OPERATOR ALGEBRAS AND THEIR REPRESENTATIONS

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Contents

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1. Algebras and Representations

By a ***-algebra**, we shall mean an algebra $\mathcal{A}$ over the complex number field $\mathbb{C}$ which is furnished with a distinguished conjugate linear transformation $* : \mathcal{A} \to \mathcal{A}$ (called a *-operation) satisfying

$$(a^*)^* = a \quad (ab)^* = b^*a^*, \quad a, b \in \mathcal{A}.$$

Exercise 1. If $\mathcal{A}$ has a unit $1_{\mathcal{A}}$ (i.e., $\mathcal{A}$ is **unital**), then $1^* = 1$.

An element a in a *-algebra $\mathcal{A}$ is said to be **hermitian** if $a^* = a$. A hermitian element p is called a **projection** if $p^2 = p$. When $\mathcal{A}$ has a unit 1, a is said to be **unitary** if $aa^* = a^*a = 1$.

A *-algebra is said to be **unitary**[1] if it is generated by unitaries.

Example 1.1. Given a *-algebra $\mathcal{A}$, the $n \times n$ matrix algebra $M_n(\mathcal{A})$ with entries in $\mathcal{A}$ is a *-algebra.

Example 1.2. Let $\mathbb{C}[X]$ be the polynomial algebra of indeterminate X and make it into a *-algebra by $(\sum_{n\geq 0} a_n X^n)^* = \sum_{n\geq 0} \overline{a_n} X^n$. Then 0 and 1 are all the projections and constant polynomials of modulus 1 are all the unitaries.

Example 1.3. Given a group G, the free vector space $\mathbb{C}G = \sum_{g\in G} \mathbb{C}g$ generated by elements in G is a *-algebra (called a group algebra) by extending the group product to the algebra multipliction and defining the *-operation so that elements in G are unitary. The group algebra $\mathbb{C}G$ is unitary.

Exercise 2. Let $\mathcal{A}$ be the vector space of functions of finite support on a group G and make it into a *-algebra (convolution algebra) by

$$(ab)(g) = \sum_{g'g''=g} a(g')b(g''), \quad a^*(g) = \overline{a(g^{-1})}.$$

The convolution algebra $\mathcal{A}$ of G is naturally isomorphic to the group algebra $\mathbb{C}G$.

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[1] Warning: This is not a common usage of terminology.

Given *-algebras $\mathcal{A}$ and $\mathcal{B}$, their direct sum $\mathcal{A}\oplus\mathcal{B}$ and tensor product $\mathcal{A}\otimes\mathcal{B}$ are again *-algebras in an obvious manner.

Exercise 3. The matrix algebra $M_n(\mathcal{A})$ is naturally identified with the tensor product $M_n(\mathbb{C})\otimes\mathcal{A}$.

Let $\mathcal{H}$ be a pre-Hilbert space; $\mathcal{H}$ is a complex vector space with a positive definite inner product $(\ |\ )$. A linear operator $T:\mathcal{H}\to\mathcal{H}$ is called the **adjoint** of a linear operator $S:\mathcal{H}\to\mathcal{H}$ (and denoted by S^*) if it satisfies $(\xi|S\eta)=(T\xi|\eta)$ (for $\xi,\eta\in\mathcal{H}$). A linear operator S on $\mathcal{H}$ is sadi to be bounded (on the unit ball) if $\|S\|=\sup\{\|S\xi\|;\xi\in\mathcal{H},\|\xi\|\leq 1\}$ is finite. Let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$ having adjoints, which is a unital *-algebra in an obvious way. The subset $\mathcal{B}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ consisting of bounded operators is a *-subalgebra. When $\mathcal{H}$ is complete, a linear operator on $\mathcal{H}$ has an adjoint if and only if it is bounded thanks to the closed graph theorem and the Riesz lemma, whence $\mathcal{L}(\mathcal{H})=\mathcal{B}(\mathcal{H})$.

By a ***-representation** of a *-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$, we shall mean an algebra-homomorphism $\pi:\mathcal{A}\to\mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^*=\pi(a^*)$ for $a\in\mathcal{A}$. When $\|\pi(a)\|<\infty$ for every $a\in\mathcal{A}$, π is said to be **bounded**. If $\mathcal{A}$ is unitary, any *-representation is automatically bounded, i.e., $\pi(\mathcal{A})\subset\mathcal{B}(\mathcal{H})$. Two *-representations $\pi_i:\mathcal{A}\to\mathcal{L}(\mathcal{H}_i)$ $(i=1,2)$ are said to be **unitarily equivalent** if we can find a unitary map $T:\mathcal{H}_1\to\mathcal{H}_2$ satisfying $T\pi_1(a)=\pi_2(a)T$ $(a\in\mathcal{A})$.

It is often convenient to regard the representation space $\mathcal{H}$ as a left $\mathcal{A}$-module by $a\xi=\pi(a)\xi$. A right $\mathcal{A}$-module structure then corresponds to a *-antirepresentation, i.e., an algebra-antihomomorphism $\pi:\mathcal{A}\to\mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^*=\pi(a^*)$, by the relation $\xi a=\pi(a)\xi$. A pre-Hilbert space $\mathcal{H}$ is called an $\mathcal{A}$-$\mathcal{B}$ bimodule ($\mathcal{B}$ being another *-algebra) if we are given a *-representation $\lambda:\mathcal{A}\to\mathcal{L}(\mathcal{H})$ and a *-antirepresentation $\rho:\mathcal{B}\to\mathcal{L}(\mathcal{H})$ satisfying $\lambda(a)\rho(b)=\rho(b)\lambda(a)$ for $a\in\mathcal{A}$ and $b\in\mathcal{B}$, i.e., $(a\xi)b=a(\xi b)$ in the module notation. An $\mathcal{A}$-$\mathcal{A}$ bimodule $\mathcal{H}$ is called a ***-bimodule** if we are given an antiunitary [2] involution ξ^* on $\mathcal{H}$ satisfying $(a\xi b)^*=b^*\xi^*a^*$ for $a,b\in\mathcal{A}$ and $\xi\in\mathcal{H}$.

A linear functional φ on a *-algebra $\mathcal{A}$ is defined to be **positive** if $\varphi(a^*a)\geq 0$ for $a\in\mathcal{A}$. A positive linear functional φ on a unital *-algebra $\mathcal{A}$ is called a **state** if $\varphi(1_{\mathcal{A}})=1$ ($1_{\mathcal{A}}$ being the unit element of $\mathcal{A}$). A linear functional τ on an algebra $\mathcal{A}$ is called a **trace** or said to be **tracial** if $\tau(ab)=\tau(ba)$ for $a,b\in\mathcal{A}$.

[2] A conjugate-linear operator J on a pre-Hilbert space $\mathcal{H}$ is called an antiunitary if it satisfies $(J\xi|J\eta)=(\eta|\xi)$ and $J\mathcal{H}=\mathcal{H}$.

Example 1.4. Given a *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a vector $\xi \in \mathcal{H}$, $\varphi(a) = (\xi|\pi(a)\xi)$ gives a positive linear functional. When ξ is a unit vector, φ is a state. This kind of state is called a vector state.

Exercise 4. Given a bounded *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a sequence $\{\xi_n\}_{n\geq 1}$ of vectors in $\mathcal{H}$ satisfying $\sum_{n\geq 1}(\xi_n|\xi_n) = 1$, observe that

$$\varphi(a) = \sum_{n=1}^{\infty}(\xi_n|\pi(a)\xi_n)$$

defines a state on $\mathcal{A}$. By specializing to $\mathcal{A} = M_2(\mathbb{C})$, recognize the difference between states of this form and vector states.

Example 1.5. Let $\mathcal{C}_0(\mathcal{H})$ be the set of finite rank operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{C}_0(\mathcal{H})$ is a *-ideal of $\mathcal{B}(\mathcal{H})$ and the ordinary trace defines a positive tracial functional tr on $\mathcal{C}_0(\mathcal{H})$.

Example 1.6. Every probability measure μ on the real line of finite moments defines a state on the polynomial algebra $\mathbb{C}[X]$ by

$$\varphi\left(\sum_n a_n X^n\right) = \sum_n a_n \int_{\mathbb{R}} t^n \mu(dt).$$

Conversely, any state arises in this way (the existence part of the Hamburger moment problem). See §X.1 in Reed-Simon for more information.

Example 1.7. In the group algebra $\mathbb{C}G$, positive linear functionals φ are one-to-one correspondence with positive definite functions on G by restriction and linear extension. The state associated to the positive definition function

$$\delta(g) = \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{otherwise} \end{cases}$$

is called the standard trace.

Exercise 5. The standard trace δ has the trace property: $\delta(ab) = \delta(ba)$ for $a, b \in \mathbb{C}G$.

Given a positive linear functional φ on a *-algebra $\mathcal{A}$, we define a *-representation as follows: The inner product $(a|b) = \varphi(a^*b)$ on $\mathcal{A}$ is positive semidefinite and the representation space is given by the associated pre-Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}$ is the quotient vector space relative to the kernel of $(\ |\ )$. The non-degenerate inner product on the quotient space is also denoted by $(\ |\ )$, whereas the quotient vector of $x \in \mathcal{A}$ in $\mathcal{H}$ is denoted by $x\varphi^{1/2}$. The inner product then looks like $(x\varphi^{1/2}|y\varphi^{1/2}) = \varphi(x^*y)$ and we introduce a representation π by

$\pi(a)(x\varphi^{1/2}) = (ax)\varphi^{1/2}$, which is well-defined in view of the Schwarz inequality

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \quad a, b \in \mathcal{A}.$$

In fact, if x is in the kernel of the inner product,

$$\varphi(x^*a^*ax) \leq \varphi(x^*a^*aa^*ax)^{1/2}\varphi(x^*x)^{1/2} = 0$$

shows that ax is in the kernel as well. Moreover, the relation

$$(x\varphi^{1/2}|\pi(a)y\varphi^{1/2}) = \varphi(x^*ay) = (\pi(a^*)x\varphi^{1/2}|y\varphi^{1/2})$$

shows that $\pi(a)^* = \pi(a^*)$, whence π is a *-representation.

The representation obtained in this way is referred to as the **GNS-representation** or its process as the GNS-construction. When $\mathcal{A}$ is unital, we have a distinguished vector $\varphi^{1/2} = 1_{\mathcal{A}}\varphi^{1/2}$ in the representation space, which is **cyclic** with respect to π in the sense that $\mathcal{H} = \pi(\mathcal{A})\varphi^{1/2}$.

Conversely, if we are given a *-representation $(\pi, \mathcal{H})$ of a *-algebra $\mathcal{A}$ and a cyclic vector $\xi \in \mathcal{H}$ for π, the formula $\varphi(a) = (\xi|\pi(a)\xi)$ defines a positive linear functional and the associated GNS-representation is unitarily equivalent to the initial one by the unitary map $a\varphi^{1/2} \mapsto \pi(a)\xi$ $(a \in \mathcal{A})$.

Remark 1. GNS is named after I.M. Gelfand, M.A. Naimark and I.E. Segal.

As a simple application of GNS-representation, we can show that tensor products of positive functionals are again positive: Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras and $\varphi : \mathcal{A} \to \mathbb{C}$, $\psi : \mathcal{B} \to \mathbb{C}$ be positive. Then

$$\begin{aligned}(\varphi \otimes \psi)(\sum_j a_j \otimes b_j)^*(\sum_k a_k \otimes b_k)) &= \sum_{j,k} \varphi(a_j^*a_k)\psi(b_j^*b_k)\\ &= (\sum_j a_j\varphi^{1/2} \otimes b_j\psi^{1/2}|\sum_k a_k\varphi^{1/2} \otimes b_k\psi^{1/2}) \geq 0.\end{aligned}$$

This can be also seen from the fact that entry-wise multiplications of positive matrices are again positive.

Example 1.8. Given a positive trace τ on a *-algebra $\mathcal{A}$, the associated GNS-representation space $\mathcal{A}\tau^{1/2}$ is made into a *-bimodule by $(a\tau^{1/2})^* = a^*\tau^{1/2}$ $(a \in \mathcal{A})$.

Proposition 1.9. Let ω be a positive functional on a unitary algebra $\mathcal{A}$ with $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ the associated GNS-representation. Then the following formula gives a one-to-one correspondence between positive functionls ω_T on $\mathcal{A}$ majorized by ω and positive operators T in the commutant $\pi(\mathcal{A})' = \{T \in \mathcal{B}(\mathcal{H}); T\pi(a) = \pi(a)T,\ \forall a \in \mathcal{A}\}$ majorized by the identity operator 1_H.

$$\omega_T(a) = (T\omega^{1/2}|\pi(a)\omega^{1/2}), \quad a \in \mathcal{A}.$$

Proof. Let φ be majorized by ω, i.e., $\varphi(a^*a) \leq \omega(a^*a)$ for $a \in \mathcal{A}$. Then by Schwarz inequality

$$|\varphi(x^*y)| \leq \varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} \leq \omega(x^*x)^{1/2}\omega(y^*y)^{1/2} = \|x\omega^{1/2}\|\,\|y\omega^{1/2}\|,$$

we see that $x\omega^{1/2} \times y\omega^{1/2} \mapsto \varphi(x^*y)$ gives a bounded sesquilinear form on the completed Hilbert space $\mathcal{H}$, whence we can find a bounded linear operator T on $\mathcal{H}$ satisfying

$$\varphi(x^*y) = (x\omega^{1/2}|T(y\omega^{1/2})) \quad x, y \in \mathcal{A}.$$

By equating $\varphi(x^*(ay))$ and $\varphi((a^*x)^*y)$, we have $T \in \pi(\mathcal{A})'$. Furthermore, the condition $0 \leq \varphi(a^*a) \leq \omega(a^*a)$ means the operator inequality $0 \leq T \leq 1_{\mathcal{H}}$.

The converse implication is immediate and the proof is left to the reader. $\square$

Definition 1.10. Given a vector η in a Hilbert space $\mathcal{H}$, the linear functional $\eta^* : \mathcal{H} \to \mathbb{C}$ is defined by $\eta^*(\xi) = (\eta|\xi)$ for $\xi \in \mathcal{H}$. By Riesz lemma, the dual space $\mathcal{H}^*$ of $\mathcal{H}$ is of the form $\mathcal{H}^* = \{\eta^*; \eta \in \mathcal{H}\}$ and it is a Hilbert space by the inner product $(\xi^*|\eta^*) = (\eta|\xi)$. The *-algebra $\mathcal{B}(\mathcal{H})$ then naturally acts on $\mathcal{H}^*$ from the right by $\eta^*a = (a^*\eta)^*$. For $\xi, \eta \in \mathcal{H}$, define a rank one operator $\xi\eta^* \in \mathcal{C}_0(\mathcal{H})$ by[3]

$$(\xi\eta^*)\zeta = (\eta|\zeta)\xi, \quad \zeta \in \mathcal{H}.$$

The notation is compatible with the multiplications by elements in $\mathcal{B}(\mathcal{H})$: $a(\xi\eta^*)b = (a\xi)(\eta^*b)$.

Example 1.11. Let tr be the ordinary trace on the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$. Then the correspondence $\xi\eta^*\mathrm{tr}^{1/2} \mapsto \xi \otimes \eta^*$ gives rise to a unitary map from the GNS-representation space $\overline{\mathcal{C}_0(\mathcal{H})\mathrm{tr}^{1/2}}$ onto $\mathcal{H} \otimes \mathcal{H}^*$.

Example 1.12. The GNS-representation associated to the state on $\mathbb{C}[X]$ realized by a probability measure μ on $\mathbb{R}$ is identified with the multiplication operator by polynomial functions on the Hilbert space $L^2(\mathbb{R}, \mu)$.

[3]According to Dirac, $\xi\eta^*$ is often denoted by $|\xi)(\eta|$.

Example 1.13. The GNS-representation of the standard trace of a group algebra $\mathbb{C}G$ is identified with the regular representation of G:

$$(a\delta^{1/2}|b\delta^{1/2}) = \delta(a^*b) = \sum_{g\in G} \overline{a_g} b_g \quad \text{for } a = \sum_{g\in G} a_g g,\ b = \sum_{g\in G} b_g g.$$

By the trace property of δ, the representation space $\ell^2(G)$ is a *-bimodule of $\mathbb{C}G$.

When G is commutative, $\ell^2(G)$ is unitarily mapped onto $L^2(\widehat{G})$ ($\widehat{G}$ being the Pontryagin dual of G) with the representation of $\mathbb{C}G$ unitarily transformed into the multiplication operator on $L^2(\widehat{G})$ given by the function

$$\widehat{G} \ni \omega \mapsto \sum_{g\in G} a_g \langle g, \omega\rangle \quad \text{for } a = \sum_{g\in G} a_g g \in \mathbb{C}G.$$

Exercise 6. For $G = \mathbb{Z}$, identify $\widehat{\mathbb{Z}}$ with $\mathbb{T}$ and the unitary map $\ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ with the Fourier expansion.

On a *-algebra $\mathcal{A}$, introduce a seminorm $\| \cdot \|_{C^*}$ by

$$\|a\|_{C^*} = \sup\{\|\pi(a)\|; \pi \text{ is a bounded *-representation}\},$$

which satisfies

$$\|ab\|_{C^*} \leq \|a\|_{C^*}\, \|b\|_{C^*}, \quad \|a^*a\|_{C^*} = \|a\|_{C^*}^2.$$

Exercise 7. Check the following: $\|a^*\|_{C^*} = \|a\|_{C^*}$ for $a \in \mathcal{A}$ and $\{a \in \mathcal{A}; \|a\|_{C^*} = 0\}$ is a *-ideal of $\mathcal{A}$.

Example 1.14. If G is commutative, we shall see later that $\|a\|_{C^*} = \|\pi(a)\|$ for $a \in \mathbb{C}G$, where π is the regular representation of G. The condition is known to be equivalent to the so-called amenability of G.

2. Gelfand Theory

Definition 2.1. An algebra A is called a **Banach algebra** if it is furnished with a complete norm satisfying $\|ab\| \leq \|a\|\,\|b\|$ for $a, b \in A$. A Banach algebra A is called a Banach *-algebra if it is further equipped with a *-operation satisfying $\|a^*\| = \|a\|$ for $a \in A$. A Banach *-algebra A is called a **C*-algebra** if the norm satisfies $\|a^*a\| = \|a\|^2$ for $a \in A$.

The completion of the quotient *-algebra $\mathcal{A}/\mathcal{I}$ relative to $\|\cdot\|_{C^*}$ ($\mathcal{I} = \{a \in \mathcal{A}; \|a\|_{C^*}\}$) is a C*-algebra. For a group algebra $\mathcal{A} = \mathbb{C}G$, $\mathcal{I} = \{0\}$ and the associated C*-algebra is called the group C*-algebra and denoted by $C^*(G)$.

Example 2.2. If we introduce a norm on the group algebra $\mathbb{C}G$ by

$$\|\sum_{g\in G} f(g)g\|_1 = \sum_{g\in G} |f(g)|,$$

then it satisfies $\|ab\|_1 \leq \|a\|_1\,\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$, whence the norm completion $\ell^1(G)$ of $\mathbb{C}G$ is a Banach *-algebra.

Example 2.3. The closure of the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$ in the operator topology on $\mathcal{B}(\mathcal{H})$ is a C*-algebra as a norm-closed *-ideal of $\mathcal{B}(\mathcal{H})$, which is referred to as the **compact operator**[4] algebra and denoted by $\mathcal{C}(\mathcal{H})$.

Exercise 8. If $\mathcal{H}_1$ is compact in the norm topology, then $\dim\mathcal{H} < \infty$.

The norm $\|a\|_2 = \|a\mathrm{tr}^{1/2}\| = \sqrt{\mathrm{tr}(a^*a)}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **Hilbert-Schmidt norm** and satisfies

$$\|ab\|_2 \leq \|a\|\|b\|_2, \quad \|b^*\|_2 = \|b\|_2 \geq \|b\|, \quad a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H}).$$

Thus the completion of $\mathcal{C}_0(\mathcal{H})$ relative to the Hilbert-Schmidt norm, which is included in $\mathcal{C}(\mathcal{H})$ and denoted by $\mathcal{C}_2(\mathcal{H})$, is a Banach *-algebra and realized as a *-ideal of $\mathcal{B}(\mathcal{H})$.

The norm $\|a\|_1 = \sup\{|\mathrm{tr}(ab)|; b \in \mathcal{C}_0(\mathcal{H}), \|b\| \leq 1\}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **trace norm** and satisfies

$$\|ab\|_1 \leq \|a\|\|b\|_1, \quad \|b^*\|_1 = \|b\|_1 \geq \|b\|_2, \quad |\mathrm{tr}(b)| \leq \|b\|_1$$

for $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H})$. Thus the completion of $\mathcal{C}_0(\mathcal{H})$ relative to the trace norm, which is included in $\mathcal{C}_2(\mathcal{H})$ and denoted by $\mathcal{C}_1(\mathcal{H})$, is a Banach *-algebra and realized as a *-ideal of $\mathcal{B}(\mathcal{H})$ with the trace functional extended to $\mathcal{C}_1(\mathcal{H})$ by continuity.

Exercise 9. Check the inequalities for the Hilbert-Schmidt and the trace norms.

Exercise 10. Show that, for a positive operator $a \in \mathcal{B}(\mathcal{H})$,

$$\mathrm{tr}(a) = \sum_j (\xi_j|a\xi_j)$$

does not depend on the choice of an orthonormal basis $\{\xi_j\}$ in $\mathcal{H}$.

Exercise 11. Show that $a \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{C}_1(\mathcal{H})$ if and only if $\mathrm{tr}(|a|) < \infty$. Here $|a| = \sqrt{a^*a}$. If this is the case, $\|a\|_1 = \mathrm{tr}(|a|)$.

[4]The terminology is based on the fact that an element T in $\mathcal{C}(\mathcal{H})$ is characterized by the property that the norm closure of $T(\mathcal{H}_1)$ ($\mathcal{H}_1$ being the unit ball in $\mathcal{H}$) is compact.

Exercise 12. Show that $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_2(\mathcal{H})\mathcal{C}_2(\mathcal{H})$ and deduce the inequality $\|ab\|_1 \leq \|a\|_2\|b\|_2$ from $|\mathrm{tr}(ab)| \leq \|a\|_2\|b\|_2$ (the Cauchy-Schwarz inequality).

Exercise 13. Any positive functional φ on $\mathcal{C}(\mathcal{H})$ is of the form $\varphi(x) = \mathrm{tr}(\rho x)$ with $0 \leq \rho \in \mathcal{C}_1(\mathcal{H})$. Hint: Consider the positive sesquilinear form $\varphi(\eta\xi^*)$ on $\mathcal{H}$.

Example 2.4. Given a compact (Hausdorff) space K, the set $C(K)$ of complex-valued continuous functions on K is a commutative C*-algebra by point-wise operations. If $F \subset K$ is a closed subset,

$$A = \{f : K \to \mathbb{C}; f \text{ is continuous and vanishing on } F\}$$

is a closed *-ideal of $C(K)$, which is unital if and only if F is open in K.

If Ω is a locally compact but non-compact space with $K = \Omega \cup \{\infty\}$ the one-point compactification and $F = \{\infty\}$, we write $A = C_0(\Omega)$. A positive functional φ on $C_0(\Omega)$ is one-to-one correspondence with a Radon measure μ on Ω of finite total mass by the formula

$$\varphi(a) = \int_\Omega a(\omega)\,\mu(d\omega).$$

The functional norm is then calculated by $\|\varphi\| = \mu(\Omega)$.

The associated GNS-representation π on $\mathcal{H}$ is identified with the multiplication operators on the Hilbert space $L^2(\Omega, \mu)$ through a natural unitary map $\mathcal{H} \to L^2(\Omega, \mu)$ specified by

$$A\varphi^{1/2} \ni a\varphi^{1/2} \mapsto a \in C_0(\Omega) \subset L^2(\Omega, \mu).$$

Exercise 14. The C*-algebra $C(K)$ has a non-trivial projection if and only if K is not connected.

Exercise 15. Let $[\mu] \subset \Omega$ be the support of μ. Then

$$\|\pi(a)\| = \sup\{|a(\omega)|; \omega \in [\mu]\}.$$

Theorem 2.5. With respect to the non-degenerate bilinear map $\mathcal{B}(\mathcal{H}) \times \mathcal{C}_1(\mathcal{H}) \ni (b, c) \mapsto \mathrm{tr}(bc) = \mathrm{tr}(cb) \in \mathbb{C}$, we have the duality relations of $\mathcal{C}(\mathcal{H})$: $\mathcal{C}(\mathcal{H})^* = \mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$.

