with the case $\varphi([v])=[a']$, we use a second reference point $[b]$ satisfying $\varphi([b])\neq\varphi([a])$. To make the definition of f compatible we need to choose a representative vector b' of $\varphi([b])$ so that $\varphi([b+v])=[b'+f(v)]$ when $\varphi([v])\notin\{[a'],[b']\}$.

To understand the situation, let us introduce a terminology: Given points p,q such that $\varphi(p)\neq\varphi(q)$, their representatives $p=[a]$, $q=[b]$, $\varphi(p)=[a']$ and $\varphi(q)=[b']$ are said to be **compatible** if $\varphi([a+b])=[a'+b']$. With this terminology, $f(v)$ is specified by the compatibility of $\{a,v,a',f(v)\}$ and

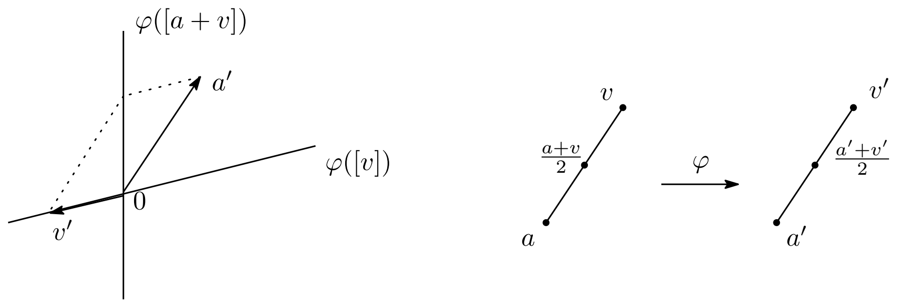

Figure3 Midpoint Positioning

the compatibility in defining f relative to another reference pair $\{b, b'\}$ is equivalent to the compatibility of $\{b, v, b', f(v)\}$.

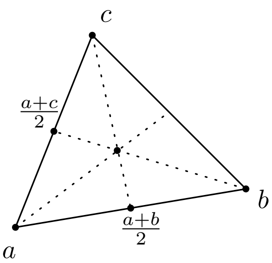

Figure4 Positioned Triangle

Lemma 3.1. Assume that $\varphi([a])$, $\varphi([b])$, $\varphi([c])$ are not collinear and let $a', b', c' \in V'$ be their representatives respectively. If $\{a, b, a', b'\}$ and $\{a, c, a', c'\}$ are compatible, then so is $\{b, c, b', c'\}$ and we have $\varphi([a+b+c]) = [a'+b'+c']$.

Proof. Notice that $\{a, b, c\}$ and $\{a', b', c'\}$ are linearly independent, whence $a+b+c \neq 0$ and $a'+b'+c' \neq 0$.

We first claim the barycentric equality $\varphi([a+b+c]) = [a'+b'+c']$. In fact, by collinearity, $[a+b+c] \in ([a+b] \vee [c]) \cap ([a+c] \vee [b])$ implies

$$\begin{aligned} \varphi([a+b+c]) &\in \varphi\Big(([a+b] \vee [c]) \cap ([a+c] \vee [b])\Big) \\ &\subset \varphi([a+b] \vee [c]) \cap \varphi([a+c] \vee [b]) \\ &\subset ([a'+b'] \vee [c']) \cap ([a'+c'] \vee [b']) \\ &= \{[a'+b'+c']\}. \end{aligned}$$

To see the third compatibility, collinearity is applied to $[b+c] \in ([b] \vee [c]) \cap ([a] \vee [a+b+c])$ to have

$$\varphi([b+c]) \in ([b'] \vee [c']) \cap ([a'] \vee [a'+b'+c']) = \{[b'+c']\}$$

and we are done. $\square$

Thus the compatibility in defining f requires the compatibility of $\{a, b, a', b'\}$, which in turn together with the compatibility of $\{a, v, a', f(v)\}$ is enough to ensure the compatibility of $\{b, v, b', f(v)\}$.