Proof. The equality $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}(\mathcal{H})^*$ is by definition and the inclusion $\mathcal{B}(\mathcal{H}) \subset \mathcal{C}_1(\mathcal{H})^*$ is a consequence of

$$\|b\| \geq \sup\{|\mathrm{tr}(bc)|; c \in \mathcal{C}_1(\mathcal{H}), \|c\|_1 \leq 1\} \geq \sup\{|(\xi|b\eta)|; \xi, \eta \in \mathcal{H}_1\} = \|b\|.$$

In view of $\|\xi\eta^*\|_1 = \|\xi\|\,\|\eta\|$ for $\xi, \eta \in \mathcal{H}$, any $\varphi \in \mathcal{C}_1(\mathcal{H})^*$ defines a bounded sesquilinear form by $\varphi(\xi\eta^*)$ and then $b \in \mathcal{B}(\mathcal{H})$ by the relation $\varphi(\xi\eta^*) = (\eta|b\xi)$. By rewriting $(\eta|b\xi) = \mathrm{tr}(b\xi\eta^*)$, we see that

$\varphi(c) = \mathrm{tr}(bc)$ for $c \in \mathcal{C}_0(\mathcal{H})$ and then, for $c \in \mathcal{C}_1(\mathcal{H})$, by the density of $\mathcal{C}_0$ and the continuity of trace. □

Any algebra $\mathcal{A}$ is enhanced to have a unit:

$$\widetilde{\mathcal{A}} = \begin{cases} \mathcal{A} & \text{if } 1 \in \mathcal{A}, \\ \mathbb{C}1 + \mathcal{A} & \text{otherwise.} \end{cases}$$

$$0 \longrightarrow \mathcal{A} \longrightarrow \widetilde{\mathcal{A}} \longrightarrow \mathbb{C} \longrightarrow 0$$

If $\mathcal{A}$ is a *-algebra, so is $\widetilde{\mathcal{A}}$ by setting $(\lambda 1 + a)^* = \overline{\lambda} 1 + a^*$. For a Banach *-algebra A without unit, the unital *-algebra $\widetilde{A}$ is a Banach *-algebra by the norm $\|\lambda 1 + a\| = |\lambda| + \|a\|$. When A is a C*-algebra, the unital *-algebra $\widetilde{A}$ is again a C*-algebra and it is in a unique way. Existence: $\widetilde{A}$ is a C*-algebra by the norm

$$\|\lambda 1 + a\| = \sup\{\|\lambda b + ab\|; b \in A, \|b\| \leq 1\}.$$

The uniqueness will be established later.

Exercise 16. Check all these other than the uniqueness.

Definition 2.6. Let a be an element in a Banach algebra A and a power series $f(z) = \sum_{n \geq 0} f_n z^n$ ($f_n \in \mathbb{C}$) be absolutely convergent at $z = \|a\|$. Then $f(a) \in \widetilde{A}$ is defined by

$$f(a) = \sum_{n=1}^{\infty} f_n a^n.$$

Note that, if $f_0 = 0$, then $f(a) \in A$.

Example 2.7.

(i) If $ab = ba$, $e^a e^b = e^{a+b}$.

(ii) If $\|a\| < |\lambda|$ with $\lambda \in \mathbb{C}$, then

$$\frac{1}{\lambda - a} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

is the inverse of $\lambda 1 - a$ in $\widetilde{A}$.

Definition 2.8. Define the **spectrum** of an element a of an algebra $\mathcal{A}$ by

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C}; \lambda 1 - a \text{ is not invertible in } \widetilde{\mathcal{A}}\}.$$

Exercise 17. If $a \in \mathcal{B}(\mathcal{H})$ is a finite rank operator, $\sigma_{\mathcal{B}(\mathcal{H})}(a) = \{\lambda \in \mathbb{C}; a\xi = \lambda\xi$ for some $0 \neq \xi \in \mathcal{H}\}$

Exercise 18. If $1 \notin \mathcal{A}$, then $0 \in \sigma_{\mathcal{A}}(a)$ for any $a \in A$.

Exercise 19. Assume that elements a and b in a Banach algebra A with unit 1 satisfy $ab - ba = 1$. Show $e^a b e^{-a} = b + 1$, $\sigma(b) = 1 + \sigma(b)$ and derive a contradiction.

Lemma 2.9. In a unital Banach algebra A, the set G of invertible elements is open in A and the map $G \ni g \mapsto g^{-1} \in G$ is analytic.

Proof. For $a \in G$, $g = a - (a - g) = a(1 - a^{-1}(a - g))$ is invertible if $\|a^{-1}(a - g)\| \leq \|a^{-1}\| \, \|a - g\| < 1$ with the inverse given by the power series

$$g^{-1} = a^{-1} \sum_{n=0}^{\infty} (a^{-1}(a - g))^n.$$

□

Theorem 2.10 (Gelfand)**.** In a Banach algebra A, $\sigma_A(a)$ is non-empty compact subset of $\mathbb{C}$ for any $a \in A$.

Proof. If $z1 - a$ is invertible for any $z \in \mathbb{C}$, $(z1 - a)^{-1}$ is a bounded holomorphic function, a contradiction. □

Corollary 2.11. If a Banach algebra A is a field, then $A = \mathbb{C}1$.

Exercise 20. Spell out the details of the proof.

Exercise 21 (Analytic Functional Calculus)**.** Let $\mathcal{A}$ be the algebra of functions which are holomorphic in a neighborhood of $\sigma_A(a)$. If we set

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(\lambda)}{\lambda - a} \, d\lambda$$

for $f \in \mathcal{A}$, where the contour integral is performed by surrounding $\sigma_A(a)$, then $\mathcal{A} \ni f \mapsto f(a) \in \widetilde{A}$ gives an algebra-homomorphism.

Theorem 2.12 (Spectral Radius Formula)**.** For an element a of a Banach algebra A,

$$\rho(a) = \sup\{|\lambda|; \lambda \in \sigma_A(a)\} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Proof. From the geometric series formula

$$(\lambda 1 - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}, \quad |\lambda| > \|a\|,$$

we have $\rho(a) \leq \|a\|$, which is combined with $\sigma(a)^n \subset \sigma(a^n)$ to get $\rho(a) \leq \|a^n\|^{1/n}$;

$$\rho(a) \leq \inf_{n \geq 1} \|a^n\|^{1/n} \leq \limsup_{n \to \infty} \|a^n\|^{1/n}.$$

Let $r > \|a\|$ and apply the Cauchy integral formula:

$$a^n = \frac{1}{2\pi i}\int_{|\lambda|=r} \lambda^n(\lambda 1 - a)^{-1}\, d\lambda.$$

The radius r is then decreased to $r > \rho(a)$ by the Caychy inttegral theorem and we obtain the estimate

$$\|a^n\| \leq r^{n+1}\max\{\|(\lambda 1 - a)^{-1}\|; |\lambda| = r\}.$$

Consequently

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r$$

for $r > \rho(a)$. □

Proposition 2.13. If A is a C*-algebra and $aa^* = a^*a$, then

$$\|a\| = \sup\{|\lambda|; \lambda \in \sigma_A(a)\}.$$

Proof. If $aa^* = a^*a$,

$$\|a^2\| = \|(aa)^*aa\|^{1/2} = \|a^*aa^*\|^{1/2} = \|a^*a\| = \|a\|^2,$$

which is used repeatedly to get $\|a^{2^n}\| = \|a\|^{2^n}$. □

Corollary 2.14.

(i) The C*-norm of a C*-algebra A is unique: $\|a\| = \sup\{|\lambda|; \lambda \in \sigma_A(a^*a)\}$ with the right hand side determined by the *-algebra structure of A.

(ii) Let $\phi : A \to B$ be a *-homomorphism from a Banach *-algebra A into a C*-algebra B. Then $\|\phi(a)\| \leq \|a\|$ for $a \in A$.

Proposition 2.15. In a Banach *-algebra, we have the following.

(i) $\sigma_A(u) \subset \mathbb{T}$ for a unitary u.

(ii) $\sigma_A(h) \subset \mathbb{R}$ for a hermitian h.

Proof. By the spectral radius formula, $\sigma(u)$ is included in the unit disk but $\sigma(u^{-1}) = \sigma(u)^{-1}$ shows that $\lambda \in \sigma(u)$ implies $|\lambda^{-1}| \leq 1$.

For $\lambda \in \mathbb{C}$, $e^{i\lambda} - e^{ih} = (\lambda - h)a = a(\lambda - h)$ with

$$a = i\sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1} + \cdots + (ih)^{n-1}}{n!}$$

shows that, if $e^{i\lambda} - e^{ih}$ is invertible, then so is $\lambda - h$ because it admits left and right inverses. Thus $\lambda \in \sigma(h)$ implies $e^{i\lambda} \in \sigma(e^{ih})$, whence $|e^{i\lambda}| = 1$, i.e., $\lambda \in \mathbb{R}$. □

Theorem 2.16. If A is a closed *-subalgebra of a Banach *-algebra B. Then $\sigma_A(a) = \sigma_B(a)$ for $a \in A$.

Proof. Since $\widetilde{A}$ is naturally regarded as a subalgebra of $\widetilde{B}$ with a common unit, we may assume that $\widetilde{A} = A$ and $\widetilde{B} = B$ with the trivial inclusion $\sigma_B(a) \subset \sigma_A(a)$ from the outset.

First assume that $a^* = a$. Since open sets $\mathbb{C} \setminus \sigma_A(a)$ and $\mathbb{C} \setminus \sigma_B(a)$ in $\mathbb{C}$ are connected thanks to $\sigma_A(a) \subset \mathbb{R}$, it suffices for the equality $\sigma_A(a) = \sigma_B(a)$ to show that $\mathbb{C} \setminus \sigma_A(a)$ is a closed subset of $\mathbb{C} \setminus \sigma_B(a)$.

Let $\lambda_n \notin \sigma_A(a)$ converge to $\lambda \notin \sigma_B(a)$. Then $\lambda_n - a$ converges in B to an invertible element $\lambda - a$, whence $(\lambda_n - a)^{-1}$ converges to $(\lambda - a)^{-1}$ in B but $(\lambda_n - a)^{-1} \in A$ implies $(\lambda - a)^{-1} \in A$, i.e., $\lambda \notin \sigma_A(a)$.

Since the non-trivial inclusion $\sigma_A(a) \subset \sigma_B(a)$ is equivalently stated that an element $c \in A$ is invertible in A if it is invertible in B, assume that $bc = cb = 1$ for some $b \in B$. Then $bb^*c^*c = bc = 1$ shows that a hermitian c^*c is invertible in B and hence it is invertible in A as well: we can find $a \in A$ so that $ac^*c = 1$. Now $b = ac^*cb = ac^* \in A$. $\square$

The above theorem allows us to simply write $\sigma(a)$ to stand for $\sigma_A(a)$ when a is an element in a Banach *-algebra A.

Definition 2.17. The **spectrum** σ_A of a commutative Banach algebra A is the set of non-zero homomorphism $A \to \mathbb{C}$.

Example 2.18. If $A = C(K)$, $\sigma_A = \{\delta_x; x \in K\}$. Here $\delta_x(f) = f(x)$.

Each $\omega \in \sigma_A$ satisfies $\|\omega\| \leq 1$. To see this, we observe that $\widetilde{\omega}(\lambda 1 + a) = \lambda + \omega(a)$ defines a homomorphism $\widetilde{A} \to \mathbb{C}$, which gives us $\omega(a) \in \sigma(a)$ and then $|\omega(a)| \leq \|a\|$ for $a \in A$ by the spectral radius formula.

Thus $\Omega = \sigma_A \cup \{0\}$ is a w*-compact subset of A_1^* and σ_A is locally compact if furnished with the relative w*-topology. Write

$$C_0(\sigma_A) = \{f : \Omega \to \mathbb{C}; f \text{ is continuous and } f(0) = 0\}.$$

Note that σ_A is compact if and only if A is unital.

For $a \in A$, the function $\sigma_A \ni \omega \mapsto \omega(a)$ is continuous and denoted by $\widehat{a}$. The correspondance $A \ni a \mapsto \widehat{a} \in C_0(\sigma_A)$ is referred to as the **Gelfand transform** on A. From definition, the Gelfand transform is multiplicative.

Let A be unital and we shall show that, given a proper ideal I of A, there exits a non-zero homomorphism $\omega : A \to \mathbb{C}$ satisfying $\omega(I) = 0$.

In fact let B be the open unit ball of A at 0. Then every element in $1 + B$ is invertible, which means that $J \cap (1 + B) = \emptyset$ if J is a proper ideal of A. Thus $\overline{J} \cap (1 + B) = \emptyset$ and any maximal ideal $J \supset I$ is closed. Since the quotient Banach algebra A/J is a field, $A/J \cong \mathbb{C}$ and a non-zero homomorphism is obtained as a composition $A \to A/J \cong \mathbb{C}$.

Theorem 2.19. Let a be an element in a Banach algebra A.

(i) If A is unital, $\sigma_A(a) = \{\omega(a); \omega \in \sigma_A\}$.
(ii) If A is non-unital, $\sigma_A(a) = \{\omega(a); \omega \in \sigma_A\} \cup \{0\}$.

Proof. Since $\omega : A \to \mathbb{C}$ is an algebra-homomorphism, $\omega(a) \in \sigma_A(a)$. If $1 \in A$ and $\lambda \in \sigma(a)$, then $A(\lambda - a)$ is a proper ideal of A, we can find an $\omega \in \sigma_A$ vanishing on $A(\lambda - a)$, whence $\lambda = \omega(a)$. □

Proposition 2.20. For a commutative Banach *-algebra A, the Gelfand transform is a *-homomorphism.

Proof. In fact, for $h = h^* \in A$ and $\omega \in \sigma_A$, $\omega(h) \in \sigma(h) \subset \mathbb{R}$. □

Corollary 2.21. If a unital Banach *-algebra A is generated by $\{1, a\}$ with $a \in A$, then $\sigma_A \ni \omega \mapsto \omega(a) \in \sigma_A(a)$ is a homeomorphism.

Proof. The compact (Hausdorff) space σ_A is continuously mapped onto $\sigma_A(a)$, which is injective because $\{1, a\}$ generates A. □

Theorem 2.22 (Gelfand)**.** If A is a commutative C*-algebra, the Gelfand transform is an isomorphism of C*-algebras between A and $C_0(\sigma_A)$.

Proof. The Gelfand transform is isometric because of

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda|; \lambda \in \sigma(a^*a)\} = \sup\{|\widehat{a}(\omega)|^2; \omega \in \sigma_A\} = \|\widehat{a}\|^2.$$

Since $\{\widehat{a}; a \in A\}$ is a *-subalgebra of $C_0(\sigma_A)$ and it separates points in σ_A, it coincides with the whole $C_0(\sigma_A)$ thanks to the Stone-Weierstrass theorem. □

Example 2.23.
(i) Let $A = C_0(\Omega)$ with Ω a locally compact space. Then $\Omega \ni x \mapsto \delta_x \in \sigma_A$ is a homeomorphism.
(ii) Let K is a compact subset of $\mathbb{R}$, $h \in C(K)$ be a continuous function defined by $h(t) = t$ $(t \in X)$ and A be the C*-subalgebra of $C(K)$ generated by h. Then σ_A is naturally identified with $K \setminus \{0\}$, whereas $\sigma(h) = K$.

Exercise 22. For a commutative C*-algebra A, σ_A is compact if and only if A has a unit.

In a unital C*-algebra A, let $a \in A$ generate a commutative *-subalgebra; $aa^* = a^*a$ and C be the C*-subalgebra of A generated by $\{1, a\}$. Then the spectrum of C is identified with $\sigma(a)$ and the Gelfand theorem enables us to define the element $f(a)$ in C for a continuous function f on $\sigma(a)$. This is referred to as the continuous **functional calculus** of a.

Exercise 23. If g is a continuous function on $\sigma(f(a)) = \{f(\lambda); \lambda \in \sigma(a)\}$, then we have $g(f(a)) = (g \circ f)(a)$.

Example 2.24. A unital C*-algebra is unitary: If $h = h^*$ with $\|h\| \leq 1$, then $u = h + i\sqrt{1 - h^2}$ is unitary and $h = (u + u^*)/2$.

Let $\mathcal{A}$ be a commutative *-algebra with A the associated C*-algebra. Any $\omega \in \sigma_A$ restricts to a non-zero *-homomorphism $\mathcal{A} \to \mathbb{C}$. Conversely, any non-zero *-homomorphism of $\mathcal{A}$ into $\mathbb{C}$ defines a bounded representation of $\mathcal{A}$, whence it is continuous with respect to the C*-seminorm and lifted to a *-homomorphism of A into $\mathbb{C}$. Thus σ_A is identified with the set of non-zero *-homomorphisms of $\mathcal{A}$ into $\mathbb{C}$. If one applies this to a commutative group algebra $\mathbb{C}G$, then σ_A is identified with the set $\widehat{G}$ of group homomorphisms of G into $\mathbb{T}$ and $C^*(G)$ with $C(\widehat{G})$. Let $\ell^2(G) \to L^2(\widehat{G})$ be the Fourier transform. Then it intertwines the left regular representation of $C^*(G)$ and the multiplication operator of the Gelfand transform of $C^*(G)$, which shows that the regular representation is norm-preserving on $C^*(G)$.

Example 2.25. The spectrum of the group C*-algebra $C^*(\mathbb{Z})$ is identified with $\widehat{\mathbb{Z}} = \mathbb{T}$ and $C^*(\mathbb{Z})$ itself with $C(\mathbb{T})$.

Exercise 24. Let G be an abelian group and $a = \sum_{g \in G} a_g g \in \mathbb{C}G$ with $a_g \geq 0$. Then the C*-norm of a is simply calculated by $\|a\| = \sum_{g \in G} a_g$.

The Gelfand theorem establishes a categorical duality between the category of compact spaces and the category of unital commutative C*-algebras. Here morphisms are continuous maps for compact spaces, unit-preserving *-homomorphisms for C*-algebras, which are in a contravariant relation. Thus a conitnuous map $f : X \to Y$ corresponds to a *-homomorphism $\phi : C(Y) \to C(X)$ by $\phi(b) = b \circ f$. Note that ϕ is injective (resp. surjective) if and only if f is surjective (resp. injective). As an immediate application of this observation, we see that ϕ is isometric if it is injective.

Theorem 2.26. Let $\phi : A \to B$ be an injective *-homomorphism between C*-algebras. Then ϕ preserves the C*-norms.

Proof. Let $\mathbb{C} \times A$ be the C*-algebra obtained by adding an external unit to A. Thus $\mathbb{C} \times A = \widetilde{A}$ if A is not unital and $\mathbb{C} \times A \cong \mathbb{C} \oplus A$ if A is unital. By extending ϕ to an injective *-homomorphism $\mathbb{C} \times A \ni (\lambda, a) \mapsto (\lambda, \phi(a)) \in \mathbb{C} \times B$, the problem is reduced to the unital case. Let $a \in A$ and we shall show that $\|\phi(a)\| = \|a\|$. By passing to the commutative C*-subalgebras $C^*(1 + a^*a)$ and $C^*(1 + \phi(a)^*\phi(a))$, the problem is further reduced to the case of commutative C*-algebras and we are done. $\square$

Related to this, the following reveals a kind of algebraic rigidity in C*-algebras. For the proof, we need the positivity of elements of the form a^*a and it will be postponed until the end of the next section.

Theorem 2.27. Any closed ideal I of a C*-algebra is a *-ideal and the quotient *-algebra A/I is a C*-algebra with the C*-norm given by the quotient norm.

Corollary 2.28. Let $\phi : A \to B$ be a *-homomorphism between C*-algebras. Then the image $\phi(A)$ is closed in B.

3. Positivity in C*-algebras

A hermitian element h in a C*-algebra A is said to be **positive** and denoted by $h \geq 0$ if $\sigma_A(h) \subset [0, \infty)$. Let A_+ be the set of positive elements in A, which is invariant under the scalar multiplication of positive reals.

Lemma 3.1. For a hermitian element h in a C*-algebra, the following conditions are equivalent.

(i) h is positive.
(ii) $\|r1 - h\| \leq r$ for some $r \geq \|h\|$.
(iii) $\|r1 - h\| \leq r$ for any $r \geq \|h\|$.

Consequently, the positive part A_+ is a closed subset of A.

Proof. Realize h as a continuous function on a compact subset of $\mathbb{R}$. □

Corollary 3.2. If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$. Thus A_+ is a convex cone.

Proof. From the positivity, $\| \|a\| - a\| \leq \|a\|$ and $\| \|b\| - b\| \leq \|b\|$, which are used to get

$$\Big\| \|a\| + \|b\| - a - b \Big\| \leq \Big\| \|a\| - a \Big\| + \Big\| \|b\| - b \Big\| \leq \|a\| + \|b\|.$$

□

Theorem 3.3 (Kelley-Vaught)**.** For any element a in a C*-algebra, $a^*a \geq 0$.

Proof. Let $a^*a = b - c$ with b, c positive and $bc = cb = 0$. Then $ca^*ac = -c^3 \leq 0$. Thus the problem is reduced to showing that $x^*x \leq 0$ implies $x^*x = 0$. Let $x = h + ik$. Then

$$xx^* = 2h^2 + 2k^2 - x^*x \geq 0$$

and $\sigma(xx^*) \subset [0, \infty)$, which is combined with the next lemma to get $\sigma(x^*x) = \{0\}$. □

Lemma 3.4. In a Banach algebra A,

$$\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$$

for $a, b \in A$.

Proof. Formally

$$(\lambda 1 - ab)^{-1} = \frac{1}{\lambda} + \frac{ab}{\lambda^2} + \frac{abab}{\lambda^3} + \dots = \frac{1}{\lambda} + \frac{1}{\lambda^2} a \left(1 + \frac{ba}{\lambda} + \frac{baba}{\lambda^2} + \dots\right) b$$
$$= \frac{1}{\lambda} + \frac{1}{\lambda} a(\lambda - ba)^{-1} b$$

for $\lambda \neq 0$ but the conclusion is true because of

$$(\lambda - ab)\left(\frac{1}{\lambda} + \frac{1}{\lambda} a(\lambda - ba)^{-1} b\right) = 1 - \frac{ab}{\lambda} + \frac{a}{\lambda}(\lambda - ba)(\lambda - ba)^{-1} b = 1$$

and

$$\left(\frac{1}{\lambda} + \frac{1}{\lambda} a(\lambda - ba)^{-1} b\right)(\lambda - ab) = 1 - \frac{ab}{\lambda} + \frac{a}{\lambda}(\lambda - ba)^{-1}(\lambda - ba) b = 1.$$

□

Definition 3.5. An order relation in the set of hermitian elements is introduced by $a \leq b \iff b - a \in A_+$.

Exercise 25. For a hermitian element h and an arbitrary element a,

$$-\|h\| a^* a \leq a^* h a \leq \|h\| a^* a.$$

Proposition 3.6. Any positive linear functional φ on a C*-algebra A is continuous and

$$\|\varphi\| = \sup\{\varphi(a); a \geq 0, \|a\| \leq 1\}.$$

Proof. Let $M = \sup\{\varphi(a); a \geq 0, \|a\| \leq 1\}$ and assume that $M = \infty$. Then we can find a sequence $\{a_n\}$ of positive elements in the unit ball such that $\varphi(a_n) \geq n$. From $\|a_n\| \leq 1$, $a = \sum_{n=1}^{\infty} \frac{1}{n^2} a_n$ defines a positive element in A and $\sum_{k=1}^{n} a_k / k^2 \leq a$ implies

$$\varphi(a) \geq \sum_{k=1}^{n} \frac{1}{k^2} \varphi(a_k) \geq \sum_{k=1}^{n} \frac{1}{k} \to \infty \ (n \to \infty),$$

a contradiction. Thus $M < \infty$.

For a hermitian $h \in A$, $-|h| \leq h \leq |h|$ imples $|\varphi(h)| \leq \varphi(|h|) \leq M\|h\|$ and, for an arbitrary a,

$$|\varphi(a)| \leq |\varphi(\frac{a + a*}{2})| + |\varphi(\frac{a - a^*}{2i})| \leq M\|\frac{a + a^*}{2}\| + M\|\frac{a - a^*}{2i}\| \leq 2M\|a\|.$$

Now, for $x, y \in A$,

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y)\varphi(x^*x) \leq M^2\|y^*y\|\,\|x^*x\|.$$

If we choose $y = \frac{xx^*}{t+xx^*}$ with $t > 0$, then

$$(x - y^*x)(x - y^*x)^* = t^2\frac{xx^*}{(t + xx^*)^2}$$

implies $\|x - y^*x\|^2 \leq t \to 0$ $(t \to +0)$ and we obtain the inequality

$$|\varphi(x)|^2 \leq M^2\|x\|^2,$$

showing $\|\varphi\| \leq M$. □

Definition 3.7. An increasing net $\{u_\alpha\}_{\alpha\in I}$ of positive elements in a unit ball of a C*-algebra A is called an **approximate unit** if $a = \lim_{\alpha\to\infty} au_\alpha$ for any $a \in A$.