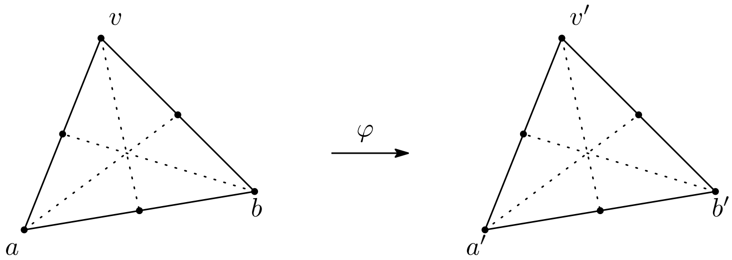

Figure5 Triangular Positioning

Here is a summary of discussions so far: Given a compatible representatives $\{a, b, a', b'\}$, a lifting map $f : V \to V'$ is well-defined by the relation $\varphi([a+v]) = [a' + f(v)]$ or $\varphi([b+v]) = [b' + f(v)]$ if $v \neq 0$ and $f(v) = 0$ if $v = 0$. Note that, from the definition, $a' = f(a)$ and $b' = f(b)$.

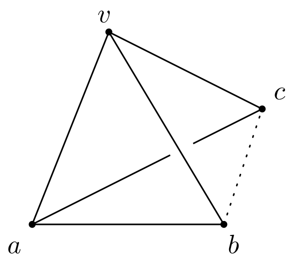

Figure6 Tetrahedral Positioning

Lemma 3.2. The map $f : V \to V'$ is additive.

Proof. To see this, we first strengthen our reference system one step further: Choose a third point $c \in \mathbb{P}(V)$ so that $\{\varphi([a]), \varphi([b]), \varphi([c])\}$ is not collinear and set $c' = f(c)$. Then, the representatives $\{a, b, c, a', b', c'\}$ are mutually compatible and the barycentric equality $\varphi([a+b+c]) = [a'+b'+c']$ holds by Lemma 3.1.

We claim that $\varphi([c+v]) = [c'+f(v)]$ whenever $\varphi([v]) \neq [c']$. In fact, $\{[a'], [c'], \varphi([v])\}$ or $\{[b'], [c'], \varphi([v])\}$ is not collinear by non-collinearity of $\{[a'], [b'], [c']\}$ and the assertion follows from $\varphi([a+v]) = [a'+f(v)]$ or $\varphi([b+v]) = [b'+f(v)]$, together with $\varphi([a+c]) = [a'+c']$ or $\varphi([b+c]) = [b'+c']$, in view of Lemma 3.1.

To see the additivity $f(v+w) = f(v) + f(w)$, we may assume that $v, w \in V \setminus \{0\}$ and discuss two cases separately.

(i) $\varphi[v]) \neq \varphi([w])$: Choose one of reference points $\{a, b, c\}$, say c, so that $[c'] \notin \varphi([v]) \vee \varphi([w])$. From the compatibility of $\{c, v, c', f(v)\}$ and $\{c, w, c', f(w)\}$, we have the barycentric equality $\varphi([c+v+w]) = [c'+f(v)+f(w)]$ for the triangle $\{[c'], \varphi([v]), \varphi([w])\}$, which is compared with $\varphi([c+v+w]) = [c'+f(v+w)]$ to get $f(v+w) = f(v) + f(w)$.

(ii) $\varphi([v]) = \varphi([w])$: At least one element in $\{a, b\}$, say a, satisfies $\varphi([v]) \neq \varphi([a])$. When $v + w \neq 0$, $\varphi([v+w]) = \varphi([v]) \neq \varphi([a])$ falls into the independent case (i) and we obtain $f(a+v+w) = f(a)+f(v+w)$, which remains valid for $v + w = 0$ obviously. On the other hand, in view of the injectivity of φ on the line $[a] \vee [v]$, the independent case (i) again works for $\varphi([a]) \neq \varphi([v])$ and $\varphi([a+v]) \neq \varphi([v]) = \varphi([w])$ to get $f(a+v+w) = f(a+v) + f(w) = f(a) + f(v) + f(w)$. $\square$

Now the proof of the theorem is completed if one checks the following.

Lemma 3.3. Let $f : V \to V'$ be an additive map whose image contains two linearly independent vectors.

(i) If $g : V \to V'$ is an additive map satisfying $g(v) \in \mathbb{K}f(v)$ for every $v \in V$, then we can find $\lambda \in \mathbb{K}$ so that $g(v) = \lambda f(v)$ $(\forall v \in V)$.