Proposition 3.8. An approximate unit exists.

Proof. As an index set, choose the directed set of finite subsets of A_+ and, for $\alpha = \{a_1, \dots, a_n\}$, set

$$u_\alpha = \frac{n(a_1 + \cdots + a_n)}{1 + n(a_1 + \cdots + a_n)}.$$

Then, for $\alpha \ni a^*a$,

$$\begin{aligned}(a - au_\alpha)^*(a - au_\alpha) &= \frac{1}{1 + n(a_1 + \cdots + a_n)}a^*a\frac{1}{1 + n(a_1 + \cdots + a_n)}\\ &\leq \frac{a_1 + \cdots + a_n}{(1 + n(a_1 + \cdots + a_n))^2}\\ &\leq \sup\{\frac{t}{(1 + nt)^2}; t \geq 0\} = \frac{1}{4n}\end{aligned}$$

reveals that $\lim_{\alpha\to\infty} au_\alpha = a$. □

Theorem 3.9. For a linear functional φ on a unital C*-algebra A, the following conditions are equivalent.

(i) φ is positive.
(ii) $\|\varphi\| = \varphi(1)$.

Proof. (i) $\Longrightarrow$ (ii) is a consequence of the previous proposition in view of $a \leq \|a\|1$ for $a \in A_+$.

(ii) $\Longrightarrow$ (i): We may assume that $\|\varphi\| = \varphi(1) = 1$. Let $h = h^*$ and $\varphi(h) = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$. Then

$$|\lambda + i(\mu + t)| = |\varphi(h + it)| \leq \|h + it\| = \sqrt{\|h\|^2 + t^2},$$

i.e., $\lambda^2 + (\mu + t)^2 \leq \|h\|^2 + t^2$ for $t \in \mathbb{R}$, which implies $\mu = 0$.

Now let $0 \leq h \leq 1$. Then

$$|1 - \varphi(h)| = |\varphi(1 - h)| \leq \|1 - h\| \leq 1.$$

Since $\varphi(h)$ is real, the above inequality requires $\varphi(h) \geq 0$. $\square$

Corollary 3.10. Let φ be a positive linear functional on a non-unital C*-algebra A and $\psi : \widetilde{A} \to \mathbb{C}$ be an extension of φ. Then ψ is positive if and only if $\psi(1) \geq \|\varphi\|$.

Proof. Assume that ψ is positive and let $a \in A_+$.

$$0 \leq \psi((t + a)^2) = \psi(1)t^2 + 2\varphi(a)t + \varphi(a^2)$$

for any $t \in \mathbb{R}$, whence

$$\varphi(a)^2 \leq \psi(1)\varphi(a^2).$$

If we restrict $\|a\| \leq 1$, this implies $\varphi(a)^2 \leq \psi(1)\|\varphi\|$ and then $\|\varphi\|^2 = \sup\{\varphi(a)^2; 0 \leq a \leq 1\} \leq \psi(1)\|\varphi\|$.

Conversely assume that $\psi(1) \geq \|\varphi\|$. Then $\psi(\lambda + a) = (\psi(1) - \|\varphi\|)\lambda + \widetilde{\varphi}(\lambda + a)$ and the positivity of ψ is reduced to that of $\widetilde{\varphi}$. Clearly $\|\widetilde{\varphi}\| \geq \|\varphi\|$ and the problem is further reduced to showing that $\|\widetilde{\varphi}\| \leq \|\varphi\|$. From $0 \leq u_\alpha a u_\alpha \leq u_\alpha^2$ for $0 \leq a \leq 1$ and $u_\alpha a u_\alpha \to a$ $(\alpha \to \infty)$, we have

$$\varphi(a) \leq \liminf \varphi(u_\alpha^2) \leq \limsup \varphi(u_\alpha^2) \leq \|\varphi\|$$

and then, by taking supremum on $0 \leq a \leq 1$, $\|\varphi\| = \lim_{\alpha\to\infty} \varphi(u_\alpha^2)$, which is used to conclude that

$$|\widetilde{\varphi}(\lambda+x)| = \lim_\alpha |\lambda\varphi(u_\alpha^2)+\varphi(xu_\alpha^2)| \leq \limsup_\alpha \|\varphi\|\,\|\lambda u_\alpha^2+xu_\alpha^2\| \leq \|\varphi\|\,\|\lambda+x\|.$$

$\square$

Exercise 26. For a bounded linear functional φ of a C*-algebra A and an approximate unit $\{u_\alpha\}$ in A, φ is positive if and only if

$$\|\varphi\| = \lim_{\alpha\to\infty} \varphi(u_\alpha).$$

Theorem 3.11. Let A be a closed *-subalgebra of a C*-algebra B. Given a positive linear functional φ on A, we can find a positive linear functional ψ on B so that $\varphi(a) = \psi(a)$ for $a \in A$ and $\|\varphi\| = \|\psi\|$.

Proof. By adding unit, we may assume that B has a unit 1. If $1 \notin A$, we first extend φ to a positive linear functional $\widetilde{\varphi}$ by putting $\widetilde{\varphi}(\lambda 1 + a) = \|\varphi\| + \varphi(a)$ (the above corollary). Thus the problem is reduced to the case $1 \in A \subset B$.

Now ψ be a Hahn-Banach extension of φ: ψ is a linear functional on B satisfying $\varphi(a) = \psi(a)$ for $a \in A$ and $\|\varphi\| = \|\psi\|$. Then $\psi(1) =$

$\varphi(1) = \|\varphi\| = \|\psi\|$ by Theorem 3.9, which guarantees the positivity of ψ again by the same theorem. □

Corollary 3.12. For any $a \in A$, we can find a positive linear functional φ on A so that $\|\varphi\| = 1$ and $\varphi(a^*a) = \|a\|^2$.

Let $\{\varphi_i\}_{i\in I}$ be a family of positive linear functionals on a C*-algebra and assume that, for each $0 \neq a \in A$, $\varphi_i(a^*a) > 0$ for some $i \in I$. Then the direct sum $\pi = \bigoplus_{i\in I} \pi_i$ of GNS representations is faithful.

In fact, if $\pi(a) = 0$, $\varphi_i(b^*a^*ab) = 0$ for $b \in A$ and $i \in I$, and then, by letting $b \to 1$, $\varphi_i(a^*a) = 0$ for all $i \in I$, whence $a = 0$.

Theorem 3.13 (Gelfand-Naimark)**.** Any C*-algebra is *-isomorphic to a closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proof of Theorem 2.27

Proof. Let $a \in I$. Then $\frac{a^*a}{t+a^*a} \in I$ for $t > 0$ and a^* is approximated by $\frac{a^*a}{t+a^*a}a^* \in I$.

Let $\{u_\alpha\}$ be an approximate unit for I. We claim that

$$\|a + I\| = \lim_\alpha \|a - au_\alpha\| = \lim_\alpha \|a - u_\alpha a\|.$$

For $x \in I$,

$$\|a+I\| \leq \|a-au_\alpha\| \leq \|(1-u_\alpha)(a+x)\|+\|x-xu_\alpha\| \leq \|a+x\|+\|x-xu_\alpha\|$$

implies

$$\|a + I\| \leq \liminf \|a - au_\alpha\| \leq \limsup \|a - au_\alpha\| \leq \|a + x\|.$$

Let $a, b \in A$. For $x, y \in I$,

$$\|ab + I\| \leq \|(a + x)(b + y)\| \leq \|a + x\| \, \|b + y\|$$

implies $\|ab + I\| \leq \|a + I\| \, \|b + I\|$.

$$\begin{aligned}
\|a + I\|^2 &= \lim \|a - au_\alpha\|^2 = \lim \|(1 - u_\alpha)a^*a(1 - u_\alpha)\| \\
&\leq \lim \|a^*a(1 - u_\alpha)\| \\
&= \|a^*a + I\| \leq \|a^* + I\| \, \|a + I\|.
\end{aligned}$$

shows that $\|a^* + I\| = \|a + I\|$ and then the equality $\|a + I\|^2 = \|a^*a + I\|$. □

The *-algebraic operatios of a C*-algebra A is now transferred into the dual Banach space A^*: For $\varphi \in A^*$ and $a \in A$, $a\varphi$, φa and φ^* are defined in A^* by

$$(a\varphi)(x) = \varphi(xa), \quad (\varphi a)(x) = \varphi(ax), \quad \varphi^*(x) = \overline{\varphi(x^*)}, \quad \text{for } x \in A$$

with the following relations

$$(a\varphi)b = a(\varphi b), \quad (a\varphi b)^* = b^*\varphi^* a^*.$$

A linear functional $\varphi \in A^*$ is said to be **hermitian** if $\varphi^* = \varphi$. Let A^*_+ be the set of positive functionals on A. Then $aA^*_+a^* \subset A^*_+$ for $a \in A$.

Lemma 3.14. Let H be the set of hermitian elements in A, which is a closed real-linear subspace of A. Then the real Banach space H^* of bounded linear functional on H is identified with the set of hermitian functionals on A.

The following is an analogue of Jordan decomposition in measure theory.

Theorem 3.15 (Grothendieck)**.** Any hermitian functional $\theta \in A^*$ is expressed by $\theta = \varphi - \psi$ with $\varphi, \psi \in A^*_+$ satisfying $\|\theta\| = \|\varphi\| + \|\psi\|$ and such an expression is unique.

Proof. The positive unit ball $A^*_{+,1}$ is compact in the weak* topology and so is the convex hull C of $A^*_{+,1} \cup (-A^*_{+,1})$. Clearly $C \subset H^*_1$ and, if $f \in H^*_1 \setminus C$, we can find $h \in H$ such that $f(h) > \sup\{g(h); g \in C\}$ by Hahn-Banach theorem (H being identified with the weak* dual of H^*).

The Gelfand transform of the C*-algebra $C^*(h)$ generated by $h \in A$ enables us to find $\omega \in \sigma_{C^*(h)}$ satisfying $|\omega(h)| = \|h\|$. Let $\varphi \in A^*_{+,1}$ be an extension of ω. Then $\pm\varphi \in C$ implies $|f(h)| \geq f(h) > |\varphi(h)| = \|h\|$, which contradicts with $\|f\| \leq 1$.

Now we express $\theta/\|\theta\| \in H^*_1$ as an element in C: $\theta/\|\theta\| = t\varphi_1 - (1-t)\psi_1$ with $0 \leq t \leq 1$ and $\varphi_1, \psi_1 \in A^*_{+,1}$. Then the choice $\varphi = t\|\theta\|\varphi_1$ and $\psi = (1-t)\|\theta\|\psi_1$ does the job for the existence part because of

$$\|\theta\| \leq \|\varphi\| + \|\psi\| = t\|\theta\|\|\varphi_1\| + (1-t)\|\theta\|\|\psi\|_1 \leq t\|\theta\| + (1-t)\|\theta\| = \|\theta\|.$$

The uniqueness will be established later on as a consequence of polar decomposition for linear functionals (see Pedersen §3.2 for a direct proof). □

4. Representations and W*-algebras

In connection with representations, the notion of W*-algebra arises in two ways: as the space of intertwiners or as a weaker notion of equivalence of representations. For the analysis of these, the norm topology turns out to be not much adequate.

As an example, consider the Schur's criterion on irreducible representations, which can be achieved by appealing to the spectral decomposition theorem. To get spectral projections starting with a hermitian operator h, a weaker notion of convergence comes into, although the process of recovering h as a limit of linear combinations of projections can be norm-convergent. Cumbersomeness here is that lots of related notions of weaker topologies arise on sets of operators.

Definition 4.1. Too many topologies on operators. Write $\ell^2 = \ell^2(\mathbb{N})$.

(i) The weak operator topology is the one described by seminorms $|(\xi|a\eta)|$ $(\xi, \eta \in \mathcal{H})$.
(ii) The strong operator topology is the one described by seminorms $\|a\xi\|$ $(\xi \in \mathcal{H})$.
(iii) The *strong operator topology is the one described by seminorms $\|a\xi\|$, $\|a^*\xi\|$ $(\xi \in \mathcal{H})$.
(iv) The σ-weak topology is the one described by seminorms $|(\xi|(a\otimes 1)\eta)|$ $(\xi, \eta \in \mathcal{H} \otimes \ell^2)$.
(v) The σ-strong topology is the one described by seminorms $\|(a\otimes 1)\xi\|$ $(\xi \in \mathcal{H} \otimes \ell^2)$.
(vi) The σ-*strong topology is the one described by seminorms $\|(a \otimes 1)\xi\|$, $\|(a^* \otimes 1)\xi\|$ $(\xi \in \mathcal{H} \otimes \ell^2)$.

Exercise 27. Given a vector $\xi \in \mathcal{H} \otimes \mathcal{H}^*$, we can find a (at most) countable orthonormal system $\{\eta_n\}$ in $\mathcal{H}$ such that $\xi \in \overline{\sum_n \mathcal{H} \otimes \eta_n^*}$.

Proposition 4.2. Regard $\mathcal{H} \otimes \mathcal{H}^*$ as a $\mathcal{B}(\mathcal{H})$-bimodule.

(i) The σ-weak topology is given by seminorms $|(\xi|a\eta)|$ $(\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*)$.
(ii) The σ-strong topology is the one described by seminorms $\|a\xi\|$ $(\xi \in \mathcal{H} \otimes \mathcal{H}^*)$.
(iii) The σ-*strong topology is the one described by seminorms $\|a\xi\|$, $\|\xi a\|$ $(\xi \in \mathcal{H} \otimes \mathcal{H}^*)$.

Proof. If $\dim \mathcal{H} = \infty$, the topologies are controlled by vectors of countable decomposition, whereas all these collapse to the single euclidean topology for a finite-dimensional $\mathcal{H}$. In the assertion (iii), note that $\|\xi a\| = \|(\xi a)^*\| = \|a^*\xi^*\|$ covers the adjoint part. □

Proposition 4.3. The operator multiplication is separately continuous in any of these topologies. The star operation is continuous in each of the weak, the σ-weak and σ-*strong topologies.

Exercise 28. Let $S : \ell^2 \to \ell^2$ be the unilateral shift operator. Then $(S^*)^n \to 0$ in the strong operator topology but not for its adjoint $\{S^n\}$.

Exercise 29. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral shift operator. Then $T^n \to 0$ $(n \to \infty)$ in the weak operator topology, but not in the strong operator topology.

Proposition 4.4. On a bounded subset B of $\mathcal{B}(\mathcal{H})$, the weak (resp. strong or *strong) operator topology is equivalent to the σ-weak (resp. σ-strong or σ-*strong) topology.

Example 4.5. Let $\mathcal{H}$ be a separable Hilbert space and choose a countable dense set $\{\xi_n\}$ in the unit sphere of $\mathcal{H}$. For $x \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_w = \sum_{m,n\geq 1} \frac{1}{2^{m+n}} |(\xi_m|x\xi_n)|$$

defines a norm weaker than the operator norm. Then the topology induced from the distance function $\|x-y\|_w$ coincides with the weak operator topology when restricted to the unit ball B of $\mathcal{B}(\mathcal{H})$. Moreover B is complete with respect to this metric.

Exercise 30. If a sequence $\{T_n\}$ of bounded operators converges to a bounded operator T in the weak operator topology, then $\{\|T_n\|\}$ is bounded and therefore $T_n \to T$ in the σ-weak topology. (Use Banach-Steinhauss theorem twice.)

Exercise 31. Let $\{\xi_n\}$ be a countable dense subset of the unit sphere of a Hilbert space $\mathcal{H}$. Consider the directed set of pairs $\alpha = (e, \epsilon)$ ($\epsilon > 0$ and e being a finite rank projection) with $(e, \epsilon) \prec (e', \epsilon')$ if and only if $e \leq e'$ and $\epsilon \geq \epsilon'$. Let $T_\alpha = \epsilon e + m^2(1-e)|\xi_m)(\xi_m|(1-e)$, where $m = \min\{n \geq 1; \|e\xi_n\|^2 \leq \epsilon\}$.

Show that $T_\alpha \to 0$ in the weak operator topology, whereas

$$\sum_n \frac{1}{n^2}|(\xi_n|T_\alpha\xi_n)| \geq \frac{1}{m^2}|(\xi_m|T_\alpha\xi_m)| \geq (1-\epsilon)^2$$

for any $\alpha = (e, \epsilon)$.

By a conjugation on a Hilbert space $\mathcal{H}$, we shall mean a conjugate-linear involution $\Gamma : \mathcal{H} \to \mathcal{H}$ satisfying $(\Gamma\xi|\Gamma\eta) = (\eta|\xi)$ for $\xi, \eta \in \mathcal{H}$. A conjugation enables us to identify $\mathcal{H}^*$ with $\mathcal{H}$ itself: $\xi^* \mapsto \Gamma\xi$ gives a unitary map between $\mathcal{H}^*$ and $\mathcal{H}$.

As an example, the Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$ is self-dual through the natural conjugation defined by $(\xi \otimes \eta^*)^* = \eta \otimes \xi^*$. In other words, the linear functional $(\xi \otimes \eta^*)^*$ on $\mathcal{H} \otimes \mathcal{H}^*$ is identified with the vector $\eta \otimes \xi^*$ in $\mathcal{H} \otimes \mathcal{H}^*$.

Proposition 4.6. The following conditions on a linear functional $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ are equivalent.

(i) φ is of the form $\varphi(a) = \mathrm{tr}(Ta)$ with $T \in \mathcal{B}(\mathcal{H})$ in the trace class.
(ii) φ is σ-weakly continuous.

(iii) φ is σ-*strongly continuous.
(iv) There exist vectors $\xi, \eta \in \mathcal{H} \otimes \ell^2$ such that $\varphi(x) = (\xi|(x \otimes 1)\eta)$ for $x \in \mathcal{B}(\mathcal{H})$.

Proof. (i) $\iff$ (ii): Write $T = x^* y$ with $x, y \in \mathcal{C}_2$ and let $\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*$ be the vector representatives of x, y. Then $\mathrm{tr}(Ta) = (\xi|a\eta)$ is σ-weakly continuous.

(ii) $\implies$ (iii) is obvious. Assume (iii). Then there exists $\xi \in \mathcal{H} \otimes \mathcal{H}^*$ such that

$$|\varphi(a)| \leq \|a\xi \oplus \xi a\|$$

whence $a\xi \oplus \xi a \mapsto \varphi(a)$ defines a bounded linear functional on $\{a\xi \oplus \xi a; a \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H} \otimes \mathcal{H}^* \oplus \mathcal{H} \otimes \mathcal{H}^*$ and, by the Riesz lemma, we can find $\eta, \zeta \in \mathcal{H} \otimes \mathcal{H}^*$ so that

$$\varphi(a) = (\eta \oplus \zeta | a\xi \oplus \xi a) = (\eta|a\xi) + (\zeta|\xi a) = (\eta|a\xi) + (\xi^*|a\zeta^*),$$

which reveals the σ-weak continuity of φ.

(i) $\iff$ (iv): If we express T as a product of two operators in the Hilbert-Schmidt class, then we see that $\varphi(x) = (\xi|(\xi \otimes 1)\eta)$ with $\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*$. Let $\{\zeta^*_{i \in I}\}$ be an orthonormal basis in $\mathcal{H}^*$, then ξ and η are supported by a countable subset $\{i_1, i_2, \dots\}$, whence ξ and η are identified with vectors in $\mathcal{H} \otimes \ell^2$. □

Exercise 32. If a linear functional $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is continuous relative to the *strong operator topology, then it is continuous relative to the weak operator topology.

Proposition 4.7. An operator $a \in \mathcal{B}(\mathcal{H})$ is in the σ-strong closure of a subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ if and only if $(a \otimes 1)\xi \in \overline{(\mathcal{S} \otimes 1)\xi}$ for any $\xi \in \mathcal{H} \otimes \ell^2$. Here $\overline{(\mathcal{S} \otimes 1)\xi}$ denotes the norm closure of $\{(s \otimes 1)\xi; s \in \mathcal{S}\}$ in $\mathcal{H} \otimes \ell^2$.

Proof. Controls at finitely many families $\{\xi_i = \oplus_j \xi_{ij}\}_{1 \leq i \leq n}$ are managed by a single vector $\xi = \xi_1 \oplus \cdots \oplus \xi_n$, which is identified with a vector in $\mathcal{H} \otimes \ell^2$ through any bijection $\mathbb{N}^n \cong \mathbb{N}$. □

The following is an analogue of the monotone convergence theorem in Lebesgue integration, which describes a completely different feature from the norm topology.

Proposition 4.8. A bounded increasing net $\{a_\alpha\}$ of positive elements in $\mathcal{B}(\mathcal{H})$ converges to a positive element in the σ-strong topology.

Proof. By the polar identity, the sesquilinear form $(\xi|a_\alpha\eta)$ converges point-wise: there is a positive element $a \in \mathcal{B}(\mathcal{H})$ such that

$$(\xi|a\eta) = \lim_{\alpha \to \infty} (\xi|a_\alpha\eta), \quad \xi, \eta \in \mathcal{H}$$

and then

$$\begin{aligned}\||(a-a_\alpha)\xi|\|^2 &\leq \|(a-a_\alpha)^{1/2}\|\,\|(a-a_\alpha)^{1/2}\xi\|^2\\ &= \|a-a_\alpha\|^{1/2}\,(\xi|(a-a_\alpha)\xi)\\ &\leq \|a\|^{1/2}(\xi|(a-a_\alpha)\xi) \to 0\ (\alpha\to\infty).\end{aligned}$$

We now apply this convergence to the net $\{a_\alpha \otimes 1\}$ in $\mathcal{B}(\mathcal{H}\otimes\ell^2)$ to find a positive operator $\widehat{a}$ on $\mathcal{H}\otimes\ell^2$. Since $a_\alpha\otimes 1$ commutes with $1\otimes p_j p_k^*$ $(j,k\geq 1)$ and the operator multiplication is seperately continuous in the strong operator topology, we see that $\widetilde{a}$ commutes with $1\otimes p_j p_k^*$ as well, whence $\widetilde{a}$ is of the form $a\otimes 1$ with $a\in\mathcal{B}(\mathcal{H})$. □

Definition 4.9. A *-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is said to be non-degenerate if $\mathcal{A}\mathcal{H}$ is dense in $\mathcal{H}$.

A **W*-algebra** on a Hilbert space $\mathcal{H}$ is a σ-weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition 4.10. The **commutant** $\mathcal{S}'$ of a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is defined to be

$$\mathcal{S}' = \{c\in\mathcal{B}(\mathcal{H}); cs = sc \text{ for every } s\in\mathcal{S}\}.$$

Note that by the separate continuity of multiplication, $\mathcal{S}' = \overline{\mathcal{S}}'$, where $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ in the weak operator topology.

Exercise 33. Show that $\mathcal{S}' = \mathcal{S}'''$.

Example 4.11. Given a subset $\mathcal{S} = \mathcal{S}^* \subset \mathcal{B}(\mathcal{H})$, its commutant $\mathcal{S}'$ is a W*-algebra. As a special case of this, given a *-representation $\pi : \mathcal{A}\to\mathcal{B}(\mathcal{H})$, the space of self intertwiners is a W*-algebra.

Example 4.12. For the left and right regular representations λ, ρ of a group G on $\ell^2(G)$, $\lambda(G)' = \rho(G)''$.

Recal that that $\ell^2(G)$ is a *-bimodule of $\mathbb{C}G$. Let $r\in\lambda(G)'$ and set $\eta = r(\delta^{1/2})\in\ell^2(G)$. Notice $r(a\delta^{1/2}) = ar(\delta^{1/2}) = a\eta$ ($a\in\mathbb{C}G$ shows that r is given by a right multiplication of η and one may expect that it can be approximated by cutting the support of ξ down to finite sets. Real life is, however, not so easy but still simple enough:

Let $l\in\rho(G)'$ and express it as a left multiplication of $\xi\in\ell^2(G)$: $l(a\delta^{1/2}) = \xi a$ for $a\in\mathbb{C}G$. Then l^* is given by the left multiplication of ξ^* and r^* by the right multiplication of η^*, which are used to see

$$\begin{aligned}(a\delta^{1/2}|lr(b\delta^{1/2})) &= (l^*(a\delta^{1/2})|r(b\delta^{1/2})) = (\xi^* a|b\eta) = (\eta^* b^*|a^*\xi)\\ &= (a\eta^*|\xi b) = (r^*(a\delta^{1/2})|l(b\delta^{1/2})) = (a\delta^{1/2}|rl(b\delta^{1/2})).\end{aligned}$$

Exercise 34. Let two subsets $A, B\subset\mathcal{B}(\mathcal{H})$ commute with each other. Then $A' = B''$ if and only if A' commutes with B'.

Lemma 4.13. Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}(\mathcal{H})$ and $p_j : \mathcal{H} \otimes \ell^2 \to \mathcal{H}$ be the projection to the j-th component.

(i) Then an element $C \in \mathcal{B}(\mathcal{H} \otimes \ell^2)$ belongs to $(\mathcal{A} \otimes 1)'$ if and only if $p_j C p_k^* \in \mathcal{A}'$ for any $j, k \geq 1$.
(ii) We have $(\mathcal{A} \otimes 1)'' = \mathcal{A}'' \otimes 1$.
(iii) A projection $e \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{A}'$ if and only if $e\mathcal{H}$ is invariant under $\mathcal{A}$.

Proof. (i) is a consequence of $p_j(a \otimes 1) = ap_j$ for $a \in \mathcal{A}$ and $j \geq 1$.

The inclusion $\mathcal{A}'' \otimes 1 \subset (\mathcal{A} \otimes 1)''$ follows from (i). To get the reverse inclusion, assume that $C \in \mathcal{B}(\mathcal{H} \otimes \ell^2)$ is in the commutant of $(\mathcal{A} \otimes 1)'$. Then C commutes with $p_j^* p_k$ for any $j, k \geq 1$, whence $p_j C p_k^* = \delta_{j,k} c$ with $c \in \mathcal{B}(\mathcal{H})$, i.e., $C = c \otimes 1$. Since C commutes with $p_j^* a' p_j$ ($a' \in \mathcal{A}'$) as well, we see that c is in the commutant of $\mathcal{A}'$.

Non-trivial is the if part: since $\mathcal{A}$ is a *-subalgebra, the invariance of $e\mathcal{H}$ under $\mathcal{A}$ implies the invariance of the orthogonal complement $(1-e)\mathcal{H}$ as well and we see that $ae\xi = a\xi = ea\xi$ for $\xi \in e\mathcal{H}$, while $ae\xi = 0 = ea\xi$ for $\xi \in (1-e)\mathcal{H}$. □

Lemma 4.14. If a *-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is non-degenerate, then $\xi \in \overline{\mathcal{A}\xi}$ for any $\xi \in \mathcal{H}$.

Proof. Passing to the norm closure, we may assume that $\mathcal{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and let $\{u_\alpha\}$ be an approximate unit in $\mathcal{A}$. Then, on a dense subspace $\mathcal{A}\mathcal{H}$,

$$\lim_{\alpha \to \infty} u_\alpha \sum_{j=1}^n a_j \xi_j = \sum_{j=1}^n \lim_{\alpha \to \infty} u_\alpha a_j \xi_j = \sum_{j=1}^n a_j \xi_j$$

shows that $\lim_{\alpha \to \infty} u_\alpha = 1$ in the strong operator topology. Particularly, $\xi = \lim_\alpha u_\alpha \xi$ belongs to the closure of $\mathcal{A}\xi$. □

Theorem 4.15 (von Neumann's Bicommutant Theorem)**.** Let $\mathcal{A}$ be a non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the bicommutant $\mathcal{A}''$ is the σ-strong closure of $\mathcal{A}$.

Proof. Since $\mathcal{A}''$ is closed in the weak operator topology, we need to show that any $a'' \in \mathcal{A}''$ is in the σ-strong colosure of $\mathcal{A}$. In fact, in view of Proposition 4.7, let $\xi \in \mathcal{H} \otimes \ell^2$ and P be the projection to $\overline{(\mathcal{A} \otimes 1)\xi}$. Then $P \in (\mathcal{A} \otimes 1)'$ commuets with $a'' \otimes 1$ by Lemma 4.13 (ii), (iii) and

$$P(a'' \otimes 1)\xi = (a'' \otimes 1)P\xi = (a'' \otimes 1)\xi.$$

Here the non-degeneracy of $\mathcal{A} \otimes 1$ on $\mathcal{H} \otimes \ell^2$ is used to ensure $P\xi = \xi$. □

Corollary 4.16. For a non-degenerate *-subalgebra M of $\mathcal{B}(\mathcal{H})$, the following conditions are equivalent.

(i) $M = M''$.
(ii) M is closed in the weak operator topology.
(iii) M is a W*-algebra.
(iv) M is closed in the σ-*strong topology.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are trivial. Assume that M is closed in the σ-*strong topology. Then it is closed in the σ-strong topology by Hahn-Banach theorem in view of Proposition 4.6, whence $M = M''$. □

Example 4.17. Let M be a W*-algebra on $\mathcal{H}$ and $a = v|a|$ be the polar decomposition of $a \in M$. Then $|a|$ and v belong to M. In fact, if u is a unitary in M', then $a = uau^* = uvu^*u|a|u^*$ and the uniqueness of polar decomposition shows that $uvu^* = v$ and $u|a|u^* = |a|$.

By a similar argument, for a hermitian $h \in M$, the spectral projections of h belong to M.

Lemma 4.18. Let M be a W*-algebra on a Hilbert space $\mathcal{H}$ and $e \in \mathcal{B}(\mathcal{H})$ be a projection. Then $e \in M$ if and only if $e\mathcal{H}$ is invariant under M'.

Corollary 4.19. The set of projections in a W*-algebra M is a complete lattice.

Proof. This is because projections in $\mathcal{B}(\mathcal{H})$ are in one-to-one correspondence with M'-invariant closed subspaces of $\mathcal{H}$. □

Exercise 35. Any σ-weakly closed left ideal of a W*-algebra M is of the form Mp with $p \in M$ a projection. If it is an ideal, then p belongs to the center $M \cap M'$ of M.

Example 4.20. The *-subalgebra of finite rank operators is dense in $\mathcal{B}(\mathcal{H})$ with respect to the σ-*strong topology.

Example 4.21. Let μ be a σ-finite measure on a Borel space Ω and regard $L^\infty(\Omega, \mu)$ as a *-subalgebra of multiplication operators on the Hilbert space $L^2(\Omega, \mu)$. Then $(L^\infty(\Omega, \mu))' = L^\infty(\Omega, \mu)$: $L^\infty(\Omega, \mu)$ is a commutative W*-algebra on $L^2(\Omega, \mu)$, which is maximally commutative in $\mathcal{B}(L^2(\Omega, \mu))$.

In fact, if $b \in \mathcal{B}(L^2(\Omega, \mu))$ is in the commutant of $L^\infty(\Omega, \mu)$ and $b_E \in L^2(\Omega, \mu)$ is defined by $b(1_E\mu^{1/2}) = b_E\mu^{1/2}$ for a Borel subset $E \subset \Omega$ satisfying $\mu(E) < \infty$, then these are patched together to a μ-measurable function b_Ω on Ω satisfying $1_E b_\Omega = b_E$ (the σ-finiteness of μ is used here). Then, for $a \in L^\infty(\Omega, \mu)$ and for E with $\mu(E) < \infty$,

$$b(a1_E\mu^{1/2}) = ab(1_E\mu^{1/2}) = b_E a1_E\mu^{1/2} = b_\Omega a1_E\mu^{1/2}.$$

Since $\cup_E L^\infty(\Omega,\mu)1_E\mu^{1/2}$ is dense in $L^2(\Omega,\mu)$, the boundedness of b compels b_Ω to be bounded and we see that b is the multiplication operator by b_Ω.

We now construct the universal representation of a commutative C*-algebra $A = C_0(\Omega)$. Consider the free A-module over the set of formal symbols $\{\varphi^{1/2}; \varphi \in A^*_+\}$ on which we introduce a positive sesquilinear form by

$$\left(\sum_\varphi x_\varphi \varphi^{1/2} \middle| \sum_\varphi y_\varphi \varphi^{1/2}\right) = \sum_{\varphi,\psi} \int_\Omega \overline{x_\varphi(\omega)} y_\psi(\omega) \sqrt{\varphi(d\omega)}\sqrt{\psi(d\omega)}.$$

Here φ (resp. ψ) is identified with the Radon measure $\varphi(d\omega)$ (resp. $\psi(d\omega)$) on Ω and the Hellinger integral is defined by

$$\int_\Omega f(\omega)\sqrt{\varphi(d\omega)}\sqrt{\psi(d\omega)} = \int_\Omega f(\omega)\sqrt{\frac{d\varphi}{d\mu}(\omega)\frac{d\psi}{d\mu}(\omega)}\,\mu(d\omega)$$

with μ any auxillirary measure satisfying $\varphi \prec \mu$ and $\psi \prec \mu$. Note that the positivity as well as the boundedness of left multiplication by elements in A follows from this expression:

$$\left(\sum_\varphi a x_\varphi \varphi^{1/2} \middle| \sum_\varphi a x_\varphi \varphi^{1/2}\right) = \int_\Omega |a(\omega)|^2 \left|\sum_\varphi x_\varphi(\omega)\sqrt{\frac{d\varphi}{d\mu}(\omega)}\right|^2 \mu(d\omega).$$

The associated Hilbert space is denoted by $L^2(A)$ on which A is represented by multiplication. Moreover $L^2(A)$ is a *-bimodule by the involution $(\sum_\varphi x_\varphi \varphi^{1/2})^* = \sum_\varphi x_\varphi^* \varphi^{1/2}$.

We shall later generalize the construction to non-commutative C*-algebras in a far-reaching way.

Theorem 4.22. A W*-algebra M is order-complete in the sense that every norm-bounded increasing net $\{a_\alpha\}$ of positive elements in M admits a least upper bound a in M and the net $\{a_\alpha\}$ converges to a in the σ-strong topology.

Proof. Let M be on $\mathcal{H}$. Then $\{a_\alpha\}$ converges to a positive element $a \in sB(\mathcal{H})$ in the σ-strong topology by Proposition 4.8, whence it converges in the weak operator topology. Since M is closed in the weak operator topology by Corollary 3.13, we have $a \in M$. Let b be an upper bound of $\{a_\alpha\}$ in M. Then

$$(\xi|a\xi) = \lim(\xi|a_\alpha\xi) \leq (\xi|b\xi) \quad \text{for } \xi \in \mathcal{H}$$

shows that $a \leq b$. Thus a is the least upper bound of $\{a_\alpha\}$ in M. $\square$

Theorem 4.23 (Kaplansky Density Theorem)**.** Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}(\mathcal{H})$ and $a \in \mathcal{B}(\mathcal{H})$ be in the closure of $\mathcal{A}$ with repsect to the weak operator topology. Then we can find a net $\{a_\alpha\}$ of elements in $\mathcal{A}$ such that $\|a_\alpha\| \leq \|a\|$ and $a = \lim_\alpha a_\alpha$ in the *strong topology.

Proof. Since $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ is in the weak operator closure of $M_2(\mathcal{A})$, the problem is reduced to approximating $a = a^*$ by $a_\alpha = a_\alpha^*$ in the strong operator topology.

Let $f(t) = 2\|a\|t/(1+t^2)$ be a $\mathbb{R}$-valued function in $C_0(\mathbb{R})$, which gives a homeomorphism between $[-1, 1]$ and $[-\|a\|, \|a\|]$ when restricted to the interval $[-1, 1]$.

Given a closed subset S of $\mathbb{R}$, let $\mathcal{B}_S(\mathcal{H})$ be the set of hermitian elements, say h, in $\mathcal{B}(\mathcal{H})$ satisfying $\sigma(h) \subset S$. Then $f(\mathcal{B}_{\mathbb{R}}(\mathcal{H})) \subset \mathcal{B}_{[-\|a\|,\|a\|]}(\mathcal{H})$ and the functional calculus by f gives a bijection between $\mathcal{B}_{[-1,1]}(\mathcal{H})$ and $\mathcal{B}_{[-\|a\|,\|a\|]}(\mathcal{H})$ with the inverse map given by $g(h)$ ($h \in \mathcal{B}_{[-\|a\|,\|a\|]}(\mathcal{H})$), where $g : [-\|a\|, \|a\|] \to [-1, 1]$ is the inverse function of $f|_{[-1,1]}$.

Now choose $b \in \mathcal{B}_{[-1,1]}(\mathcal{H})$ so that $a = f(b)$. Then, for a unitary $u \in \mathcal{A}'$, $ubu^* = f^{-1}(uau^*) = b$ shows that $b \in \mathcal{A}'$ and we can find a net $\{b_\alpha = b_\alpha^*\}_{\alpha \in I}$ in $\mathcal{A}''$ so that $b = \lim_{\alpha\to\infty} b_\alpha$ in the strong operator topology. From the expression $f(t) = \frac{\|a\|}{i+t} + \frac{\|a\|}{-i+t}$,

$$f(b_\alpha) - f(b) = \frac{\|a\|}{i + b_\alpha}\Big(b - b_\alpha\Big)\frac{1}{i + b} + \frac{\|a\|}{-i + b_\alpha}\Big(b - b_\alpha\Big)\frac{1}{-i + b}$$

reveals that $f(b_\alpha) \to f(b)$ in the strong operator topology. $\square$

Example 4.24. Let $\mathcal{H}$ be a separable Hilbert space and $\|x-y\|_w$ be the complete metric on the unit ball B of $\mathcal{B}(\mathcal{H})$ discussed in Example 4.5. The complete metric space B is then separable. In fact, each operator in B is σ-weakly approximated by finite rank operators in B, which in turn are approximated in norm by finite rank operators of rational entries.

At the closing of this section, we introduce standard terminologies on *-representations. To simplify the description, it is convenient to view representations as left A-modules; $\pi(a)\xi$ is simply denoted by $a\xi$. Let ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ be two *-representations of A. A bounded linear map $T : \mathcal{H} \to \mathcal{K}$ is called an **intertwiner** between them if it satisfies $T(a\xi) = aT(\xi)$ for $a \in A$ and $\xi \in \mathcal{H}$. Regarding the space $\mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{K})$ of such intertwiners as a hom-set, we have the category ${}_A\mathcal{M}od$ of *-representations of A.

Clearly $\mathrm{End}({}_A\mathcal{H}) = \mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{H})$ is a W*-algebra on $\mathcal{H}$ and each hom-vector space as an off-diagonal part of the block presentation of the W*-algebra

$$\mathrm{End}({}_A(\mathcal{H}\oplus\mathcal{K})) = \begin{pmatrix} \mathrm{End}(\mathcal{H}) & \mathrm{Hom}(\mathcal{K},\mathcal{H}) \\ \mathrm{Hom}(\mathcal{H},\mathcal{K}) & \mathrm{End}(\mathcal{K}) \end{pmatrix}.$$

A closed A-submodule ${}_A\mathcal{H}'$ of ${}_A\mathcal{H}$ is called a **subrepresentation** of ${}_A\mathcal{H}$. Let e be the projection to $\mathcal{H}' \subset \mathcal{H}$. Then $e \in \mathrm{End}({}_A\mathcal{H})$ and there is a one-to-one correspondence between subrepresentations of ${}_A\mathcal{H}$ and projections in $\mathrm{End}({}_A\mathcal{H})$. To make the commutativity with the left action of A visible, let M be the oppositive W*-algebra[5] of $\mathrm{End}({}_A\mathcal{H})$, which acts on $\mathcal{H}$ from the right: $\xi a^\circ = a\xi$ for $a \in \mathrm{End}({}_A\mathcal{H})$ and $\xi \in \mathcal{H}$. Then two subrepresentations ${}_A\mathcal{H}e$ and ${}_A\mathcal{H}f$ with e, f projections in M are unitarily equivalent if and only if e, f are **equivalent** in M in the sense that there exists a partial isometry $v \in M$ such that $v^*v = e$ and $vv^* = f$.

Lemma 4.25 (Bernstein)**.** Assume that e is equivalent to a subprojection f' of f and conversely f is equivalent to a subprojection e' of e. Then e and f are equivalent in M.

Proof. We just imitate the set-theoretical proof: Let u and v be partial isometries satisfying $u^*u = e$, $v^*v = f$, $uu^* \leq f$ and $vv^* \leq e$.

Then

$$u, vu, uvu, vuvu, \cdots \quad \text{and} \quad v, uv, vuv, uvuv, \cdots$$

are sequence of partial isometries with their initial projections satisfying

$$\begin{aligned} e &\geq u^*u \geq u^*v^*vu \geq u^*v^*u^*uvu \geq \cdots, \\ f &\geq v^*v \geq v^*u^*uv \geq v^*u^*v^*vuv \geq \cdots. \end{aligned}$$

Let $e = e_0 + e_1 + \cdots + e_\infty$ and $f = f_0 + f_1 + \cdots + f_\infty$ be the acompanied decomposition, where $e_0 = e - u^*u$, $e_1 = u^*u - u^*v^*vu$ and so on. Since $v^*e_0v = f_1$, $u^*f_0u = e_1$, $v^*e_1v = f_2$, $u^*f_1u = e_2$ and so on, partial isometries defined by

$$u_0 = u(e_0 + e_2 + \cdots), \quad v_0 = v(f_0 + f_2 + \cdots), \quad u_\infty = ue_\infty$$

satisfy

$$\begin{aligned} u_\infty^*u_\infty &= e_\infty, \quad u_\infty u_\infty^* = f_\infty, \\ u_0^*u_0 &= e_0 + e_2 + \cdots, \quad u_0u_0^* = f_1 + f_3 + \cdots, \\ v_0^*v_0 &= f_0 + f_2 + \cdots, \quad v_0v_0^* = e_1 + e_3 + \cdots. \end{aligned}$$

[5]If N is a W*-algebra on a Hilbert space $\mathcal{H}$, then the opposite algebra N° is a W*-algebra on $\mathcal{H}^*$: $a^\circ\xi^* = (a^*\xi)^*$ for $a \in N$ and $\xi \in \mathcal{H}$.

Thus the partial isometry $w = u_\infty + u_0 + v_0^*$ gives an equivalence between e and f: $w^*w = e$ and $ww^* = f$. □

A *-representation ${}_A\mathcal{H}$ is said to be **irreducible** if $\mathrm{End}({}_A\mathcal{H}) = \mathbb{C}1_{\mathcal{H}}$. A positive functional is said to be **pure** if the associated GNS-representation is irreducible. A family $\{{}_A\mathcal{H}_j\}$ of *-representations of A is said to be **disjoint** if $\mathrm{Hom}({}_A\mathcal{H}_j, {}_A\mathcal{H}_k) = \{0\}$ for $j \neq k$. Two *-representations of A, π on $\mathcal{H}$ and σ on $\mathcal{K}$, are said to be **quasi-equivalent** if the correspondance $\pi(a) \mapsto \sigma(a)$ $(a \in A)$ extends to a *-isomorphism of $\pi(A)''$ onto $\sigma(A)''$. Two positive functionals φ and ψ of A are said to be **disjoint** (resp. **quasi-equivalent**) if the associated (left) GNS representations are disjoint (resp. quasi-equivalent).

Theorem 4.26.

(i) A positive functional φ is pure if and only if any positive functional ψ satisfying $\psi \leq \varphi$ is proportional to φ.
(ii) A *-representation ${}_A\mathcal{H}$ is irreducible if and only if $\overline{A\xi} = \mathcal{H}$ for any $0 \neq \xi \in \mathcal{H}$.
(iii) Two *-representations ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are not disjoint if and only if we can find non-zero subrepresentations ${}_A\mathcal{H}' \subset {}_A\mathcal{H}$ and ${}_A\mathcal{K}' \subset {}_A\mathcal{K}$ such that ${}_A\mathcal{H}'$ and ${}_A\mathcal{K}'$ are unitarily equivalent.
(iv) Two *-representations ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are quasi-equivalent if and only if there are sets X, Y such that ${}_A\mathcal{H} \otimes \ell^2(X)$ and ${}_A\mathcal{K} \otimes \ell^2(Y)$ are unitarily equivalent.

Corollary 4.27. The set of pure states of a C*-algebra A is invariant under *-automorphisms of A.

Exercise 36. The following conditions on a family $\{\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)\}$ of *-representations of a *-algebra $\mathcal{A}$ are equivalent.

(i) The representations π_i $(i \in I)$ are mutually disjoint.
(ii) $(\bigoplus_i \pi_i)(\mathcal{A})' = \bigoplus_i \pi_i(\mathcal{A})'$.
(iii) $(\bigoplus_i \pi_i)(\mathcal{A})'' = \bigoplus_i \pi_i(\mathcal{A})''$.

5. Linear Functionals on W*-algebras

The L^p-duality is usually established via Radon-Nikodym derivatives and in a measure theoretical way (see Rudin's Real and Complex Analysis Chapter 6 for example.) Try to give a functional analytic proof based on the Riesz lemma, i.e., by modifying the von Neumann's proof of the Radon-Nikodym theorem.

A positive linear functional φ of a W*-algebra M is said to be completely additive if $\varphi(\sum_{\alpha \in I} e_\alpha) = \sum_{\alpha \in I} \varphi(e_\alpha)$ for any family $\{e_\alpha\}_{\alpha \in I}$

of pair-wise orthogonal projections in M. It is said to be **normal**[6] if $\varphi(\sup_{\alpha\in I} a_\alpha) = \sup_{\alpha\in I}\varphi(a_\alpha)$ for any bounded increasing net $\{a_\alpha\}_{\alpha\in I}$ of positive elements in M, where $\sup_{\alpha\in I} a_\alpha$ denotes the least upper bound of $\{a_\alpha\}_{\alpha\in I}$. Clearly complete additivity follows from normality.

Theorem 5.1 (Dixmier)**.** The following conditions on a positive functional φ of a W*-algebra are equivalent.

(i) φ is σ-weakly continuous.
(ii) φ is normal.
(iii) φ is completely additive.

Proof. The implication (i) $\Longrightarrow$ (ii) is a consequence of σ-weak convergence of $\sup_\alpha a_\alpha$ and (ii) $\Longrightarrow$ (iii) is obvious.

We first show that, given any projection $0 \neq p \in M$, we can find a subprojection $0 \neq p' \leq p$ such that $M \ni x \mapsto \varphi(xp')$ is σ-strongly continuous. To see this, choose a positive $\psi \in M_*$ so that $\varphi(p) < \psi(p)$. Since both of φ and ψ are completely additive, we can find a subprojection $p'' \leq p$ which is maximal among subprojections satisfying $\varphi(p'') \geq \psi(p'')$. From $\varphi(p) < \psi(p)$, $p' = p - p'' \neq 0$ and, by maximality, any subprojection e of p' has the property $\varphi(e) \leq \psi(e)$, which implies $\varphi(a) \leq \psi(a)$ for $a \in p'M_+p'$ by spectral decomposition. Now $M \ni x \mapsto \varphi(xp')$ is σ-strongly continuous becuase of

$$|\varphi(xp')| \leq \varphi(1)^{1/2}\varphi(p'x^*xp')^{1/2} \leq \varphi(1)^{1/2}\psi(p'x^*xp')^{1/2}.$$

To complete the implication (iii) $\Longrightarrow$ (i), choose a maximal family $\{p_\alpha\}$ of pairwise orthogonal projections satisfying $p_\alpha\varphi \in M_*$. Then $\sum_\alpha p_\alpha = 1$ by the previous step and, for a finite subset $J \subset I$,

$$\left|\varphi(x(1-\sum_{j\in J} p_j))\right| \leq \varphi(xx^*)^{1/2}\varphi(1-\sum_{j\in J} p_j)^{1/2} \leq \|\varphi\|^{1/2}\|x\|\varphi(1-\sum_{j\in J} p_j)^{1/2}$$

shows that

$$\|\sum_{j\in J} p_j\varphi - \varphi\| \leq \|\varphi\|^{1/2}\varphi(1-\sum_{j\in J} p_j)^{1/2} \to 0 \ (J \nearrow I).$$

□

Proposition 5.2. Let M_* be the set of σ-weakly continuous linear functionals on a W*-algebra M. Then M_* is closed in M^*, M is the dual of M_* and the σ-weak topology on M is equal to the weak* topology as a dual of M_*.

[6]This is a standard but not illuminating terminology; order-continuity would have been much better.

Proof. Since $M \subset \mathcal{B}(\mathcal{H})$ is σ-weakly closed, this follows from $\mathcal{C}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ and the polar relations. □

The predual M_* is now intrinsically characterized as the linear span of normal states. This follows from the fact that every trace class operator is a linear combination of four density operators.