(ii) If f satisfies $f(\mathbb{K}v) \subset \mathbb{K}f(v)$ for every $v \in V$, then f is semilinear.

Proof. (i) We need to show that the $\mathbb{K}$-valued function λ_v on $V \setminus \ker f$ defined by $g(v) = \lambda_v f(v)$ is constant.

If $f(v)$ and $f(w)$ are linearly independent, $f(v + w) = f(v) + f(w) \neq 0$ and

$$\lambda_v f(v) + \lambda_w f(w) = g(v) + g(w) = g(v + w) = \lambda_{v+w} f(v + w) = \lambda_{v+w} f(v) + \lambda_{v+w} f(w)$$

implies $\lambda_v = \lambda_{v+w} = \lambda_w$. When $\mathbb{K}f(v) = \mathbb{K}f(w)$, choose $a \in V$ so that $f(a) \notin \mathbb{K}f(v) = \mathbb{K}f(w)$ (here the assumption $\dim \mathbb{K}f(V) \geq 2$ being used) and the constancy in the independent case is used twice to get $\lambda_v = \lambda_a = \lambda_w$.

(ii) To see the semilinearity of f, for $\lambda \in \mathbb{K}$, an additive map defined by $g(v) = f(\lambda v)$ meets the condition in (i) and there exists an element $\sigma(\lambda) \in \mathbb{K}$ satisfying the semilinearity. □

4 Wigner's Theorem

Let V be a complex vector space with a positive definite inner product $(\ |\ )$. For two points $[v], [w]$ in $\mathbb{P}(V)$, the transition probability between them is defined by

$$P_{[v],[w]} = \frac{|(v|w)|^2}{(v|v)(w|w)}.$$

Note that $P_{[v],[w]}$ is in the range $[0, 1]$ (the Schwarz' inequality). Two points are then said to be **orthogonal** and denoted by $[v] \perp [w]$ if $P_{[v],[w]} = 0$, i.e., if $(v|w) = 0$. Given a subset S of $\mathbb{P}(V)$, its orthogonal complement is defined by $S^\perp = \{p \in \mathbb{P}(V); p \perp q, \forall q \in S\}$, which is always a projective subspace. It is immediate to see that $S \subset T$ implies $T^\perp \subset S^\perp$ and $(S^\perp)^\perp \supset S$. When $S = \mathbb{P}(X)$ with $X \subset V$ a linear subsapce, $\mathbb{P}(X)^\perp = P(X^\perp)$ and the condition $(S^\perp)^\perp = S$ is equivalent to $X = (X^\perp)^\perp$. Consequently, $(\mathbb{P}(X)^\perp)^\perp = \mathbb{P}(X)$ for any finite-dimensional X thanks to the Gram-Schmidt orthogonalization.

Theorem 4.1. Suppose that $\dim V \geq 3$. Let $\varphi : \mathbb{P}(V) \to \mathbb{P}(V)$ be an orthogonality-preserving bijection, i.e., $\varphi(p) \perp \varphi(q)$ if and only if $p \perp q$. Then we can find a unitary or an antiunitary operator f on V so that $\varphi = [f]$.

Proof. Since φ preserves orthogonality, $\varphi(S^\perp) = \varphi(S)^\perp$ for any subset S of $\mathbb{P}(V)$. In view of the expression $p \vee q = ((p \vee q)^\perp)^\perp = (p^\perp \cap q^\perp)^\perp$,

$$\varphi(p \vee q) = \varphi(p^\perp \cap q^\perp)^\perp = (\varphi(p)^\perp \cap \varphi(q)^\perp)^\perp = \varphi(p) \vee \varphi(q)$$

shows that φ is strongly collinear. Thus, under the assumption that $\dim \mathbb{P}(V) \geq 2$ (i.e., $\dim V \geq 3$), we can find a semilinear bijection $f : V \to V$ with twist σ satisfying $\varphi = [f]$.

Let $v, w \in V$ be orthogonal unit vectors. Then, for $\lambda \in \mathbb{R}$, $\lambda v + w$ and $v - \lambda w$ are non-zero orthogonal vectors as well, whence

$$0 = (f(\lambda v + w)|f(v - \lambda w)) = (\sigma(\lambda)f(v) + f(w)|f(v) - \sigma(\lambda)f(w)) = \overline{\sigma(\lambda)}(f(v)|f(v)) - \sigma(\lambda)(f(w)|f(w))$$

implies, in view of $(f(v)|f(v)) > 0$ and $(f(w)|f(w)) > 0$, that $\sigma(\lambda) \in \mathbb{R}$ and $(f(v)|f(v)) = (f(w)|f(w))$. Thus σ gives an endomorphism of $\mathbb{R}$ and $(f(v)|f(v)) = (f(w)|f(w))$.

Since the identity is the only unit-preserving endomorphism of $\mathbb{R}$, we see that σ is the identity or the complex conjugation. Moreover $\|f\|^2 = (f(v)|f(v))$ does not depend on the choice of a unit vector v: If v' is another unit vector, we can find a unit vector w so that $(v|w) = 0 = (v'|w)$ thanks to $\dim V \geq 3$, which is utilized to conclude that $(f(v)|f(v)) = (f(w)|f(w)) = (f(v')|f(v'))$.

Replacing f with $\|f\|^{-1}f$, f turns out to be a norm-preserving linear or conjugate-linear isomosrphism of V. □

Corollary 4.2 (Wigner's Theorem on Quantum Symmetry)**.** Suppose that $\dim V \geq 3$. Let $\varphi : \mathbb{P}(V) \to \mathbb{P}(V)$ be a bijection which preserves transition probability. Then we can find a unitary or an antiunitary operator f on V so that $\varphi = [f]$.

Exercise 4. Show that the identity is the only unit-preserving endomorphism of $\mathbb{R}$.

5 Minkowski Space

By a Minkowski vector space, we shall mean an $(n+1)$-dimensional real vector space V furnished with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature $(1, n)$. A **Minkowski space** is then an affine space based on a Minkowski vector space. To simplify the notation, V is used to express the associated Minkowski space. Physically, the Minkowski space for $n = 3$ is used to describe the spacetime from various inertia systems. The geometrical cone $C_a = \{x \in V; (x - a, x - a) = 0\}$ then expresses the trajectory of lights which are emitted or absorbed at a spacetime point $a \in V$. We shall regard a Minkowski space as a coned affine space in what follows. An affine line $\ell = a + \mathbb{R}v$ in V is called a **light line** if $(v, v) = 0$, i.e., if $\ell \subset C_a$ for some point $a \in V$. Here are standard notations for groups related with the inner product space V:

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V); (gv, gw) = (v, w), \forall v, w \in V\}, \quad \mathrm{SO}(V) = \{g \in \mathrm{O}(V); \det(g) = 1\}.$$

One-dimensional subspaces of V are classified into three portions according to the signature of the restricted inner product. The group $\mathrm{SO}(V)$ as well as $O(V)$ acts transitively on each of three; $\mathbb{P}(V)/\mathrm{SO}(V) = \mathbb{P}(V)/O(V)$ is a three-point set.

Two-dimensional subspaces of V are also classified into three similarly: If the restricted inner product is degenerate, it is negative semidefinite (parabolic). If the restricted inner product is non-degenerate, it is indefinite (hyperbolic) or negative definite (elliptic). These are distinguished by observing how many light directions are there: An elliptic subspace contains no light directions, a parabolic subspace contains one light direction, and a hyperbolic subspace contains two light directions. Note that a parabolic subspace is tangential to the cone C_0.

Exercise 5. Check and picturize these.

By a **light automorphism** of V, we shall mean a bijection $\phi : V \to V$ satisfying $\phi(C_a) = C_{\phi(a)}$ for any $a \in V$. Clearly affine automorphisms of V which preserve cones fall into this class.

Theorem 5.1 (Alexandrov)**.** Assume $\dim V \geq 3$. Then any light automorphism is an affine map. Thus the coned space structure is enough to recover the affineness in Minkowski spaces.

Remark 2. The theorem is also referred to as Alexandrov-Ovchinnikova's because the full proof appeared in their joint paper first and was later proved independently by E.C. Zeeman.

Lemma 5.2. Let $v, w \in V \setminus \{0\}$ satisfy $(v, v) = 0 = (v, w)$. Then $(w, w) \leq 0$ and the equality holds if and only if w is proportional to v.