Definition 5.3. Let $M \subset \mathcal{B}(\mathcal{H})$ and $N \subset \mathcal{B}(\mathcal{K})$ be W*-algebras. A *-homomorphism $\phi : M \to N$ is said to be **normal** if it is order-continuous: if $\{a_\alpha\}$ is a norm-bounded increasing net of positive elements in M, then

$$\phi\left(\sup_\alpha a_\alpha\right) = \sup_\alpha \phi(a_\alpha).$$

A *-representation π of M on a Hilbert space $\mathcal{H}$ is said to be normal if $\pi : M \to \mathcal{B}(\mathcal{H})$ is normal.

Proposition 5.4. The following conditions on a *-homomorphism between W*-algebras are equivalent.

(i) ϕ is normal.
(ii) If a positive functional $\omega : M \to \mathbb{C}$ is normal, then so is $\omega \circ \phi$.
(iii) $N_* \circ \phi \subset M_*$.

Proof. (i) $\Longrightarrow$ (ii) is trivial.
(ii) $\Longrightarrow$ (iii): Any $\psi \in N_*$ is a difference of two normal functionals.
(iii) $\Longrightarrow$ (i) follows from the σ-weak convergence of suprema. □

Corollary 5.5. If $\phi : M \to N$ is a *-isomorphism between W*-algebras, ϕ and its inverse ϕ^{-1} are σ-weakly continuous.

Theorem 5.6. If $\phi : M \to N$ is a normal *-homomorphism of W*-algebras, then $\phi(M)$ is σ-weakly closed in N.

Proof. Since ϕ is continuous with respect to the weak* topologies and the unit ball $B \subset M$ is σ-weakly compact, $\phi(B) \subset N$ is σ-weakly compact and therefore it is σ-weakly closed in N. If b is in the σ-weak closure of $\phi(M)$, then it is in the σ-weak closure of $\|b\|\phi(B)$.

Note that $(1+\epsilon)\phi(B)$ contains the unit ball of the C*-algebra $\phi(M)$ for any $\epsilon > 0$. □

Example 5.7.

(i) An ampliation $M \ni x \mapsto x \otimes 1 \in M \otimes 1$ is an injective normal *-homomorphism.
(ii) A unitary map $\mathcal{H} \to \mathcal{K}$ induces an *-isomorphism of W*-algebras $M \ni x \mapsto UxU^* \in UMU^*$.
(iii) Let $\mathcal{K} = e'\mathcal{H}$ be M-invariant with $e' \in M'$ a projection. Then $M \ni x \mapsto xe' \in Me'$ is a surjective normal *-homomorphism.

These are simple examples of normal *-homomorphisms but they are enough to describe general ones.

Theorem 5.8 (Dixmier). Let M on $\mathcal{H}$ and N on $\mathcal{K}$ be W*-algebras, $\phi : M \to N$ be a normal *-homomorphism and suppose that $N = \phi(M)$. Then we can find an index set I, a projection $e \in (M \otimes 1_{\ell^2(I)})'$ and a unitary map $U : e(\mathcal{H} \otimes \ell^2(I)) \to \mathcal{K}$ so that

$$U(a \otimes 1)U^* = \phi(a) \quad \text{for } a \in M.$$

Proof. Choose vectors $\{\eta_j\}_{j\in J}$ in $\mathcal{K}$ so that $\mathcal{K} = \bigoplus_{j\in J} \overline{N\eta_j}$ and let φ_j be positive normal functionals on M defined by $\varphi_j(a) = (\eta_j|\phi(a)\eta_j)$. Then we can find vectors ξ_j in $\mathcal{H} \otimes \ell^2$ satisfying $\varphi_j(a) = (\xi_j|(a \otimes 1)\xi_j)$ $(a \in M)$, which allows us to introduce unitary maps $U_j : \overline{(M \otimes 1)\xi_j} \to \overline{N\eta_j}$ by $U_j((a \otimes 1)\xi_j) = \phi(a)\eta_j$. Note that $N = \phi(M)$ is used here to guarantee the surjectivity of U_j.

Since $\bigoplus_{j\in J} \overline{(M \otimes 1)\xi_j}$ is identified with an $M \otimes 1$-invariant closed subspace of $\mathcal{H} \otimes \ell^2 \otimes \ell^2(J)$, the projection e to the subspace belongs to $(M \otimes 1)'$.

Now, with the choice $I = \mathbb{N} \times J$ and the identification $\ell^2(I) = \ell^2 \otimes \ell^2(J)$, the unitary map $U : e(\mathcal{H} \otimes \ell^2(I)) \to \mathcal{K}$ defined by $U = \bigoplus_{j\in J} U_j$ meets the desired property. □

As a simple application of this theorem, we shall show that the tensor product of W*-algebras has a space-free meaning.

Let M_j (resp. N_j) be W*-algebras on $\mathcal{H}_j$ (resp. $\mathcal{K}_j$) for $j = 1, 2$ and $\alpha : M_1 \to M_2$, $\beta : N_1 \to N_2$ be *-isomorphisms. Then the correspondence $x \otimes y \mapsto \alpha(x) \otimes \beta(y)$ is (uniquely) extended to a *-isomorphism of $M_1 \otimes N_1$ (acting on $\mathcal{H}_1 \otimes \mathcal{K}_1$) onto $M_2 \otimes N_2$ (acting on $\mathcal{H}_2 \otimes \mathcal{K}_2$).

Exercise 37. Describe the details in the proof of the assertion.

In view of the intrinsic nature of σ-weakly continuous linear functionals, it is natural to expect a Jordan type decomposition for $\varphi = \varphi^* \in M_*$.

Definition 5.9. Given a normal linear functional φ on a W*-algebra M, its left and right supports $[\varphi]_l$ and $[\varphi]_r$ are the largest projection e and f in M satisfying $e\varphi = \varphi$ and $\varphi f = \varphi$ respectively.

Note that, if $\varphi^* = \varphi$, left and right supports coincide and are denoted by $[\varphi]$. If φ is positive, $\|\varphi\| = \varphi(1) = \varphi([\varphi])$ and φ is faithful on $[\varphi]M[\varphi]$ in the sense that $\varphi(a) = 0$ implies $a = 0$ for any positive $a \in [\varphi]M[\varphi]$ (apply φ to the spectral projections of a).

Proposition 5.10. A projection e in a W*-algebra M is of the form $e = [\varphi]$ for some $\varphi \in M_*^+$ if and only if it σ-finite in the sense that an orthogonal decomposition $\sum_{i\in I} e_i$ of e in M is possible only for a countable index set I.

Theorem 5.11 (Sakai)**.** Each normal linear functional φ on a W*-algebra M is of the form $u\,|\varphi|$, where $|\varphi|$ is a positive normal linear functional and u is a partial isometry in M satisfying $u^*u = [\varphi]_r = [|\varphi|]$ and the pair $(u, |\varphi|)$ is uniquely determined by these properties. Moreover we have $uu^* = [\varphi]_l$.

Proof. By replacing φ with $\varphi/\|\varphi\|$, we may assume that $\|\varphi\| = 1$. Let $e = [\varphi]_r$ and $[\varphi]_l = f$.

We first check the uniqueness. Let (u, ω) be such a pair. From $\|\varphi\| \leq \|u\|\,\|\omega\| = \|\omega\|$ and $\|\omega\| \leq \|u^*\|\,\|\varphi\| = \|\varphi\|$, we see $\|\omega\| = \|\varphi\| = 1$, whence $1 = \omega(1) = \varphi(u^*)$.

Assume that $\varphi(a) = \|\varphi\| = 1$ with a in the unit ball of eMf. Then $1 = \omega(b)$ with $b = au$ in the unit ball of eMe and

$$1 = \omega(b) \leq \sqrt{\omega(1)}\sqrt{\omega(b^*b)} = \sqrt{\omega(b^*b)} \leq 1$$

shows that $\omega(b^*b) = 1 = \omega(1) = \omega(e)$. Since b^*b is in the unit ball of eMe and ω is faithul on eMe, $\omega(e - b^*b) = 0$ implies $e = b^*b$. Likewise, starting with $\omega(b^*) = 1$, we obtain $e = bb^*$. Thus b is a unitary in the unital C*-algebra eMe and $te + (1-t)b$ $(0 < t < 1)$ can be identified with the continuous function $t + (1-t)z$ of $z \in \sigma(b) \subset \mathbb{T}$. Let μ be the probability measure in $\sigma(b)$ associated with the state ω. Then

$$\begin{aligned} 1 = \omega(te + (1-t)b) &= \int_{\sigma(b)} (t + (1-t)z)\,\mu(dz) \\ &\leq \int_{\sigma(b)} |t + (1-t)z|\,\mu(dz) \\ &\leq \int_{\sigma(b)} (t + (1-t)|z|)\,\mu(dz) = 1. \end{aligned}$$

Since $|t + (1-t)z| < 1$ for $1 \neq z \in \sigma(b)$, μ must be supported by a single point $\{1\}$ and we have

$$\omega((e-b)^*(e-b)) = \int_{\sigma(b)} |1-z|^2\,\mu(dz) = 0,$$

which means $b = e$ by the faithfulness of ω on eMe, i.e., $a = u^*$. Thus the partial isometry part $u \in fMe$ is uniquely determined by $\varphi(u^*) = 1$ and so is $\omega = u^*\varphi$.

Now we establish the existence. Since the unit ball B of M is σ-weakly compact, the function $B \ni x \mapsto |\varphi(x)|$ is σ-weakly continous, and since $1 = \|\varphi\| = \sup\{|\varphi(x)|; x \in B\}$, we can find $x \in B$ such that $|\varphi(x)| = 1$ and then $a = x/\varphi(x) \in B$ satisfies $\varphi(a) = 1$. Replacing a with eaf, we may further assume that a belongs to the unit ball of eMf. Let $a^* = uh$ be the polar decomposition of a^* with $0 \leq h \leq 1$ and $p \leq e$ be the support projection of h. Then the linear functional $\omega = u^*\varphi$ satisfies $\|\omega\| = 1$ and $\omega(h) = \varphi(a) = 1$.

We claim that ω is positive. In fact, $\|h + e^{i\theta}(1-h)\| \leq 1$ and, for the choise of $\theta \in \mathbb{R}$ satisfying $e^{i\theta}\omega(1-h) \geq 0$,

$$1 \leq 1 + e^{i\theta}\omega(1-h) = \omega(h + e^{i\theta}(1-h)) \leq \|h + e^{i\theta}(1-h)\| \leq 1$$

$\omega(1-h) = 0$, i.e., $\omega(1) = \omega(h) = 1$ with $\|\omega\| = 1$, whence ω is positive.

Now the existence is proved by checking $p = e$, i.e., $\varphi p = \varphi$. If not, $\varphi(1-p) \neq 0$ and we can find $b \in B$ satisfying $(1-p)b = b$ and $\varphi(b) > 0$. Then $a^*b = a^*p(1-p)b = 0$ and, for $t > 0$,

$$\|a + tb\|^2 = \|(a+tb)^*(a+tb)\| = \|a^*a + t^2b^*b\| \leq 1 + t^2$$

and therefore

$$1 + t\varphi(b) = \varphi(a + tb) \leq \|a + tb\| \leq \sqrt{1+t^2},$$

which is impossible for $\varphi(b) > 0$. □

6. Tomita-Takesaki Theory

Mainly we follow the presentation in [Bratteli-Robinson 1].

A positive normal functional φ on a W*-algebra M is said to be **faithful** if $\varphi(a) = 0$ for a positive $a \in M$ implies $a = 0$. A positive normal functional φ is faithful if and only if $[\varphi] = 1$ and any positive normal functional φ is faithful when restricted to $[\varphi]M[\varphi]$.

Let M_*^+ be the set of positive normal functionals. Then M_*^+ and M_+ are in the polarity relation: $M_+ = \{a \in M; \omega(a) \in \mathbb{R}_+ \ \forall \omega \in M_*^+\}$ and, if $a \in M_+$ satisfies $\omega(a) = 0$ for all $\omega \in M_*^+$, then $a = 0$.

Given a faithful $\varphi \in M_*^+$, we shall identify the left and right GNS representation spaces $\overline{M\varphi^{1/2}}$ and $\overline{\varphi^{1/2}M}$ in such a way that the identified Hilbert space $L^2(M)$ allows a *-bimodule structure satisfying $(\varphi^{1/2})^* = \varphi^{1/2}$ and the closed convex hull of $\{a\varphi^{1/2}a^*; a \in M\}$ gives a positive cone in $L^2(M)$, i.e., $(\varphi^{1/2}a|a\varphi^{1/2}) = (\varphi^{1/2}|a\varphi^{1/2}a^*) \geq 0$ for any $a \in M$ and $\varphi \in M_*^+$.

To get a hint for the construction, think of a (possibly unbounded) operator $\Delta^{1/2}$ formally defined by $\Delta^{1/2}(a\varphi^{1/2}) = \varphi^{1/2}a$, which is positive by the positive cone assumption, and introduce the notation J to stand for the conjugate-linear isometric involution $\xi \mapsto \xi^*$ ($\xi \in L^2(M)$). Then the combination $J\Delta^{1/2}$ satisfies

$$J\Delta^{1/2}(a\varphi^{1/2}) = a^*\varphi^{1/2}.$$

In other words, if we introduce a (possibly unbounded) conjugate-linear involution S on the left GNS space $\mathcal{H} = \overline{M\varphi^{1/2}}$ by $S(a\varphi^{1/2}) = a^*\varphi^{1/2}$, then J and $\Delta^{1/2}$ can be captured as parts of the polar decomposition of S. Now we regard $M \subset \mathcal{B}(\mathcal{H})$ and introduce the *-operation as well as the right multiplication of $a \in M$ on $\mathcal{H}$ by $\xi^* = J\xi$ and $\xi a = Ja^*J\xi$ for $\xi \in \mathcal{H}$. At this point, we have $(\varphi^{1/2})^* = \varphi^{1/2}$ because $\varphi^{1/2}$ is invariant under S.

The condition $(a\xi)b = a(\xi b)$ is then equivalent to $JMJ \subset M'$, which turns out to be enough to ensure $(a\xi b)^* = b^*\xi^*a^*$ for $a, b \in M$. We also need to show the inequality $(a^*\varphi^{1/2}|J(a\varphi^{1/2})) \geq 0$ $(a \in M)$ to realize the positive cone assumption.

Under these backgrounds, we introduce two conjugate-linear involutions S_0 and F_0 by

$$S_0(a\varphi^{1/2}) = a^*\varphi^{1/2}, \quad F_0(a'\varphi^{1/2}) = (a')^*\varphi^{1/2}, \quad a \in M, a' \in M'$$

Lemma 6.1. Both of S_0 and F_0 are closable with their closure S and F being adjoints of F_0 and S_0 respectively. Moreover, the following conditions on $\xi, \eta \in \mathcal{H}$ are equivalent.

(i) $\xi \in D(F)$ and $\eta = F\xi$.
(ii) There is a closed operator ρ affiliated[7] with M' satisfying $\rho\varphi^{1/2} = \xi$ and $\rho^*\varphi^{1/2} = \eta$.

A similar statement holds for S and M.

Proof. Inclusions $S_0 \subset F_0^*$ and $F_0 \subset S_0^*$ are immediate.

For $\xi \in \mathcal{H}$, a densely defined operator ρ_ξ in $\mathcal{H}$ is set to be $\rho_\xi(a\varphi^{1/2}) = a\xi$ for $a \in M$. If $\xi \in D(S_0^*)$ and $\eta = S_0^*\xi$, i.e., if

$$(a\varphi^{1/2}|\eta) = (\xi|a^*\varphi^{1/2}) \quad \text{for all } a \in M,$$

then

$$(a\varphi^{1/2}|\rho_\xi(b\varphi^{1/2})) = (b^*a\varphi^{1/2}|\xi) = (\eta|a^*b\varphi^{1/2}) = (\rho_\eta(a\varphi^{1/2})|b\varphi^{1/2})$$

shows that $\rho_\xi \subset \rho_\eta^*$ and $\rho_\eta \subset \rho_\xi^*$. Let $\rho = \rho_\xi^{**}$ be the closure of ρ_ξ with $v|\rho|$ the polar decomposition of ρ (cf. Reed-Simon §VIII.9).

If $u \in M$ is a unitary, $\rho_\xi(ua\varphi^{1/2}) = ua\xi = u\rho_\xi(a\varphi^{1/2})$ shows that $u^*\rho_\xi u = \rho_\xi$ and hence $u^*\rho u = \rho$. By the uniqueness of the polar decomposition, $v \in M'$ and the spectral projections of $|\rho|$ belong to M', i.e., ρ is affiliated to M'. Let $e_n' \in M'$ be the one associated to the

[7] If $v|\rho|$ denotes the polar decomposition of ρ, this means that v and spectral projections of $|\rho|$ belong to M'.

interval $[0,n]$ and set $\rho_n = v|\rho|e'_n \in M'$. Then, the convergence

$$\rho_n\varphi^{1/2} = ve'_n v^*\rho_\xi\varphi^{1/2} = ve'_n v^*\xi \to \xi,$$
$$\rho_n^*\varphi^{1/2} = e'_n\rho_\xi^*\varphi^{1/2} = e'_n\eta \to \eta$$

shows that $\xi \oplus \eta$ is in the closure of the graph of F_0. $\square$

Let $S = J\Delta^{1/2}$ be the polar decomposition. Recall that $\Delta^{1/2}$ is a positive self-adjoint operator specified by $\|\Delta^{1/2}\xi\|^2 = \|S\xi\|^2$ with $D(\Delta^{1/2}) = D(S)$ and the antiunitary operator J by $J : \Delta^{1/2}\xi \mapsto S\xi$ for $\xi \in D(S)$. Since $S = S^{-1}$ as the closure of $S_0 = S_0^{-1}$, we have $J\Delta^{1/2} = \Delta^{-1/2}J^{-1} = J^{-1}J\Delta^{-1/2}J^{-1}$ and the uniqueness of the polar decomposition gives

$$J^{-1} = J, \quad J\Delta^{1/2}J = \Delta^{-1/2}.$$

Thus the polar decomposition of F is given by $S^* = \Delta^{1/2}J = J\Delta^{-1/2}$ and the positive self-adjoint operator $FS = SF$ is equal to the square Δ of $\Delta^{1/2}$.

Lemma 6.2 (Fundamental Lemma)**.** Let $a \in M$, $\lambda \in \mathbb{C} \setminus [0,\infty)$ and set $\xi = \frac{1}{\lambda - \Delta^{-1}}a\varphi^{1/2}$. Then an element ρ_λ in M' is defined by

$$\rho_\lambda(x\varphi^{1/2}) = x\xi \quad \text{for } x \in M$$

with an estimate

$$\|\rho_\lambda\| \leq \frac{\|a\|}{\sqrt{2|\lambda| - \lambda - \lambda^*}}.$$

Proof. Since $\xi = \frac{1}{\lambda - \Delta^{-1}}a\varphi^{1/2}$ is in the domain $D(S^*) = D(\Delta^{-1/2})$, we can find a closed operator ρ_λ affiliated to M' satisfying

$$\rho_\lambda(x\varphi^{1/2}) = x\xi \quad \text{for } x \in M$$

and the problem is reduced to showing the estimate on $\|\rho_\lambda\|$.

Let $x \in M$. In the expression

$$\|x\xi\|^2 = (x^*x\xi|\xi) = \left(\frac{1}{\lambda^* - \Delta^{-1}}x^*x\xi \middle| a\varphi^{1/2} \right),$$

we use the fact that $\frac{1}{\lambda^* - \Delta^{-1}}x^*x\xi$ is in the domain of F to find a closed operator ρ affiliated to M' and satisfying

$$\rho\varphi^{1/2} = \frac{1}{\lambda^* - \Delta^{-1}}x^*x\xi, \quad \rho^*\varphi^{1/2} = F\left(\frac{1}{\lambda^* - \Delta^{-1}}x^*x\xi \right).$$

Let $v|\rho|$ be the polar decomposition of ρ. Then

$$\|x\xi\|^2 = (\rho\varphi^{1/2}|a\varphi^{1/2}) = \left(|\rho|^{1/2}\varphi^{1/2} \middle| a|\rho|^{1/2}v^*\varphi^{1/2} \right)$$
$$\leq \|a\|\, \||\rho|^{1/2}\varphi^{1/2}\|\, \||\rho|^{1/2}v^*\varphi^{1/2}\|.$$

Since $\||\rho|^{1/2}v^*\varphi^{1/2}\|^2 = (\rho\varphi^{1/2}|v\varphi^{1/2})$ and $\||\rho|^{1/2}v^*\varphi^{1/2}\|^2$ is equal to

$$(v^*\varphi^{1/2}|\rho^*\varphi^{1/2}) = (F(v\varphi^{1/2})|F(\rho\varphi^{1/2})) = (\rho\varphi^{1/2}|\Delta^{-1}v\varphi^{1/2}),$$

we observe that

$$\begin{aligned}\left|\lambda\||\rho|^{1/2}\varphi^{1/2}\|^2 - \||\rho|^{1/2}v^*\varphi^{1/2}\|^2\right| &= \left|((\lambda^* - \Delta^{-1})\rho\varphi^{1/2}|v\varphi^{1/2})\right| \\ &= \left|(x^*x\xi|v\varphi^{1/2})\right| = \left|(x\xi|vx\varphi^{1/2})\right| \\ &\leq \|x\xi\|\,\|x\varphi^{1/2}\|,\end{aligned}$$

which is combined with the quadratic inequality

$$|\lambda s - t|^2 \geq |\lambda s - t|^2 - (|\lambda|s - t)^2 = (2|\lambda| - \lambda - \lambda^*)st$$

for the choice $s = \||\rho|^{1/2}\varphi^{1/2}\|^2$, $t = \||\rho|^{1/2}v^*\varphi^{1/2}\|^2$ to get

$$\begin{aligned}\sqrt{2|\lambda| - \lambda - \lambda^*}\|x\xi\|^2 &\leq \sqrt{2|\lambda| - \lambda - \lambda^*}\|a\|\||\rho|^{1/2}\varphi^{1/2}\|\,\||\rho|^{1/2}v^*\varphi^{1/2}\| \\ &\leq \|a\|\,\|x\xi\|\,\|x\varphi^{1/2}\|.\end{aligned}$$

□

Lemma 6.3 (Fundamental Relation)**.** As a sesquilinear form relation, we have

$$JaJ = \lambda\Delta^{-1/2}\rho_\lambda^*\Delta^{1/2} - \Delta^{1/2}\rho_\lambda^*\Delta^{-1/2}$$

on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$.

Proof. We start with the relation $\rho_\lambda\varphi^{1/2} = (\lambda - \Delta^{-1})^{-1}a\varphi^{1/2}$ in the form

$$(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) = \lambda(x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) - (x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2})$$

for $x', y' \in M'$ and rewrite each of three terms as follows:

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) &= ((y')^*\varphi^{1/2}|a(x')^*\varphi^{1/2}) = (F(y'\varphi^{1/2})|aF(x'\varphi^{1/2})) \\ &= (JaJ\Delta^{-1/2}x'\varphi^{1/2}|\Delta^{-1/2}y'\varphi^{1/2}),\end{aligned}$$

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) &= ((y')^*\varphi^{1/2}|(x')^*\rho_\lambda\varphi^{1/2}) = (F(y'\varphi^{1/2})|F(\rho_\lambda^*x'\varphi^{1/2})) \\ &= (\Delta^{-1/2}\rho_\lambda^*x'\varphi^{1/2}|\Delta^{-1/2}y'\varphi^{1/2})\end{aligned}$$

and

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2}) &= (F(\rho_\lambda\varphi^{1/2})|F(x'(y')^*\varphi^{1/2})) \\ &= (\rho_\lambda^*\varphi^{1/2}|y'(x')^*\varphi^{1/2}) = ((y')^*\rho_\lambda^*\varphi^{1/2}|(x')^*\varphi^{1/2}) \\ &= (F(\rho_\lambda y'\varphi^{1/2})|F(x'\varphi^{1/2})) \\ &= (\Delta^{-1/2}x'\varphi^{1/2}|\Delta^{-1/2}\rho_\lambda y'\varphi^{1/2})\end{aligned}$$

Note here that we need to be alert on the domain of undounded operators.