Proof. In a coordinate system, $v_0^2 = \mathbf{v}^2 > 0$ and $v_0 w_0 = \mathbf{v} \cdot \mathbf{w}$ ($\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and the dot denotes the canonical inner product in $\mathbb{R}^n$). By the inner product inequality,

$$v_0^2 w_0^2 = (\mathbf{v} \cdot \mathbf{w})^2 \leq \mathbf{v}^2 \mathbf{w}^2 = v_0^2 \mathbf{w}^2,$$

whence $(w, w) = w_0^2 - \mathbf{w}^2 \leq 0$ and the equality holds if and only if $\mathbf{w} = \lambda \mathbf{v}$ with $\lambda \in \mathbb{R}$. Then $v_0 w_0 = \mathbf{v} \cdot \mathbf{w} = \lambda \mathbf{v}^2 = \lambda v_0^2$ and therefore $w = \lambda v$. □

Corollary 5.3. Let $a \neq b$. Then the following are equivalent.

(i) $a \vee b$ is a light line, i.e., $(a - b, a - b) = 0$.
(ii) $b \in C_a$.
(iii) $a \vee b = C_a \cap C_b$.

With this characterization of light lines, a light automorphism ϕ maps light lines onto light lines.

Proof. (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is clear.

Assume that $(a-b, a-b) = 0$ and, for $c \in C_a \cap C_b$, set $v = c - a$ and $w = c - b$. Then $(v, v) = 0 = (w, w)$ and $(v - w, v - w) = (b - a, b - a) = 0$, whence $(v, w) = 0$. Therefore v and w are proportional, showing $c \in a \vee b$. □

Lemma 5.4. Let ℓ and m be light lines satisfying $\ell \cap m = \emptyset$. Then $l \nparallel m$ if and only if there exists exactly one point $p \in \ell$ satisfying $C_p \cap m = \emptyset$.

When $\ell \parallel m$, we have the following alternatives.

(i) $C_p \cap m \not= \emptyset$ for any $p \in \ell$. In this case, $\ell \vee m$ is hyperbolic.
(ii) $C_p \cap m = \emptyset$ for any $p \in \ell$. In this case, $\ell \vee m$ is parabolic.

Proof. Let $\ell = a + \mathbb{R}v$ and $m = b + \mathbb{R}w$ with $(v, v) = 0 = (w, w)$. Note that $\ell \nparallel m$ if and only if $(v, w) \neq 0$.

We seek for a point $p = a + sv$ such that $(p - (b + tw), p - (b + tw)) = 0$ has no solutions for $t \in \mathbb{R}$. The equation is of the form

$$0 = (p - b - tw, p - b - tw) = (p - b, p - b) - 2t(p - b, w),$$

which admits a solution if and only if $(p-b,w) \neq 0$ or $(p-b,w) = 0 = (p-b,p-b)$. Since the latter condition implies the proportionality of $p-b$ and w, $p \in b+\mathbb{R}w$, which contradicts with $\ell \cap m = \emptyset$. Thus there is no solution for the choice $(p-b,w) = 0$, i.e., $s = (b-a,w)/(v,w)$.

Now let $v = w$. Then $C_p \cap m = \emptyset$ if and only if $(p-b,v) = 0 \iff (a-b,v) = 0$. By Lemma, this implies $(a-b,a-b) \leq 0$. If $(a-b,a-b) = 0$, $a-b$ is proportional to v, which contradicts with $\ell \cap m = \emptyset$. Thus $(a-b,a-b) < 0$ and $(a-b,v)$, which is equivalent to the parbolicity of the subspace $\mathbb{R}(a-b)+\mathbb{R}v$.

Otherwise, we can find $p \in \ell$ satisfying $C_p \cap m \neq \emptyset \iff 0 \neq (p-b,v)$ which is equivalent to $0 \neq (a-b,v)$. Thus, if this is the case, $C_p \cap m$ is a one-point set for any $p \in \ell$ in view of the equation

$$0 = (p-b-tv, p-b-tv) = (a-b,a-b) + 2(s-t)(a-b,v)$$

for $t \in \mathbb{R}$. Since $\ell \vee m$ admits two light directions inside, it is hyperbolic. $\square$

Corollary 5.5. A light automorphism ϕ maps parallel light lines onto parallel light lines.