To get rid of these nuisances, we again apply the fundamental lemma. For any $x \in M$, $(1+\Delta^{-1})^{-1}x\varphi^{1/2}$ is of the form $x'\varphi^{1/2}$ with $x' \in M'$ and similarly for y'. With this special choice of x' and y', we obtain the following bounded version:

$$(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) = \left(\frac{\Delta^{-1/2}}{1+\Delta^{-1}}JaJ\frac{\Delta^{-1/2}}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)$$
$$(x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) = \left(\frac{\Delta^{-1}}{1+\Delta^{-1}}\rho_\lambda^*\frac{1}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)$$
$$(x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2}) = \left(\frac{1}{1+\Delta^{-1}}\rho_\lambda^*\frac{\Delta^{-1}}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)$$

Since $x, y \in M$ are arbitrary, these relations lead us to

$$\frac{\Delta^{-1/2}}{1+\Delta^{-1}}JaJ\frac{\Delta^{-1/2}}{1+\Delta^{-1}} = \lambda\frac{\Delta^{-1}}{1+\Delta^{-1}}\rho_\lambda^*\frac{1}{1+\Delta^{-1}} - \frac{1}{1+\Delta^{-1}}\rho_\lambda^*\frac{\Delta^{-1}}{1+\Delta^{-1}}$$

and, as a sesquilinear form relation, we conclude[8] that

$$JaJ = \lambda\Delta^{-1/2}\rho_\lambda^*\Delta^{1/2} - \Delta^{1/2}\rho_\lambda^*\Delta^{-1/2}$$

on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$. □

Theorem 6.4 (Tomita-Takesaki)**.** We have $\Delta^{it}JMJ\Delta^{-it} = M'$ for $t \in \mathbb{R}$.

Proof. We shall prove $\Delta^{it}JMJ\Delta^{-it} \subset M'$ for $t \in \mathbb{R}$ in a series of arguments below. Then the inclusion for M' gives $\Delta^{-it}JM'J\Delta^{it} \subset M'' = M$ for $t \in \mathbb{R}$ in view of $J' = J$ and $\Delta' = \Delta^{-1}$, whence the equality holds. □

Before giving a proof, we discuss a general fact on one-parameter automorphism group of the form $\mathrm{Ad}\Delta^{it}$.

Let Δ^{it} be a one-parameter group of unitaries on $\mathcal{H}$ and $C \in \mathcal{B}(\mathcal{H})$. Let $\xi, \eta \in \mathcal{H}$ be entirely analytic vectors for Δ^z $(z \in \mathbb{C})$ and consider a holomorphic function of the form $G(z) = g(z)(\Delta^{\overline{z}}\xi|C\Delta^{-z}\eta)$, where $g(z)$ is a holomorphic function in the punctured strip domain $\{|\mathrm{Re}(z)| \leq 1/2\} \setminus \{0\}$ with a residue r at $z = 0$, and try to have the following form of Cauchy's formula:

$$2\pi r(\xi|C\eta) = \int_{-\infty}^{\infty} (G(it+1/2) - G(it-1/2))\, dt,$$

[8]Recall that the range of $\frac{1}{\Delta^{1/2}+\Delta^{-1/2}}$ is a core for $\Delta^{1/2}$ and $\Delta^{-1/2}$.

where the right hand side is equal to

$$\int_{-\infty}^{\infty} dt\,(\xi|\Delta^{it}\left(g\Big(it+\frac{1}{2}\Big)\Delta^{1/2}C\Delta^{-1/2} - g\Big(it-\frac{1}{2}\Big)\Delta^{-1/2}C\Delta^{1/2}\right)\Delta^{-it}\eta).$$

To tie the integrand to the fundamental relation in Lemma 6.3, we require

$$g\Big(it-\frac{1}{2}\Big) = \lambda g\Big(it+\frac{1}{2}\Big) \quad \text{for } t \in \mathbb{R}.$$

Then, by the choice $\mu = \log\lambda$, the function $g(z)e^{\mu z}$ is periodic of period 1 with simple poles at integer points. Thus it is reasonable to set $g(z)e^{\mu z} = 1/\sin(\pi z)$, i.e., $g(z) = e^{-\mu z}/\sin(\pi z)$.

With this choice, $g(it+s)$ ($t \in \mathbb{R}$, $-1/2 \leq s \leq 1/2$) is rapidly decreasing as $t \to \pm\infty$ if and only if $-\pi < \operatorname{Im}\mu < \pi$. Thus, for μ in this range, the above integral formula in fact holds and takes the form

$$2C = e^{-\mu/2}\int_{-\infty}^{\infty} dt\,\frac{e^{-i\mu t}}{\cosh(\pi t)}\Delta^{it}\left(\Delta^{1/2}C\Delta^{-1/2} - \lambda\Delta^{-1/2}C\Delta^{1/2}\right)\Delta^{-it}.$$

as sesquilinear forms on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$.

Now apply the formula just established for the choice $C = JaJ$ to get

$$-2e^{\mu/2}\rho_\lambda^* = \int_{-\infty}^{\infty} dt\,\frac{e^{-i\mu t}}{\cosh(\pi t)}\Delta^{it}JaJ\Delta^{-it}.$$

Here the range of μ is further restricted to $0 < |\operatorname{Im}\mu| < \pi$ to meet the condition $\lambda \notin [0,\infty)$.

Finally, let $b \in M$. Then, for $\xi, \eta \in \mathcal{H}$, we have

$$\int_{-\infty}^{\infty} dt\,\frac{e^{-i\mu t}}{\cosh(\pi t)}(\xi|[\Delta^{it}JaJ\Delta^{-it}, b]\eta) = 0$$

first for $0 < |\operatorname{Im}\mu| < \pi$ and then for $\mu \in \mathbb{R}$ by continuity on the parameter μ. Thus $\Delta^{it}JaJ\Delta^{-it}$ commutes with every $b \in M$ and we are done.

Corollary 6.5. We have $JMJ = M'$ and $\Delta^{it}M\Delta^{-it} = M$ for $t \in \mathbb{R}$.

Exercise 38. It is instructive to see what is goin on in the case $M = \mathcal{B}(\mathcal{H})$.

Let N be another W*-algebra and $\psi \in N_*^+$ be faithful. In view of the expression

$$\varphi(x)\psi(y) = (\varphi^{1/2}\otimes\psi^{1/2}|(x\otimes y)(\varphi^{1/2}\otimes\psi^{1/2}), \quad x \in M, y \in N$$

$M \otimes N \ni x \otimes y \mapsto \varphi(x)\psi(y)$ defines a normal positive functional $\varphi\otimes\psi$ of $M \otimes N$. Note that $\varphi\otimes\psi$ is faithful because $M' \otimes N' \subset M \otimes N$ and $(M'\otimes N')(\varphi^{1/2}\otimes\psi^{1/2})$ is dense in $\overline{M\varphi^{1/2}}\otimes\overline{N\psi^{1/2}}$ in view of

$\overline{M'\varphi^{1/2}} = \overline{M\varphi^{1/2}}$ and $\overline{N'\psi^{1/2}} = \overline{N\psi^{1/2}}$. Since the algebraic tensor product of M and N is σ-weakly dense in $M \otimes N$, Kaplansky densitiy theorem ensures that $M\varphi^{1/2} \otimes N\psi^{1/2}$ is a core of $S_{\varphi\otimes\psi}$ and we have

$$J_{\varphi\otimes\psi} = J_\varphi \otimes J_\psi, \quad \Delta_{\varphi\otimes\psi} = \Delta_\varphi \otimes \Delta_\psi.$$

Now, on the Hilbert space $\overline{M\varphi^{1/2}} \otimes \overline{N\psi^{1/2}}$, we have

$$(M \otimes N)' = J_{\varphi\otimes\psi}(M \otimes N)J_{\varphi\otimes\psi} = (J_\varphi M J_\varphi) \otimes (J_\psi N J_\psi) = M' \otimes N'$$

Theorem 6.6 (Tomita)**.** Let $M \subset \mathcal{B}(\mathcal{H})$ and $N \subset \mathcal{B}(\mathcal{K})$ be W*-algebras. Then $(M \otimes N)' = M' \otimes N'$ on $\mathcal{H} \otimes \mathcal{K}$.

Proof. We use a general fact that, given a projection $e \in M$, $(eMe)' = M'e$. By the previous discussion, we know

$$(M\otimes N)'(e\otimes f) = ((e\otimes f)(M\otimes N)(e\otimes f))' = (eMe\otimes fNf)' = M'e\otimes N'f$$

if $e = [\varphi]$, $f = [\psi]$ with $\varphi \in M_*^+$, $\psi \in N_*^+$ and the theorem follows from the next lemma. □

Lemma 6.7. For the directed set structure in M_*^+, we have

$$1 = \lim_{\varphi \nearrow \infty} [\varphi]$$

with respect to the σ-weak convergence in M.

Proof. This is a consequence of the fact that, for $a \in M_+$, $\varphi(a) = 0$ for all $\varphi \in M_*^+$ implies $a = 0$, which in turn follows from $M = (M_*)^*$ and $M_* = M_*^+ - M_*^+ + iM_*^+ - iM_*^+$. □

7. Standard Hilbert Spaces

We now identify the left GNS space $\overline{M\varphi^{1/2}}$ and the right GNS Hilbert space $\overline{\varphi^{1/2}M}$ by the unitary map $J(x\varphi^{1/2}) \mapsto \varphi^{1/2}x^*$ with the identified Hilbert space denoted by $L^2(M, \varphi)$ as a non-commutative analogue of $L^2(\Omega, \mu)$. On the Hilbert space $L^2(M, \varphi)$, we have $J(x\varphi^{1/2}) = \varphi^{1/2}x^* = (x\varphi^{1/2})^*$; J gives a star operation on $L^2(M, \varphi)$.

Let $\rho(b)$ be the right multiplication operator by $b \in M$. Then

$$\rho(b)J(x\varphi^{1/2}) = \varphi^{1/2}x^*b = J(b^*x\varphi^{1/2}) = Jb^*JJ(x\varphi^{1/2})$$

shows that $\rho(b) = Jb^*J$ on the left GNS space and the associativity $(a\xi)b = a(\xi)b$ for $a, b \in M$ and $\xi \in L^2(M, \varphi)$ follows from the commutativity $JMJ \subset M'$. The compatibility $(a\xi b)^* = b^*\xi^*a^*$ is also reduced to the commutativity $Ja(Jb^*J) = b^*(JaJ)J$. The positivity assumption is a consequence of the positivity of $\Delta^{1/2}$: $(a^*\varphi^{1/2}|J(a\varphi^{1/2})) = (a\varphi^{1/2}|\Delta^{1/2}a\varphi^{1/2}) \geq 0$. Note that another positivity holds also: For

$a, b \in M_+$, $(a\varphi^{1/2}|\varphi^{1/2}b) = (\varphi^{1/2}|aJbJ\varphi^{1/2}) \geq 0$ because $aJbJ$ is a positive operator on $L^2(M, \varphi)$.

Though the J-relation is enough to identify left and right GNS Hilbert spaces, it is of fundamental importance for further analysis of φ to utilize the fact $\Delta^{it}M\Delta^{-it} = M$.

Definition 7.1. The modular automorphism group[9] $\{\sigma_t\}_{t\in\mathbb{R}}$ of M associated to a faithful $\varphi \in M_*^+$ is defined by $\sigma_t(a) = \Delta^{it}a\Delta^{-it}$ $(a \in M)$.

Let $\mathcal{M}$ be the set of entirely analytic elements for $\{\sigma_t\}$. Then $\mathcal{M}$ is a (weakly) dense *-subalgebra of M. In fact, if $f(z) = \sigma_z(x)$ and $g(z) = \sigma_z(y)$ are analytic continuations of $\sigma_t(x)$ and $\sigma_t(y)$ for $x, y \in \mathcal{M}$, then $f(\overline{z})^*$ and $f(z)g(z)$ are analytic continuations of $\sigma_t(x^*)$ and $\sigma_t(xy)$ respectively. Moreover, thanks to the Gaussian regularization and the Kaplansky density theorem, $\{xx^*; x \in \mathcal{M}, \|x\| \leq 1\}$ is strongly dense in the operator interval $\{a \in M_+; \|a\| \leq 1\}$.

When $\varphi \in M_*^+$ is not faithful, we set $L^2(M, \varphi) = L^2([\varphi]M[\varphi], \varphi)$.

For a natural number $n \geq 2$, let $M_n(M) = M \otimes M_n(\mathbb{C})$ be the matrix ampliation of M. Given a finite family $\{\omega_j\}$ in M_*^+, let $\omega \in M_n(M)_*^+$ be defined by

$$\omega(\{a_{jk}\}) = \sum_{j=1}^{n} \omega_j(a_{jj}).$$

Then $[\omega] = \mathrm{diag}([\omega_1], \cdots, [\omega_n])$ and $[\omega]M_n(M)[\omega]$ is of the form

$$\begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix} \quad \text{with } M_{jk} = [\omega_j]M[\omega_k].$$

The left and right GNS spaces are naturally identified with

$$\overline{[\omega]M_n(M)\omega^{1/2}} = \begin{pmatrix} \overline{M_{11}\omega_1^{1/2}} & \overline{M_{12}\omega_2^{1/2}} & \dots & \overline{M_{1n}\omega_n^{1/2}} \\ \overline{M_{21}\omega_1^{1/2}} & \overline{M_{22}\omega_2^{1/2}} & \dots & \overline{M_{2n}\omega_n^{1/2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{M_{n1}\omega_1^{1/2}} & \overline{M_{n2}\omega_2^{1/2}} & \dots & \overline{M_{nn}\omega_n^{1/2}} \end{pmatrix}$$

[9]Though the natural notation is definitely $\varphi^{it}(\cdot)\varphi^{-it}$, it is customary to use the same symbol σ with the spectrum.

and

$$\overline{\omega^{1/2}M_n(M)[\omega]} = \begin{pmatrix} \overline{\omega_1^{1/2}M_{11}} & \overline{\omega_1^{1/2}M_{12}} & \dots & \overline{\omega_1^{1/2}M_{1n}} \\ \overline{\omega_2^{1/2}M_{21}} & \overline{\omega_2^{1/2}M_{22}} & \dots & \overline{\omega_2^{1/2}M_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\omega_n^{1/2}M_{n1}} & \overline{\omega_n^{1/2}M_{n2}} & \dots & \overline{\omega_n^{1/2}M_{nn}} \end{pmatrix}$$

respectively. Here $\omega^{1/2}$ is identified with the diagonal matrix

$$\begin{pmatrix} \omega_1^{1/2} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \omega_n^{1/2} \end{pmatrix}$$

to facilitate the matrix structure in $L^2(M_n(M), \omega)$.

Since $\overline{[\omega]M_n(M)\omega^{1/2}} = L^2(M_n(M), \omega) = \overline{\omega^{1/2}M_n(M)[\omega]}$, we have a natural identification $\overline{M_{jk}\omega_k^{1/2}} = \overline{\omega_j^{1/2}M_{jk}}$, we recall the explicit procedure of this in the present context:

A densely defined conjugate-linear map of $\overline{M_{jk}\omega_k} \to \overline{M_{kj}\omega_j^{1/2}}$

$$a_{jk}\omega_k^{1/2} \mapsto a_{jk}^*\omega_j^{1/2}$$

is closable with its closure S_{jk} satisfying $S_{jk}^{-1} = S_{kj}$. Let $S_{jk} = J_{jk}\Delta_{jk}^{1/2}$ be the polar decomposition[10]. $J_{jk} : \overline{M_{jk}\omega_k^{1/2}} \to \overline{M_{kj}\omega_j^{1/2}}$ is antiunitary and the identification is given by

$$\overline{M_{jk}\omega_k^{1/2}} \ni \xi \mapsto (J_{jk}\xi)^* \in \overline{\omega_j^{1/2}M_{jk}}$$

The unitaries Δ_{jk}^{it} on $\overline{M_{jk}\omega_k^{1/2}}$ induces a σ-weakly continuous one-parameter group σ_t^{jk} of isometries on M_{jk} so that

$$\Delta_{jk}^{it}a_{jk}\omega_k^{1/2} = \sigma_t^{jk}(a_{jk})\omega_k^{1/2}$$

and, for entirely analytic elements of σ^{jk}, the identification is also specified by

$$a_{jk}\omega_k^{1/2} = \omega_j^{1/2}\sigma_{i/2}^{jk}(a_{jk})$$

We know, in particular, that the identification depends only on the pair ω_j and ω_k, with the modular automorphism group of ω given by

$$\sigma_t \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \sigma_t^{11}(a_{11}) & \sigma_t^{12}(a_{12}) & \dots & \sigma_t^{1n}(a_{1n}) \\ \sigma_t^{21}(a_{21}) & \sigma_t^{22}(a_{22}) & \dots & \sigma_t^{2n}(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{n1}(a_{n1}) & \sigma_t^{n2}(a_{n2}) & \dots & \sigma_t^{nn}(a_{nn}) \end{pmatrix}.$$

[10] $\Delta_{j,k}$ are called relative modular operators.

The positivity of $\sqrt{\varphi_L\varphi_R}(a^*a, bb^*)$ is highly non-trivial. Any counter example?

For each $\varphi \in M_*^+$, let $M \otimes \varphi^{1/2} \otimes M$ be a dummy of the algebraic tensor product $M \otimes M$, which is an M-M bimodule in an obvious way with a compatible *-operation defined by the relation $(a \otimes \varphi^{1/2} \otimes b)^* = b^* \otimes \varphi^{1/2} \otimes a^*$. On the algebraic direct sum

$$\bigoplus_{\varphi \in M_*^+} M \otimes \varphi^{1/2} \otimes M$$

of these *-bimodules, introduce a sesquiliear form by

$$\left(\bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \middle| \bigoplus x_k' \otimes \omega_k^{1/2} \otimes y_k' \right) \\ = \sum_{j,k} ([\omega_k](x_k')^* x_j \omega_j^{1/2} | \omega_k^{1/2} y_k' y_j^* [\omega_j]),$$

which is positive because of

$$\sum_{j,k} ([\omega_k] x_k^* x_j \omega_j^{1/2} | \omega_k^{1/2} y_k y_j^* [\omega_j]) = (X\omega^{1/2} | \omega^{1/2} Y) \\ = (X^{1/2}\omega^{1/2}Y^{1/2} | X^{1/2}\omega^{1/2}Y^{1/2}) \geq 0.$$

Here

$$X = [\omega] \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} [\omega] \quad \text{and} \quad Y = [\omega] \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \begin{pmatrix} y_1^* & \dots & y_n^* \end{pmatrix} [\omega]$$

are positive elments in $[\omega]M_n(M)[\omega]$. Note that $[\omega] = \mathrm{diag}([\omega_1], \dots, [\omega_n])$.

The associated Hilbert space is denoted by $L^2(M)$ and the quotient element of $a \otimes \varphi^{1/2} \otimes b$ by $a\varphi^{1/2}b$. Here the notation is compatible with the one for $L^2(M, \varphi)$ because

$$[\varphi]M[\varphi] \otimes \varphi^{1/2} \otimes [\varphi]M[\varphi] \ni a \otimes \varphi^{1/2} \otimes b \mapsto a\varphi^{1/2}b \in L^2(M, \varphi)$$

gives an isometric map by the way of the definition of the inner product. Similar remarks are in order for left and right GNS spaces.

The left and right actions of M are compatible with taking quotients and they are bounded: For $a \in M$,

$$\left\| \bigoplus_j a x_j \otimes \omega_j^{1/2} \otimes y_j \right\|^2 = (\omega^{1/2} | ZJYJ\omega^{1/2})$$

with

$$0 \leq Z = [\omega] \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix} a^* a \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} [\omega] \leq \|a\|^2 X.$$

Moreover, these actions give *-representations of M: $(a\xi|\eta) = (\xi|a^*\eta)$ and $(\xi a|\eta) = (\xi|\eta a^*)$ for $\xi, \eta \in L^2(M)$ and $a \in M$, which is clear from the definition of the inner product.

The *-operation on $L^2(M)$ is also compatible with the inner product:

$$\begin{aligned} \left\| \left(\bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \right)^* \right\|^2 &= \left\| \bigoplus_j y_j^* \otimes \omega_j^{1/2} \otimes x_j^* \right\|^2 = (Y\omega^{1/2}|\omega^{1/2}X) \\ &= ((\omega^{1/2}X)^*|(Y\omega^{1/2})^*) = (X\omega^{1/2}|\omega^{1/2}Y) \\ &= \left\| \bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \right\|^2. \end{aligned}$$

So far, we have constructed a *-bimodule $L^2(M)$ of M in such a way that $L^2(M, \varphi) \subset L^2(M)$ for each $\varphi \in M_*^+$ and the closed subspaces $\overline{M\varphi^{1/2}}$, $\overline{\varphi^{1/2}M}$ in $L^2(M)$ are naturally identified with the left and right GNS spaces of φ respectively. Moreover, for $\varphi, \psi \in M_*^+$, we have $[\varphi]\overline{M\psi^{1/2}} = \overline{\varphi^{1/2}M}[\psi]$ in $L^2(M)$, which is just a reflection of the fact that the same identification inside $L^2(M_n(M), \omega)$ is used in the definition of the inner product.

Lemma 7.2. Given a countable family $\{a_j\}$ in M and a countable family $\{\varphi_j\}$ in M_*^+, let $\varphi \in M_*^+$ be defined by

$$\varphi = \sum_{j,k \geq 1} \frac{1}{2^{j+k}} \frac{a_j \varphi_k a_j^*}{\varphi_k(a_j^* a_j)}.$$

Then $[\varphi] a_j \varphi_k^{1/2} = a_j \varphi_k^{1/2}$ for $j, k \geq 1$.

Proof. In fact, $0 = \varphi(1 - [\varphi]) = \sum_{j,k} \frac{1}{2^{j+k}} \|(1 - [\varphi]) a_j \varphi_k^{1/2}\|^2 / \|a_j \varphi_k\|^2$. □

Proposition 7.3. Let $\varphi \in M_*^+$ and p be the central support[11] of φ. Then we have

$$\overline{\varphi^{1/2}M} = [\varphi]L^2(M), \quad L^2(M, \varphi) = [\varphi]L^2(M)[\varphi], \quad \overline{M\varphi^{1/2}M} = pL^2(M).$$

[11] p is the minimal projection in $M \cap M'$ satisfying $p\varphi = \varphi$.

Proof. The first equality follows from

$$[\varphi]\sum_j a_j\omega_j^{1/2}b_j \in \sum_j [\varphi]\overline{M\omega_j^{1/2}}b_j = \sum \overline{\varphi^{1/2}M}[\omega_j]b_j \subset \overline{\varphi^{1/2}M}.$$

The second equality is a consequence of the first equality by

$$L^2(M,\varphi) = \overline{\varphi^{1/2}[\varphi]M[\varphi]} = \overline{\varphi^{1/2}M}[\varphi].$$

Let $\mathfrak{U}$ be the set of unitaries in M. Then $p = \bigvee_{u\in\mathfrak{U}} u[\varphi]u^*$ and

$$pL^2(M) = \overline{\sum_{u\in\mathfrak{U}} u[\varphi]u^*L^2(M)} = \overline{\sum_{u\in\mathfrak{U}} u[\varphi]L^2(M)} = \overline{\sum_{u\in\mathfrak{U}} u\overline{\varphi^{1/2}M}} \subset \overline{M\varphi^{1/2}M}.$$

□

Corollary 7.4. The algebraic sum $\sum_{\varphi\in M_*^+} M\varphi^{1/2}$ is dense in $L^2(M)$. Consequently the left and right representations of M on $L^2(M)$ are σ-weakly continuous.

Proof. By the previous lemma, $[\varphi] \nearrow 1$ as $\varphi \nearrow \infty$, which is used to have

$$L^2(M) = \lim_{\varphi\nearrow\infty}[\varphi]L^2(M) = \lim_{\varphi\nearrow\infty}\overline{\varphi^{1/2}M}.$$

□

Lemma 7.5. Let $\varphi,\psi \in M_*^+$ satisfy $\varphi \leq \psi$. Then there exists exactly one $a \in M$ satisfying $\varphi^{1/2} = a\psi^{1/2}$ and $a[\psi] = a$.

Proof. If $c\psi^{1/2} = 0$ with $c[\psi] = c$, then $c^*c \in [\psi]M[\psi]$ and $\psi(c^*c) = 0$ imply $c^*c = 0$, showing the uniqueness of a. In particular, a satisfies $[\varphi]a = a$, whence $a \in [\varphi]M[\psi] \subset [\psi]M[\psi]$. Thus, replacing M with $[\psi]M[\psi]$, we may assume that ψ is faithful for the existence.

The map $\psi^{1/2}M \ni \psi^{1/2}x \mapsto \varphi^{1/2}x \in L^2(M)$ is contractive and it gives a bounded linear operator a on $L^2(M)$ by the density of $\psi^{1/2}M$ in $L^2(M)$. Clearly a commutes with the right action of M and therefore it belongs to M. □

Proposition 7.6. Let $\varphi = \sum_{n\geq 1}\varphi_n$ with $\varphi_n \in M_*^+$. Then

$$\overline{M\varphi^{1/2}} = \overline{\sum_{n\geq 1} M\varphi_n^{1/2}}.$$

Moreover $M(\varphi_1 + \cdots + \varphi_n)^{1/2}$ is increasing in $n \geq 1$ and their union is dense in $M\varphi^{1/2}$.