Lemma 5.6. A light automorphism ϕ maps hyperbolic planes onto hyperbolic planes.

Proof. Consider a hyperbolic plane in V generated by two (intersecting) non-parallel light lines ℓ_j ($j = 1,2$). Since two non-parallel light lines $\phi(\ell_j)$ intersects at a point in $\phi(\ell_1 \cap \ell_2)$, they generate a hyperbolic plane.

Since any point $p \in \ell_1 \vee \ell_2$ is expressed as an intersect of two light lines $\ell'_j \subset \ell_1 \vee \ell_2$ ($j = 1,2$) parallel to ℓ_j respectively, $\phi(p) = \phi(\ell'_1) \wedge \phi(\ell'_2)$. Since ℓ'_j intersects with ℓ_{3-j}, so does $\phi(\ell'_j)$ with $\phi(\ell_{3-j})$. Since $\phi(\ell'_j)$ is parallel to $\phi(\ell_j)$, $\phi(\ell'_j) \subset \phi(\ell_1) \vee \phi(\ell_2)$ and therefore $\phi(p) \in \phi(\ell_1) \vee \phi(\ell_2)$.

Thus $\phi(\ell_1 \vee \ell_2) \subset \phi(\ell_1) \vee \phi(\ell_2)$. By replacing ϕ with ϕ^{-1} and ℓ_j with $\phi(\ell_j)$, we obtain the reverse inclusion $\phi^{-1}(\phi(\ell_1) \vee \phi(\ell_2)) \subset \ell_1 \vee \ell_2$, showing that $\phi(\ell_1 \vee \ell_2) = \phi(\ell_1) \vee \phi(\ell_2)$ is a hyperbolic plane. $\square$

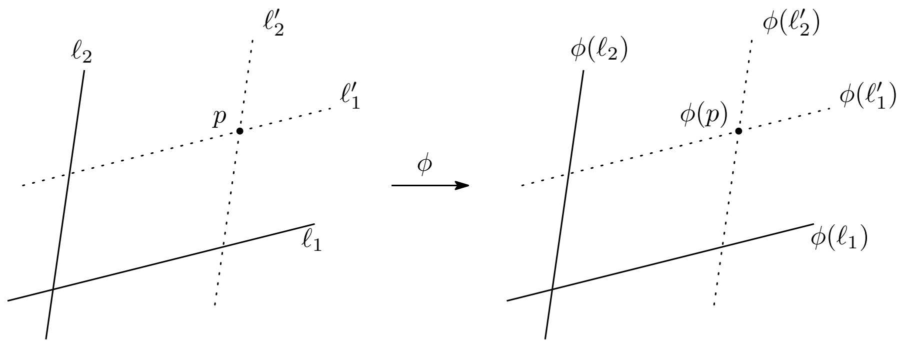

Figure7 Parallelogram Positioning

Now the following ends up the proof of Alexandrov's theorem thanks to the affine space version of the fundamental theorem of projective geometry.

Lemma 5.7. A light automorphism ϕ maps affine lines onto affine lines.

Proof. Let ℓ be an affine line in V. Since $\dim V \geq 3$, any one-dimensional subspace is expressed as an intersection of two hyperbolic planes containing 0 and we can find hyperbolic planes H_j ($j = 1,2$) so that $\ell = H_1 \cap H_2$. By the previous lemma, $\phi(H_j)$ are again planes. Since $\phi(H_1) \neq \phi(H_2)$ follows from $H_1 \neq H_2$, their intersection $\phi(H_1) \cap \phi(H_2) = \phi(H_1 \cap H_2) = \phi(\ell)$ is an affine subspace of V containing at least two points and different from $\phi(H_j)$; $\phi(\ell)$ is an affine line in V. $\square$

Remark 3. An affine automorphism ϕ of V preserves cones if and only if the linear transformation part of ϕ is a scalar multiplication of an element in $\mathrm{O}(V)$.

Exercise 6. Let a linear transformation $T : V \to V$ satisfy $T(C_0) \subset C_0$. Then $T = \lambda g$ with $\lambda \in \mathbb{R}$ and $g \in \mathrm{O}(V)$.