Proof. First note that there is a one-to-one correspondence closed M-submodules of ${}_ML^2(M)$ and projections in M: Given a prohjection $e \in M$, $L(M)e$ is a closed submodule and any closed submodule is of this form. Consequently, $L^2(M)(\bigvee_{i\in I} e_i) = \overline{\sum_{i\in I} L^2(M)e_i}$.

Now $[\varphi] = \bigvee_{n\geq 1}[\varphi_n]$ shows that

$$\overline{\sum_{n\geq 1} M\varphi_n^{1/2}} = \overline{\sum_{n\geq 1} L^2(M)[\varphi_n]} = L^2(M)(\bigvee_{n\geq 1}[\varphi_n]) = L^2(M)[\varphi] = \overline{M\varphi^{1/2}}.$$

Finally, by the previous lemma, $\varphi_j^{1/2} = a_j(\varphi_1 + \cdots + \varphi_n)^{1/2}$ for some $a_j \in M$ $(1 \leq j \leq n)$ shows that

$$M\varphi_1^{1/2} \subset M(\varphi_1 + \varphi_2)^{1/2} \subset \cdots \subset M\varphi^{1/2}$$

and

$$\sum_{j=1}^n M\varphi_j^{1/2} = \sum_{j=1}^n Ma_j(\varphi_1 + \cdots + \varphi_n)^{1/2} \subset M(\varphi_1 + \cdots + \varphi_n)^{1/2},$$

which give the density in question. □

Corollary 7.7. Let $\varphi_1, \dots, \varphi_n \in M_*^+$. Then

$$\overline{M(\varphi_1^{1/2} + \cdots + \varphi_n^{1/2})} = \overline{M\varphi_1^{1/2} + \cdots + M\varphi_n^{1/2}} = \overline{M(\varphi_1 + \cdots + \varphi_n)^{1/2}}.$$

Lemma 7.8. Let $f \in M$ be a projection of the form $f = [\omega]$ for some $\omega \in M_*^+$. Then each $T \in \mathrm{End}({}_ML^2(M)f)$ is realized by the right multiplication of a uniquely determined element in fMf.

Proof. Let $T \in \mathcal{B}(L^2(M)f)$ commute with the left action of M. Let $\varphi \in M_*^+$ satisfy $[\varphi] \geq f$ and $T_\varphi \in \mathcal{B}([\varphi]L^2(M)[\varphi])$ be defined by

$$T_\varphi(\xi) = [\varphi]T(\xi f) \quad \text{for } \xi \in [\varphi]L^2(M)[\varphi].$$

Clearly T_φ commutes with the left action of $[\varphi]M[\varphi]$ and it is realized by the right multiplication of a uniquely determined element $a_\varphi \in [\varphi]M[\varphi]$. Since the range of T_φ is included in $L^2(M)f$ and T_φ vanishes on $[\varphi]L^2(M)([\varphi]-f)$, a_φ belongs to fMf. Now let $\psi \in M_*^+$ be another functional satisfying $[\psi] \geq f$. Since $\omega^{1/2} \in fL^2(M)f$ is separating for fMf, the equality

$$\omega^{1/2}a_\varphi = T\omega^{1/2} = \omega^{1/2}a_\psi$$

implies $a_\varphi = a_\psi$. Thus, writing a for the common a_φ,

$$[\varphi]T[\varphi]\xi = \xi a \quad \text{for } \xi \in L^2(M)f$$

and then, by taking the limit $\varphi \nearrow \infty$, we conclude that $T\xi = \xi a$ for $\xi \in L^2(M)f$. □

Theorem 7.9. The left and right actions of M on $L^2(M)$ give the commutants of each other.

Proof. Let T commute with the left action of M on $L^2(M)$. For $\varphi \in M_*^+$, let $[\varphi]' \in \mathcal{B}(L^2(M))$ be given by the right multiplication of $[\varphi] \in M$. Then $[\varphi]'T[\varphi]'$ on $L^2(M)[\varphi]$ is realized by $a_\varphi \in [\varphi]M[\varphi]$:

$$[\varphi]'T[\varphi]'\xi = \xi a_\varphi.$$

If $[\psi] \geq [\varphi]$ with $\psi \in M_*^+$, then by the uniqueness we see $a_\varphi = [\varphi]a_\psi[\varphi]$. Thus, if $a \in M$ is defined so that $a_\varphi = [\varphi]a[\varphi]$, then

$$T\xi = \lim_{\varphi \nearrow \infty} \xi a_\varphi = \xi a$$

for $\xi \in L^2(M)$. □

Definition 7.10. Let $L^2_+(M)$ be the closed convex cone generated by the set $\{a\varphi^{1/2}a^*; a \in M, \varphi \in M_*^+\}$.

Lemma 7.11. Let $\varphi, \psi \in M_*^+$ and $a \in M$. Then

$$(\varphi^{1/2}|a\psi^{1/2}a^*) \geq 0.$$

Proof.

$$0 \leq \left(\begin{pmatrix} \varphi^{1/2} & 0 \\ 0 & \psi^{1/2} \end{pmatrix} \middle| \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi^{1/2} & 0 \\ 0 & \psi^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a^* & 0 \end{pmatrix} \right) = (\varphi^{1/2}|a\psi^{1/2}a^*).$$

□

For $\xi, \eta \in L^2(M)$, let $\xi\eta \in M_*$ be defined by

$$\langle x, \xi\eta \rangle = (\eta^*|x\xi), \quad x \in M.$$

Clearly $\|\xi\eta\| \leq \|\xi\| \, \|\eta\|$ and the bilinear map $\xi \times \eta \mapsto \xi\eta$ is compatibile with the *-bimodule sutructure of $L^2(M)$: $(\xi b)\eta = \xi(b\eta)$, $(a\xi)\eta = a(\xi\eta)$ and $(\xi\eta)^* = \eta^*\xi^*$.

Given $\xi \in L^2(M)$, its left (resp. right) support is the maximal projection e (resp. f) in M satisfying $(1-e)\xi = 0$ (resp. $\xi(1-f) = 0$). When $\xi^* = \xi$, these projections coincide and are denoted by $[\xi]$.

Theorem 7.12 (Polar Decomposition)**.** Each $\xi \in L^2(M)$ has one and only one expression of the form $\xi = v|\xi|$, where $|\xi| \in L^2_+(M)$ and $v \in M$ satisfies $v^*v = [|\xi|]$. Moreover the unique $|\xi|$ is equal to $(\xi^*\xi)^{1/2}$ with $\xi^*\xi \in M_*^+$.

Proof. Let e and f be the left and right supports of ξ. Clearly $\varphi \equiv \xi^*\xi \in M_*^+$ and

$$\varphi^{1/2}x \mapsto \xi x, \quad x \in M$$

defines an isometry of $fL^2(M)$ onto $eL^2(M)$, which commutes with the right action of M and we can find a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$ and $\xi = v\varphi^{1/2}$.

Now assume that $\xi \in L^2_+(M)$ and we shall show that $v = e = f$.

First, from the invariance $\xi^* = \xi$, $e = f$. By replacing M with the reduced W*-algebra eMe, we may assume that φ is faithul and v is a unitary.

Then the densely defined conjugate-linear map $\varphi^{1/2}x \mapsto \xi x^* = v\varphi^{1/2}x^*$ has the closure $vF = v\Delta^{1/2}J$ with its adjoint given by $J\Delta^{1/2}v^*$. Note here that, by the unitarity of v, both of $v\Delta^{1/2}$ and $\Delta^{1/2}v^*$ are closed and adjoints of each other. Now, for $x \in M$,

$$\begin{aligned} v\Delta^{1/2}J(\varphi^{1/2}x) &= v(\varphi^{1/2}x^*) = \varphi^{1/2}v^*x^* = \Delta^{1/2}J(\varphi^{1/2}xv) \\ &= \Delta^{1/2}J(Jv^*J)(\varphi^{1/2}x) = \Delta^{1/2}v^*J(\varphi^{1/2}x), \end{aligned}$$

where we used $v\varphi^{1/2} = (v\varphi^{1/2})^* = \varphi^{1/2}v^*$ at the middle of the first line. Since $J(\varphi^{1/2}M) = M\varphi^{1/2}$ is a core for $v\Delta^{1/2}$ and $\Delta^{1/2}v^*$, see that $v\Delta^{1/2} = \Delta^{1/2}v^*$ is self-adjoint.

Finally we take the positivity of $\xi \in L^2(M)_+$ into account to have

$$\begin{aligned} 0 \leq (\xi|x\varphi^{1/2}x^*) &= (x^*\varphi^{1/2}v^*|\varphi^{1/2}x^*) = (JvJ(x^*\varphi^{1/2})|\varphi^{1/2}x^*) \\ &= (Jv\Delta^{1/2}(x\varphi^{1/2})|\varphi^{1/2}x^*) = (x\varphi^{1/2}|v\Delta^{1/2}(x\varphi^{1/2})) \end{aligned}$$

for $x \in M$. Since $M\varphi^{1/2}$ is a core for the self-adjoint operator $v\Delta^{1/2}$, this implies $v\Delta^{1/2} \geq 0$ and we conclude that $v = 1$ thanks to the uniqueness of polar decomposition. □

Corollary 7.13 (Jordan Decomposition)**.**

(i) Each $\xi \in L^2_+(M)$ is of the form $\varphi^{1/2}$ with a unique $\varphi \in M_*^+$.
(ii) Each $\xi = \xi^*$ in $L^2(M)$ has a unique decomposition $\xi = \xi_+ - \xi_-$, where $\xi_\pm \in L^2_+(M)$ satisfies $(\xi_+|\xi_-) = 0$.

Proof. Let $\xi = v|\xi|$ be the polar decomposition. Then $\xi = \xi^* = |\xi|v^* = v^*(v|\xi|v^*)$ gives another polar decomposition and the uniqueness implies $v = v^*$ and $v|\xi|v^* = |\xi|$. Let $v = p_+ - p_-$ be the spectral decomposition. Since $v|\xi| = |\xi|v$, $v^*|\xi| = |\xi|v^*$ and $p_\pm$ are σ-weak limit of polynomials of v, v^*, we see that $p_\pm$ commute with $|\xi|$ and $\xi_\pm = p_\pm|\xi| = |\xi|p_\pm$ give the decomposition.

Let $\xi = \eta_+ - \eta_-$ be another Jordan decomposition. Then $\xi_+ - \eta_+ = \xi_- - \eta_-$ and the uniqueness follows from

$$\|\xi_+ - \eta_+\|^2 = (\xi_+ - \eta_+|\xi_- - \eta_-) = -(\xi_+|\eta_-) - (\eta_+|\xi_-) \leq 0.$$

□

Example 7.14. Let θ be a *-sutomorphism of a W*-algebra M. Then $\Theta\varphi^{1/2} = (\varphi \circ \theta)^{1/2}$. Conversely, given Θ,

Exercise 39. Let $\varphi \in M_*^+$ and $v \in M$ satisfy $v^*v = [\varphi]$. Then $(v\varphi v^*)^{1/2} = v\varphi^{1/2}v^*$.

Exercise 40. For $\xi, \eta \in L^2_+(M)$, the following conditions are equivalent. (i) $(\xi|\eta) = 0$. (ii) $[\xi][\eta] = 0$. (iii) $\xi\eta = 0$. (Hint: the construction and the uniqueness of the Jordan decomposition of $\xi - \eta$.)

As applications of this basic fact, we record here two further results.

Theorem 7.15 (Powers-Störmer-Araki)**.** For $\varphi, \psi \in M_*^+$,

$$\|\varphi^{1/2} - \psi^{1/2}\|^2 \leq \|\varphi - \psi\| \leq \|\varphi^{1/2} + \psi^{1/2}\| \, \|\varphi^{1/2} - \psi^{1/2}\|.$$

Proof. We first remark that

$$\varphi(a) - \psi(a) = \frac{1}{2}\Big((\varphi^{1/2} + \psi^{1/2}|a(\varphi^{1/2} - \psi^{1/2})) + (\varphi^{1/2} - \psi^{1/2}|a(\varphi^{1/2} + \psi^{1/2}))\Big),$$

from which the second inequality follows immediately.

Let $\varphi^{1/2} - \psi^{1/2} = \xi - \eta$ ($\xi, \eta \in L^2_+(M)$, $(\xi|\eta) = 0$) be a Jordan decomposition. Then, for the choice $a = [\xi] - [\eta]$,

$$\begin{aligned}
\|\varphi - \psi\| \geq \varphi(a) - \psi(a) &= \mathrm{Re}(\varphi^{1/2} - \psi^{1/2}|a(\varphi^{1/2} + \psi^{1/2})) \\
&= \mathrm{Re}(\xi - \eta|a(\varphi^{1/2} + \psi^{1/2})) \\
&= (\xi + \eta|\varphi^{1/2} + \psi^{1/2}) \\
&\geq (\xi - \eta|\varphi^{1/2}) - (\xi - \eta|\psi^{1/2}) \\
&= \|\varphi^{1/2} - \psi^{1/2}\|^2.
\end{aligned}$$

□

Theorem 7.16. $L^2_+(M)$ is a self-dual cone in the sense that

$$L^2_+(M) = \{\xi \in L^2(M); (\xi|\eta) \geq 0 \ \forall \eta \in L^2_+(M)\}.$$

Proof. Assume that $\zeta \in L^2(M)$ is evaluated with elements in $L^2_+(M)$ to have positive reals. Then, in terms of the four sum decomposition of $\zeta \in L^2(M)$, $\zeta = \xi_+ - \xi_- + i(\eta_+ - \eta_-)$, $0 = \mathrm{Im}(\eta_\pm|\zeta) = \pm\|\eta_\pm\|^2$ implies $\eta_\pm = 0$ and then $0 \leq (\xi_-|\zeta) = -\|\xi_-\|^2$ shows that $\xi_- = 0$. □

Let $L^2_+(M, \varphi)$ be the closure of $\{x\varphi^{1/2}x^*; x \in [\varphi]M[\varphi]\}$ in $L^2(M, \varphi)$.

Lemma 7.17. Let $\varphi \in M_*^+$ be faithful. Then we have $\overline{\Delta^{1/4}(M_+\varphi^{1/2})} = L^2_+(M, \varphi) = \overline{\Delta^{-1/4}(M'_+\varphi^{1/2})}$.

Proof. Let $a \in \mathcal{M}$. Then the relation $\Delta^{it}(aa^*\varphi^{1/2}) = \sigma_t(a)\sigma_t(a)^*\varphi^{1/2}$ is analytically continued to

$$\begin{aligned}\Delta^{1/4}(aa^*\varphi^{1/2}) &= \sigma_{-i/4}(a)(\sigma_{i/4}(a))^*\varphi^{1/2} = \sigma_{-i/4}(a)J\Delta^{1/2}(\sigma_{i/4}(a)\varphi^{1/2}) \\ &= \sigma_{-i/4}(a)J(\sigma_{-i/4}(a)\varphi^{1/2}) = \sigma_{-i/4}(a)\varphi^{1/2}\sigma_{-i/4}(a)^*.\end{aligned}$$

Since $\sigma_{-is}(\mathcal{M}) = \mathcal{M}$ is σ-weakly dense in M, the Kaplansky density theorem shows that each $b \in M$ is boundedly approximated by elements in $\mathcal{M}$ in the strong operator topology and we see that $L^2_+(M,\varphi) \subset \overline{\Delta^{1/4}M_+\varphi^{1/2}}$.

Conversely, we use the Kaplansky's density theorem again to approximate $a \in M_+$ in the strong operator topology by a sequence $a_n = b_n b_n^*$ with $b_n \in \mathcal{M}$. Since $J\Delta^{1/2}(a_n\varphi^{1/2}) = a_n\varphi^{1/2} \to a\varphi^{1/2} = J\Delta^{1/2}(a\varphi^{1/2})$,

$$\|\Delta^{1/4}((a_n - a)\varphi^{1/2})\|^2 = ((a_n - a)\varphi^{1/2}|\Delta^{1/2}(a_n - a)\varphi^{1/2}) \to 0.$$

Thus $\Delta^{1/4}(a\varphi^{1/2})$ is approximated by $\Delta^{1/4}(b_n b_n^*\varphi^{1/2}) \in L^2_+(M,\varphi)$. $\square$

Exercise 41. Show $\overline{M_+\varphi^{1/2}} \subset D(\Delta^{1/4})$ and $\Delta^{1/4}\overline{M_+\varphi^{1/2}} \subset L^2_+(M,\varphi)$.

Lemma 7.18. Let $\varphi_j \in M_*^+$ be faithful for $j = 1, 2$. Then $L^2_+(M,\varphi_1) = L^2_+(M,\varphi_2)$.

Proof. Let $\begin{pmatrix}\mathcal{M}_{11} & \mathcal{M}_{12}\\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{pmatrix}$ be the set of entirely analytic elements of the modular automorphism group σ_t associated to a faithful $\varphi = \mathrm{diag}(\varphi_1,\varphi_2)$ on $M_2(M)$: If we introduce σ-weakly continuous one-parameter groups $\{\sigma_t^{j,k}\}$ of isometries on M by

$$\sigma_t\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix}\sigma_t^{1,1}(a_{11}) & \sigma_t^{1,2}(a_{12})\\ \sigma_t^{2,1}(a_{21}) & \sigma_t^{2,2}(a_{22})\end{pmatrix},$$

$\mathcal{M}_{j,k}$ is the set of entirely analytic elements of M for $\sigma_t^{j,k}$. The modular operator Δ is also split into four parts $\Delta_{j,k}$ so that

$$\Delta = \begin{pmatrix}\Delta_{1,2} & \Delta_{1,2}\\ \Delta_{2,1} & \Delta_{2,2}\end{pmatrix},$$

where positive self-adjoint operators $\Delta_{j,k}$ on $L^2(M)$ are specified by $\Delta_{j,k}^{1/2}(x\varphi_k^{1/2}) = \varphi_j^{1/2}x$ $(x \in M)$.

Let $a \in M_2(M)$ with $a_{j,k} \in \mathcal{M}_{j,k}$. As in the proof of the previous lemma, we have the relation $\Delta^{1/4}(aa^*\varphi^{1/2}) = \sigma_{-i/4}(a)\varphi^{1/2}\sigma_{-i/4}(a)^*$ and then, from the choice $a = \begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}$,

$$\Delta_{1,1}^{1/4}(xx^*\varphi_1^{1/2}) = \sigma_{-i/4}^{1,2}(x)\varphi_2^{1/2}\sigma_{-i/4}^{1,2}(x)^*.$$

Since $\begin{pmatrix}\mathcal{M}_{1,1} & \mathcal{M}_{1,2}\\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2}\end{pmatrix}$ is σ-weakly dense in $M_2(M)$, the Kaplansky density theorem shows that each $y \in M$ is boundedly approximated by elements in $\sigma^{1,2}_{-i/4}(\mathcal{M}_{1,2}) = \mathcal{M}_{1,2}$ in the *strong topology and we see that $y\varphi_2^{1/2}y^*$ is in the weak closure of $\Delta_{1,1}^{1/4}(M_+\varphi_1^{1/2}) \subset L^2_+(M,\varphi_1)$. Since $L^2_+(M,\varphi_1)$ is a convex set, this means that $y\varphi_2^{1/2}y^* \in L^2_+(M,\varphi_1)$. $\square$

Recall that $L^2_+(M,\varphi)$ is the norm closure of $\{a\varphi^{1/2}a^*; a \in [\varphi]M[\varphi]$ in $L^2(M)$.

Theorem 7.19.

(i) We have $L^2_+(M,\varphi) = [\varphi]L^2_+(M)[\varphi]$ and the unitary Δ^{it} on $[\varphi]L^2(M)[\varphi]$ leaves $L^2_+(M,\varphi)$ invariant globally.

(ii) Let p be the central support of $\varphi \in M_*^+$. Then the closed convex hull C of $\{a\varphi^{1/2}a^*; a \in M\}$ in $L^2(M)$ is equal to $pL^2_+(M) = L^2_+(M)p$.

Proof. (i) Let $\omega^{1/2} \in [\varphi]L^2_+(M)[\varphi]$. Then $\omega + \epsilon\varphi$ with $\epsilon > 0$ is faithful on $[\varphi]M[\varphi]$ and we see $(\omega+\epsilon\varphi)^{1/2} \in L^2_+(M,\omega+\epsilon\varphi) = L^2_+(M,\varphi)$, whence $\omega^{1/2} = \lim_{\epsilon\to+0}(\omega+\epsilon\varphi)^{1/2} \in L^2_+(M,\varphi)$.

(ii) Let $\omega \in pM_*^+$. Since $p = \bigvee_{u\in\mathcal{U}} u[\varphi]u^*$, we can find a sequence $\{u_n\}$ of unitaries in M such that $[\omega] = \bigvee_{n\geq 1} u_n[\varphi]u_n^*$. Then $[\omega] = [\psi]$ with $\psi^{1/2} = \sum_{n\geq 1} 2^{-n}u_n\varphi^{1/2}u_n^*$ implies $\omega^{1/2} \in L^2_+(M,\psi) \subset C$. $\square$

Corollary 7.20. $L^2_+(M,\varphi)$ is a self-dual cone in $L^2(M,\varphi)$.

Exercise 42. Identify the standard Hilbert space of $M = \mathcal{B}(\mathcal{H})$ with $\mathcal{H}\otimes\mathcal{H}^*$ and describe its positive cone.

8. Universal Representations

Let A be a C*-algebra and we shall construct a *-bimodule $L^2(A)$ of A in such a way that it generalizes the commutative case discussed in §4.

Consider the Hilbert space direct sum $\mathcal{U} = \bigoplus_{\varphi\in A_+^*} \overline{A\varphi^{1/2}}$ of left GNS spaces on which A is represented by left multiplication. Thanks to the Gelfand-Naimark theorem, the representation is faithful and we regard A as a C*-subalgebra of $\mathcal{B}(\mathcal{U})$. Let $M = A''$ be the W*-algebra on $\mathcal{U}$ generated by A. Since A is a weakly dense *-subalgebra of M, M_* is identified with a subspace of A^* by restriction. Note that $A_+^* \subset M_*^+$ by the way of our construction. Since elements in A^* are linear combinations of positive functionals (Theorem 3.15), the equality $M_* = A^*$ holds as a linear space.

We claim that $M_* = A^*$ as a Banach space. This follows from the Kaplansky density theorem: Let $\phi \in M_*$, $a \in M$ and choose a net $\{a_\alpha\} \subset A$ so that $\|a_\alpha\| \leq \|a\|$ and $a = \lim_{\alpha\to\infty} a_\alpha$ in the σ-weak topology of M. Then

$$|\phi(a)| = \lim_{\alpha\to\infty} |\phi(a_\alpha)| \leq \limsup_{\alpha\to\infty} \|\phi\|\|a_\alpha\| \leq \|\phi\|\|a\|$$

shows that the norm of ϕ in M_* is equal to that in A^*. Thus M is identified with the second dual A^{**} of A.

Now we set $L^2(A) = L^2(M)$. For an index set I, let

$${}_AL^2(A)^{\oplus I} = \bigoplus_{i\in I} {}_AL^2(A) = {}_AL^2(A) \otimes \ell^2(I).$$

The opposite algebra of $\mathrm{End}({}_AL^2(A)^{\oplus I})$ is naturally identified with the matrix ampliation $M_I(M)$ of M, which acts on $L^2(A)^{\oplus I}$ from the right. Any *-representation ${}_A\mathcal{H}$ of A is unitarily equivalent to ${}_AL^2(A)^{\oplus I}e$, where e is a projection in $M_I(M)$ and the cardinality of I is specified by the existence of an orthogonal decomposition of the form $\mathcal{H} = \bigoplus_{i\in I} \overline{A\xi_i}$ (see Theorem 5.8).

Denote the associated *-representation of A by π_e and let e' be the projection onto the subspace $L^2(A)^{\oplus I}e$ with p the central support of e'. Note that p belongs to the center of $M_I(M) = M \otimes \mathcal{B}(\ell^2(I))$, i.e., $p \in (M \cap M') \otimes 1 \cong M \cap M'$. By this natural identification, p is realized by the right multiplication of $\bigvee_{u\in U_I(M)} ueu^*$ as an element in $(M \cap M') \otimes 1)$ and by the left multiplication as an element in $M \cap M'$.

We claim that $\pi_e(A)'' = Me'$ is isomorphic to Mp. In fact, for $x \in M$, $xe' = 0$, i.e., $xL^2(A)^{\oplus I}e = 0$ if and only if $x \sum_{u\in U_I(M)} L^2(A)^{\oplus I}eu = 0$, i.e., $xp = 0$.