The arguments so far are essentially algebraic and the scalar field can be relaxed to any commutative field if one admits the conclusion in Lemma 5.2 as an assumption: $(v, v) = 0 = (w, w)$ and $(v, w) = 0$ imply linear dependence of $\{v, w\}$.

The original formulation of Alexandrov as well as Zeeman is based on the order structure of real numbers in the form of so-called causality. Consider an open subset $D = \{v \in V; (v, v) > 0\}$ in V, which has two connected components with respect to the euclidean topology. The causality is then specified by choosing a connected component D^+ of D: Events at $a \in V$ have effect on those at $b \in V$ only if $b-a \in \overline{D^+}$. Here $\overline{D^+}$ denotes the closure of D^+. Since $\overline{D^+}$ is a convex cone satisfying $\overline{D^+} \cap (-\overline{D^+}) = \{0\}$, the **causality** is equally described by the order reltion $a \leq b$ in D which is defined by $b - a \in \overline{D^+}$. A Minkowski space is called **causal** if it is furnished with a causality.

By a **light causality**, we shall mean a choice C^+ from the connected components of $C_0 \setminus \{0\}$. Clearly there is a one-to-one correspondence between a causality $\overline{D^+}$ and a light causality by the relation $C^+ \cup \{0\} = C_0 \cap \overline{D^+}$.

From the viewpoint of causality, the trajectry of lights emitted at $a \in V$ is given by $C_a^+ = \{b \in V; b-a \in C^+\}$, not $C_a \setminus \{0\}$. It is therefore reasonable to impose on ϕ the condition that $b \in C_a^+$ if and only if $\phi(b) \in C_{\phi(a)}^+$, which menas that lights can be emitted at $a \in V$ and absorbed at $b \in V$ if and only if the statement holds by replacing a, b by $\phi(a)$, $\phi(b)$ respectively. Since $C_a \setminus \{a\} = C_a^+ \cup \{b \in V; a \in C_b^+\}$, the bijection ϕ necessarily satisfies $\phi(C_a) = C_{\phi(a)}$. In this sense, ϕ may be called a causal light automorphism, whereas causal automorphism is reserved for ϕ which keeps the order relation $a \leq b$.

For the causality on massive objects, we can also think of a finer order relation $a \prec b$, which is defined by $b - a \in D^+ \cup \{0\}$.

The massive and light causalities can produce each other in a simple manner:

Lemma 5.8 (Zeeman)**.** The causality relation $a \leq b$ is equivalent to each of the following conditions.

(i) For $c \in V$, $b \prec c$ implies $a \prec c$.
(ii) There exists $c \in V$ such that $c - a, b - c \in C^+ \cup \{0\}$; two-step light causality is nough to get the whole causality.

Corollary 5.9. If a bijection $\phi : V \to V$ and its inverse[†] preserve the order relation $\prec$, then they preserve light cones.

[†] One-way implication is enough to have the reverse implication, see [Cacciafesta].

Proof. Use the expression $C_a^+ = \{b \in V; a \leq b, a \not\asymp b\}$ to see $\phi(C_a^+) = C_{\phi(a)}^+$. $\square$

Conversely, massive and light causalities follow from a causality. To see this, let us introduce **causal interval**[‡] by $[a,b] = \{c \in V; a \leq c \leq b\}$ for $a \leq b$.

Lemma 5.10 (Benz)**.** The causal interval $[a,b]$ is linearly ordered if and only if $(a-b, a-b) = 0$.

Proof. By translation, we may assume that $a = 0$. Express $b = (b_0, b_1, 0, \dots, 0)$ in a coordinate system. Then $c = (c_0, \dots, c_n) \in V$ belongs to $[0,b]$ if and only if

$$\sqrt{c_1^2 + \cdots + c_n^2} \leq c_0 \leq b_0 - \sqrt{(c_1 - b_1)^2 + c_2^2 + \cdots + c_n^2}.$$

When $0 \not\leq b$, we have $b_0 \leq 0$ in a suitable coordinate system and the above inequalities are reduced to

$$c_1^2 + \cdots + c_n^2 = c_0 = b_0 = (c_1 - b_1)^2 + c_2^2 + \cdots + c_n^2 = 0 \iff b_j = c_j = 0 \ \forall j.$$

When $0 \leq b$ and $(b,b) = 0$, we may further assume that $b_0 = b_1 = 1$ and the condition on c has an expression

$$\sqrt{c_1^2 + \cdots + c_n^2} \leq c_0 \leq 1 - \sqrt{(c_1 - 1)^2 + c_2^2 + \cdots + c_n^2},$$

which is reduced to solving the inequality

$$\sqrt{x^2 + \lambda^2} \leq 1 - \sqrt{(x-1)^2 + \lambda^2} \quad \text{with} \quad \lambda^2 = c_2^2 + \cdots + c_n^2.$$

In view of

$$1 - \sqrt{(x-1)^2 + \lambda^2} < 1 - |x-1| \leq |x| < \sqrt{x^2 + \lambda^2}$$

for $\lambda \neq 0$, the inequality in question has no solutions unless $c_2 = \cdots = c_n = 0$, whereas for the choice $\lambda = 0$ the inequality $|x| \leq 1 - |x-1|$ is equivalent to $0 \leq x \leq 1$ with the equality $|x| = 1 - |1-x|$ satisfied. Thus, the condition $c \in [a,b]$ is equivalent to $c_0 = c_1$ for $0 \leq c_1 \leq 1$, i.e., $[0,b]$ is the line interval connecting 0 and b.

When $0 \leq b$ and $(b,b) > 0$, we may assume both of $b_1 = 0$ and $b_0 > 0$ in a suitable coordinate system and $c \in [a,b]$ satisfies the inequalities

$$\sqrt{c_1^2 + \cdots + c_n^2} \leq c_0 \leq b_0 - \sqrt{c_1^2 + c_2^2 + \cdots + c_n^2}.$$

If $c_1 = \cdots = c_n = 0$ and c is different from $\{a,b\}$, then c belongs to the open region $\sqrt{c_1^2 + \cdots + c_n^2} < c_0 < b_0 - \sqrt{c_1^2 + c_2^2 + \cdots + c_n^2}$ and there are plenty of $c' \in [a,b]$ satisfying $(c - c', c - c') < 0$. Otherwise, we may assume that $c_1 \neq 0$ and we seek for a point of the form $c' = (c_0, c_1 + \epsilon, c_2, \dots, c_n)$, which clearly satisfies $(c - c', c - c') < 0$ for $\epsilon \neq 0$ with the condition $c' \in [a,b]$ given by

$$\sqrt{c_1^2 + \cdots + c_n^2 + 2c_1\epsilon + \epsilon^2} \leq c_0 \leq b_0 - \sqrt{c_1^2 + \cdots + c_n^2 + 2c_1\epsilon + \epsilon^2}.$$

In view of the inequalities satisfied by c, the above inequalities for c' remain valid for small ϵ as far as $c_1\epsilon < 0$. $\square$

[‡] We may call it causal diamond in view of its geometric shape.

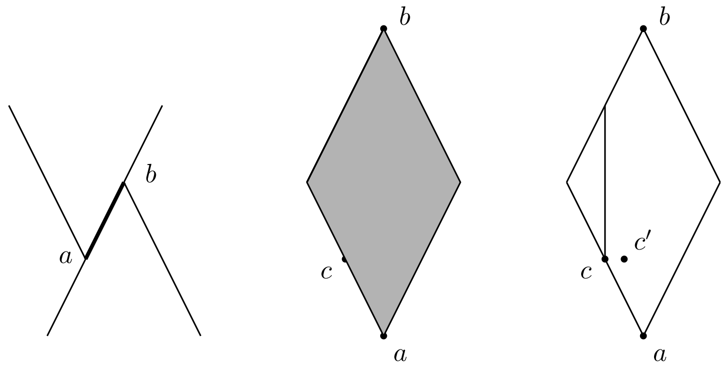

Figure8 Causal Interval

Corollary 5.11. Let V be a causal Minkowski space. If a bijection $\phi : V \to V$ preserves the causality (i.e., the order relation $\leq$), then it preserves the light causality as well. Consequently, such a bijection is composed of a translation, a chronological Lorentz transformation and a scalar dilation by a positive real.