Let $f \in M_I(M)$ be another projection with π_f the associated *-representation of A. Then π_e and π_f are quasi-equivalent if and only if e and f have the same central support in $M_I(M)$. From

$$\mathrm{Hom}({}_AL^2(A)^{\oplus I}e, {}_AL^2(A)^{\oplus I}f) = fM_I(M)e,$$

π_e and π_f are disjoint if and only if the central supports of e and f are orthogonal.

In other words, if we denote the central support of ${}_A\mathcal{H}$ by $[{}_A\mathcal{H}] \in A^{**}$, then ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are disjoint (resp. quasi-equivalent) if and only if $[{}_A\mathcal{H}][{}_A\mathcal{K}] = 0$ (resp. $[{}_A\mathcal{H}] = [{}_A\mathcal{K}]$).

Theorem 8.1. Let φ and ψ be positive functionals on a C*-algebra A.

(i) φ and ψ are disjoint if and only if $A\varphi^{1/2}A$ and $A\psi^{1/2}A$ are orthogonal. When $\overline{A\varphi^{1/2}} = \overline{\varphi^{1/2}A}$, this is further equivalent to $(\varphi^{1/2}|\psi^{1/2}) = 0$.

(ii) φ and ψ are quasi-equivalent if and only if $\overline{A\varphi^{1/2}A} = \overline{A\psi^{1/2}A}$.

(iii) φ is pure if and only if $\overline{A\varphi^{1/2}} \cap \overline{\varphi^{1/2}A} = \mathbb{C}\varphi^{1/2}$.

Proof. (ii) and the first statement of (i) follow from $[{}_A\overline{A\varphi^{1/2}}]L^2(A) = \overline{A\varphi^{1/2}A}$. From the identity $(\varphi^{1/2}|\psi^{1/2}) = \|\varphi^{1/4}\psi^{1/4}\|^2$, the vanishing of transition probability is equivalent to $\varphi^{1/2}\psi^{1/2} = 0$, i.e., the orthogonality of $A\varphi^{1/2}$ and $A\psi^{1/2}$. When $\overline{A\varphi^{1/2}} = \overline{\varphi^{1/2}A}$, this implies

$L^4(M)$ calculus here!

$$(A\varphi^{1/2}A|A\psi^{1/2}A) \subset (A\varphi^{1/2}A|A\psi^{1/2}) \subset (\overline{A\varphi^{1/2}}|A\psi^{1/2}) = \{0\}.$$

Let e be the support of φ in A^{**}. Then the identity

$$\overline{A\varphi^{1/2}} \cap \overline{\varphi^{1/2}A} = L^2(A^{**})e \cap eL^2(A^{**}) = L^2(eA^{**}e)$$

shows that the condition in (iii) is equivalent to $eA^{**}e = \mathbb{C}e$, i.e., the purity of φ. Note that $\mathrm{End}({}_AL^2(A)e) \cong eA^{**}e$. □

Given a *-representation π of A on a Hilbert space $\mathcal{H}$, let

$$A^*_\pi = \{\phi \circ \pi; \phi \in \pi(A)''_*\},$$

which is an A-biinvariant closed subspace of A^* and the Hilbert space $L^2_\pi(A) = \overline{\sum_{0\leq\varphi\in A^*_\pi} A\varphi^{1/2}A}$ is naturally isomorphic to $L^2(\pi(A)'')$. Now let A be commutative and choose a family $\{\varphi_i\}_{i\in I}$ of states in A^*_π so that they have mutually disjoint supports and $L^2_\pi(A) = \bigoplus_{i\in I} \overline{A\varphi_i^{1/2}}$ with $A\varphi_i^{1/2} = \varphi_i^{1/2}A$. Then the Radon measure μ_i on $\Omega = \sigma_A$ associated to φ_i have mutually disjoint supports Ω_i and, if we define a measure μ on Ω by $\mu|_{\Omega_i} = \mu_i$, then $L^2_\pi(A) = L^2(\Omega,\mu)$ on which $\pi(A)''$ is identified with $L^\infty(\Omega,\mu)$. Note that μ is σ-finite if and only if I is a countable set.

Example 8.2. Let $A = C(\Omega)$ be commutative with expressions

$$\varphi(a) = \int_\Omega a(\omega)\,\mu(d\omega), \quad \psi(a) = \int_\Omega a(\omega)\,\nu(d\omega).$$

(i) φ and ψ are disjoint if and only if we can find Borel subsets Ω_μ and Ω_ν such that $\Omega_\mu \cap \Omega_\nu = \emptyset$ and $\mu(\Omega\setminus\Omega_\mu) = 0 = \nu(\Omega\setminus\Omega_\nu)$,

In fact, in the expression

$$(\varphi^{1/2}|\psi^{1/2}) = \int_\Omega \sqrt{\frac{d\mu}{d(\mu+\nu)}(\omega)\frac{d\nu}{d(\mu+\nu)}(\omega)}(\mu+\nu)(d\omega),$$

let Ω_μ and Ω_ν be the (Borel) supports of $d\mu/d(\mu+\nu)$ and $d\nu/d(\mu+\nu)$ respectively. If $(\varphi^{1/2}|\psi^{1/2}) = 0$, Ω_μ and Ω_ν can be chosen disjoint by adjustment up to $(\mu+\nu)$-negligible sets.

(ii) $\varphi^{1/2} \in \overline{A\psi^{1/2}}$ if and only if μ is absolutely continuous relative to ν. Consequently, φ and ψ are quasi-equivalent if and only if μ and ν are equivalent measures.

In fact, if $\mu \prec \nu$, $\mu(d\omega) = f(\omega)\nu(d\omega)$ with $0 \leq f \in L^1(\Omega,\nu)$ (Radon-Nykodym theorem), whence $\varphi^{1/2} = \sqrt{f(\omega)}\sqrt{\nu(d\omega)} \in L^2(\Omega,\nu) = \overline{A\psi^{1/2}}$. Conversely, unless $\mu \prec \nu$, $\mu(\Omega \setminus \Omega_\nu) > 0$ and, if $\mu^{1/2} \in L^2(\Omega,\nu)$, i.e., $\sqrt{\mu(d\omega)} = g(\omega)\sqrt{\nu(d\omega)}$ with $g \in L^2(\Omega,\nu)$, then

$$0 < \int_{\Omega\setminus\Omega_\nu} \mu(d\omega) = \int_{\Omega\setminus\Omega_\nu} |g(\omega)|^2\,\nu(d\omega) = 0,$$

a contradiction.

Exercise 43. There is a one-to-one correspondence between closed A-subbimodules in $L^2(A)$ and closed A-subbimodules in A^*

Let ω be a state of a C*-algebra A and $\{\tau_t \in \mathrm{Aut}(A)\}_{t\in\mathbb{R}}$ be a one-parameter group of *-isomorphisms. Recall that ω and $\{\tau_t\}$ satisfy the **KMS-condition**[12] if the following property holds: Given $x, y \in A$, the function $\mathbb{R} \ni t \mapsto \omega(x\tau_t(y))$ is analytically extended to a continuous function on the strip $\{\zeta \in \mathbb{C}; -1 \leq \Im\zeta \leq 0\}$ so that $\omega(x\tau_t(y))|_{t=-i} = \omega(yx)$.

If one replaces y with $\tau_s(y)$ and x with 1, then the condition takes the form $\omega(\tau_{s-i}(y)) = \omega(\tau_s(y))$ for $s \in \mathbb{R}$ and we see that the analytic function $\omega(\tau_z(y))$ is periodically extended to an entirely analytic function. Thus $\omega(\tau_t(y))$ is a constant function of t; the automorphisms τ_t make ω invariant.

Lemma 8.3. If ω satisfies the KMS-condition, then $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$.

Proof. We argue as in BR2 Corollary 5.3.9:

By the invariance of ω, a unitary operator $v(t)$ on $\overline{A\omega^{1/2}}$ is defined by $v(t)(x\omega^{1/2}) = \tau_t(x)\omega^{1/2}$, which is continuous in t from the continuity assumption on the function $\omega(x\tau_t(y))$. Moreover, the function $\mathbb{R} \ni t \mapsto v(t)(x\omega^{1/2})$ is analytically continued to the strip $\{-1 \leq \Im\zeta \leq 0\}$. By the Kaplansky's density theorem and the three line theorem, the same property holds for $x \in A^{**}$ and the KMS-condition takes the form

$$(x\omega^{1/2}|v(t)(y\omega^{1/2}))|_{t=-i} = (\omega^{1/2}x|\omega^{1/2}y) \quad \text{for } x, y \in A^{**}.$$

[12]fomulated by R. Kubo (1957), P.C. Martin and J. Schwinger (1959) as a characterisitc property of thermally equilibrium states.

Let c be the central support of ω in A^{**} and assume that $a \in cA^{**}$ satisfies $a\omega^{1/2} = 0$. Then, $xa\omega^{1/2} = 0$ for $x \in A^{**}$ and therefore $(\omega^{1/2}(xa)|\omega^{1/2}y) = 0$ for any $y \in A^{**}$ by analytic continuation, whence $\omega^{1/2}xa = 0$ for $x \in A^{**}$. Thus $cL^2(A^{**})a = 0$ and we have $a = 0$.

In this way, we have proved that $\omega^{1/2}$ is separating for cA^{**}, which implies $\overline{cA^{**}\omega^{1/2}} = \overline{\omega^{1/2}cA^{**}}$. □

Conversely suppose that $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$. Then the support projection $[\omega] \in A^{**}$ of ω is central and the modular automorphism group $\{\sigma_t\}$ for ω satisfies the KMS-condition: Let $\mathcal{M}$ be the dense *-subalgebra of entirely analytic elements for $\{\sigma_t\}$. Then for $a, b \in \mathcal{M}$

$$\begin{aligned}\omega(a\sigma_{-i}(b)) &= (a^*\omega^{1/2}|\Delta(b\omega^{1/2}))\\ &= (J\Delta^{1/2}(b\omega^{1/2})|J\Delta^{1/2}(a^*\omega^{1/2}))\\ &= (b^*\omega^{1/2}|a\omega^{1/2}) = \omega(ba).\end{aligned}$$

If $x, y \in M$ are boundedly approximated by elements in $\mathcal{M}$ with respect to the *strong topology, the three line theorem shows that the function $\omega(x\tau_t(y))$ has the KMS property as a boundary function of a uniform limit of $\omega(a\sigma_z(b))$.

Let $\{u(t)\}$ and $\{v(t)\}$ be one-parameter groups of unitaries on $\overline{A\omega^{1/2}}$ which induce automorphism groups of $[\omega]A^{**}$ and satisfy the KMS-condition for ω.

Let $x \in M$ be entirely analytic for σ_t and $y \in M$ be entirely analytic for τ_t. Then we have

$$(u(-i)(x\omega^{1/2})|y\omega^{1/2}) = (\omega^{1/2}x|\omega^{1/2}y) = (x\omega^{1/2}|v(-i)(y\omega^{1/2})).$$

Since $\{x\omega^{1/2}\}$ and $\{y\omega^{1/2}\}$ are cores for $u(-i)$ and $v(-i)$ respectively by Example A.9, we see that $u(-i)$ and $v(-i)$ are adjoints of each other; $u(-i) = v(-i)$, whence $u(t) = v(t)$ for $t \in \mathbb{R}$.

Theorem 8.4 (Takesaki)**.** If $\omega \in A^*_+$ satisfies $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$, then the support projection $[\omega] \in A^{**}$ of ω is central and ω satisfies the KMS-condition for the modular automorpshim group $\{\sigma_t \in \mathrm{Aut}([\omega]A^{**})\}$. Moreover, the modular automorphism group for ω is characterized as the one satisfying the KMS condition.

9. Reduction Theory

The von Neumann's disintegration theory on representations. Dixmier, Pedersen, Reed-Simon §IV.5, Bratteli-Robinson §4.4.1.

9.1. **Commutative Ampliations.** We shall start with describing $L^2(\Omega,\mu)\otimes\mathcal{H}$ in terms of $\mathcal{H}$-valued measurable functions.

Each $\xi \in L^2(\Omega,\mu)\otimes\mathcal{H}$ has an expression

$$\xi = \lim_{n\to\infty} \sum_{j=1}^{N_n} f_j^{(n)} \otimes \delta_j^{(n)}$$

with $f_j^{(n)} \in L^2(\Omega,\mu)$ and $\delta_j^{(n)} \in \mathcal{H}$, which shows that the range of partial evaluation $\{\langle\xi\rangle_{f\in L^2(\Omega,\mu)}\}$ is included in a separable closed subspace $\mathcal{H}_\xi$ of $\mathcal{H}$, i.e., $\xi \in L^2(\Omega,\mu)\otimes\mathcal{H}_\xi$. Let $\{\delta_j\}$ be an orthonormal basis in $\mathcal{H}_\xi$ and write $\xi = \sum_j f_j \otimes \delta_j$ with $f_j \in L^2(\Omega,\mu)$. Since

$$\|\xi\|^2 = \sum_j \|f_j\|^2 = \sum_j \int_\Omega |f_j(\omega)|^2\,\mu(d\omega) < \infty,$$

we see that $\{f_j(\omega)\}_{j\geq 1} \in \ell^2$ for μ-a.e. $\omega \in \Omega$ and ξ is represented by an $\mathcal{H}_\xi$-valued function of $\omega \in \Omega$ defined by

$$\xi(\omega) = \sum_j f_j(\omega)\delta_j.$$

As an $\mathcal{H}$-valued function on Ω, $\xi(\xi)$ is weakly μ-measurable in the sense that the function $\Omega \ni \omega \mapsto (\alpha|\xi(\omega))$ is μ-measurable for every $\alpha \in \mathcal{H}$.

Conversely, given an $\mathcal{H}$-valued function $\xi(\cdot)$ on Ω which is weakly μ-measurable and satisfies the separability condition on the range of ξ,

$$\|\xi(\omega)\|^2 = \sum_{j\geq 1} (\xi(\omega)|\delta_j)(\delta_j|\xi(\omega))$$

is a μ-measurable function of $\omega \in \Omega$ and the square-integrability condition

$$\int_\Omega \|\xi(\omega)\|^2\,\mu(d\omega) < \infty$$

has a meaning.

The set of $\mathcal{H}$-valued functions on Ω satisfying the weak μ-measurability, the range separability and the square-integrability is an inner product space $L^2(\Omega,\mu;\mathcal{H})$ with the inner product given by

$$(\xi|\eta) = \int_\Omega (\xi(\omega)|\eta(\omega))\,\mu(d\omega).$$

Proposition 9.1. The inner product space $L^2(\Omega,\mu;\mathcal{H})$ is complete and the correspondence $f\otimes\alpha \mapsto \xi$ with $\xi(\omega) = f(\omega)\alpha$ is extended to a unitary map of $L^2(\Omega,\mu)\otimes\mathcal{H}$ onto $L^2(\Omega,\mu;\mathcal{H})$.

From various reasons, it is reasonable to impose the separability condition on relevant Hilbert spaces for doing measure theoretical analysis further. So we shall assume the σ-finiteness on measures and the separability on Hilbert spaces in what follows.

Choose an orthonormal basis $\{\delta_j\}_{j\geq 1}$ in $\mathcal{H}$ and set $\mathcal{D} = \sum_{j\geq 1}(\mathbb{Q} + i\mathbb{Q})\delta_j$.

For $\alpha, \beta \in \mathcal{H}$, a σ-weakly continuous linear map $\langle \alpha, \beta \rangle : \mathcal{B}(L^2(\Omega, \mu)\otimes \mathcal{H}) \to \mathcal{B}(L^2(\Omega, \mu))$ is defined by the relation

$$(f|\langle \alpha, a\beta \rangle g) = (f \otimes \alpha | a(g \otimes \beta)).$$

When $a \in (L^\infty(\Omega, \mu) \otimes 1)'$, the operator $\langle \alpha, a\beta \rangle$ on $L^2(\Omega, \mu)$ commutes with $L^\infty(\Omega, \mu)$ and hence it belongs to $L^\infty(\Omega, \mu)$ (Example 4.21). Let $a_{j,k}$ be a μ-measurable function which represents $\langle \delta_j, a\delta_k \rangle$. Then, for $\alpha, \beta \in \mathcal{D}$,

$$\sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta)$$

represents $\langle \alpha, a\beta \rangle \in L^\infty(\Omega, \mu)$ and the inequality $\|\langle \alpha, a\beta \rangle\| \leq \|a\| \, \|\alpha\| \, \|\beta\|$ implies that the set

$$N_{\alpha,\beta} = \left\{ \omega \in \Omega; \left| \sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta) \right| > \|a\| \, \|\alpha\| \, \|\beta\| \right\}$$

is μ-negligible and so is their countable union $N = \cup_{\alpha,\beta\in\mathcal{D}} N_{\alpha,\beta}$. Now the μ-measurable function

$$a_{\alpha,\beta}(\omega) = \begin{cases} \sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta) & \text{if } \omega \notin N, \\ 0 & \text{otherwise,} \end{cases}$$

which is a representative of $\langle \alpha, a\beta \rangle$, depends on α, β in a sesquilinear fashion and satisfies

$$|a_{\alpha,\beta}(\omega)| \leq \|a\| \, \|\alpha\| \, \|\beta\| \quad \text{for any } \omega \in \Omega \text{ and } \alpha, \beta \in \mathcal{D}.$$

Now express each $\alpha, \beta \in \mathcal{H}$ in the form $\alpha = \lim_n \alpha_n$, $\beta = \lim_n \beta_n$ with $\alpha_n, \beta_n \in \mathcal{D}$. Then

$$\begin{aligned} |a_{\alpha_m,\beta_m}(\omega) - a_{\alpha_n,\beta_n}(\omega)| &\leq |a_{\alpha_m - \alpha_n,\beta_m}(\omega)| + |a_{\alpha_n,\beta_m-\beta_n}(\omega)| \\ &\leq \|a\| \|\beta_m\| \|\alpha_m - \alpha_n\| + \|a\| \|\alpha_n\| \|\beta_m - \beta_n\| \end{aligned}$$

shows that the sequence of functions $\{a_{\alpha_n,\beta_n}(\cdot)\}$ converges uniformly on Ω to a function $a_{\alpha,\beta}$, which represents $\langle \alpha, a\beta \rangle$, satisfies the inequality $|a_{\alpha,\beta}(\omega)| \leq \|a\| \, \|\alpha\| \, \|\beta\|$ for $\alpha, \beta \in \mathcal{H}$ and $\omega \in \Omega$. Moreover, $\alpha \times \beta \mapsto a_{\alpha,\beta}(\omega)$ gives a sesquilinear form at every $\omega \in \Omega$. By Riesz' lemma, we obtain a family of bounded operators $\{a(\omega)\}$ by the relation $a_{\alpha,\beta}(\omega) = (\alpha|a(\omega)\beta)$ for $\alpha, \beta \in \mathcal{H}$ with the obvious bound $\|a(\omega)\| \leq \|a\|$.

Conversely, given a uniformly bounded $\mathcal{B}(\mathcal{H})$-valued function $a(\omega)$ such that $(\alpha|a(\omega)\beta)$ is μ-measurable for any $\alpha, \beta \in \mathcal{H}$, the $\mathcal{H}$-valued function $a(\omega)\xi(\omega) \in \mathcal{H}$ is weakly μ-measurable for any $\xi \in L^2(\Omega, \mu; \mathcal{H})$ in view of the expression

$$(\alpha|a(\omega)\xi(\omega)) = \sum_j (\alpha|a(\omega)\delta_j)(\delta_j|\xi(\omega))$$

for $\alpha \in \mathcal{H}$, and the inequality

$$\int_\Omega \|a(\omega)\xi(\omega)\|^2 \, \mu(d\omega) \leq \|a\|^2 \int_\Omega \|\xi(\omega)\|^2 \, \mu(d\omega)$$

shows that it is square-integrable. Here the function

$$\|a(\omega)\| = \sup\{|(\alpha|a(\omega)\beta)|; \alpha, \beta \in \mathcal{D}, \|\alpha\| \leq 1, \|\beta\| \leq 1\}$$

is μ-measurable and $\|a\|$ is equal to its essential supremum.

Thus the totality $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ of such functions $a(\omega)$ (two $\mathcal{B}(\mathcal{H})$-valued functions $a(\omega)$ and $b(\omega)$ satisfying $\|a - b\| = 0$ being identified) is identified with the commutant of $L^\infty(\Omega, \mu) \otimes 1_\mathcal{H}$ on $L^2(\Omega, \mu) \otimes \mathcal{H} = L^2(\Omega, \mu; \mathcal{H})$.

Remark 2. As already appeared in the above discussion, the following measure-theoretical fact will be repeatedly used without explicit qualification: Let $P_j(\omega)$ be a sequence of propositions on $\omega \in \Omega$. If, for each $j \geq 1$, $P_j(\omega)$ holds at almost every $\omega \in \Omega$, then $\wedge_{j \geq 1} P_j(\omega)$ holds at almost every $\omega \in \Omega$.

Definition 9.2. Given a W*-algebra M on a separable Hilbert space $\mathcal{H}$, let $L^\infty(\Omega, \mu; M)$ be a *-subalgebra of $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ consisting of M-valued μ-measurable functions.

Proposition 9.3. The commutant of $L^\infty(\Omega, \mu; M)$ on $L^2(\Omega, \mu; \mathcal{H})$ is equal to $L^\infty(\Omega, \mu; M')$.

Proof. Let M be generated by a sequence $\{u_n\}$ of unitaries (cf. Example 4.24 and Example 2.24). Then each $a' \in L^\infty(\Omega, \mu; M)'$ belongs to $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ and satisfies $u_n a' u_n^* = a'$ for $n \geq 1$. Then we can find a representative $a'(\omega)$ so that $u_n a'(\omega) u_n^* = a'(\omega)$ for any ω and any $n \geq 1$. Thus a' is in $L^\infty(\Omega, \mu; M')$, showing

$$L^\infty(\Omega, \mu; M)' \subset L^\infty(\Omega, \mu; M').$$

Since the reverse inclusion is obvious, we have the equality. □

Corollary 9.4.

$$(L^\infty(\Omega, \mu) \otimes M)' = L^\infty(\Omega, \mu; M') = L^\infty(\Omega, \mu) \otimes M'.$$

Proof. Since the W*-algebra $L^\infty(\Omega,\mu) \otimes M$ is generated by $L^\infty \otimes 1$ and $1 \otimes M$ and since M is generated by $\{u_n\}$, the above proof shows that $(L^\infty \otimes M)' \subset L^\infty(\Omega,\mu;M')$ and then the equality $(L^\infty \otimes M)' = L^\infty(\Omega,\mu;M')$ because the reverse inclusion is trivial. □

Remark 3. The equality $(L^\infty(\Omega,\mu) \otimes M)' = L^\infty(\Omega,\mu) \otimes M'$ is also a special case of the Tomita's commutant theorem.

9.2. **Measurable Fields.** Assume that a commutative W*-algebra $L^\infty(\Omega,\mu)$ is faithfully represented in a separable Hilbert space $\mathcal{H}$. The representation is then unitarily equivalent to a subrepresentation of $L^\infty(\Omega,\mu)$ on $L^2(\Omega,\mu) \otimes \ell^2$ by Theorem 5.8. The projection e realizing this subrepresentation is in $L^\infty(\Omega,\mu;\mathcal{B}(\ell^2))$ and $\mathcal{H}$ is unitarily isomorphic to $e(L^2(\Omega) \otimes \ell^2)$. Thanks to a function realization $e(\omega)$ of e, we obtain a family of separable Hilbert spaces $\{\mathcal{H}_\omega = e(\omega)\ell^2\}$, which is μ-measurable in the sense that there is a sequence of sections $\{\xi_n\}_{n\geq 1}$ satisfying the following conditions.

(i) $\{\xi_n(\omega)\}_{n\geq 1}$ is total in $\mathcal{H}_\omega$ at almost every $\omega \in \Omega$.
(ii) functions $\omega \mapsto (\xi_m(\omega)|\xi_n(\omega))$ $(m,n \geq 1)$ are μ-measurable.

We call a section $\{\xi(\omega) \in \mathcal{H}_\omega\}_{\omega\in\Omega}$ measurable with respect to the family $\{\xi_n\}_{n\geq 1}$ if $\omega \mapsto (\xi_n(\omega)|\xi(\omega))$ is measurable for every $n \geq 1$. Each $\xi \in e(L^2(\Omega,\mu;\ell^2))$ is then characterised as a measurable section satisfying

$$\int_\Omega \|\xi(\omega)\|^2 \,\mu(d\omega) < \infty.$$

More generally, given a family $\{\mathcal{H}_\omega\}$ of separable Hilbert spaces, a sequence of sections $\{\xi_n\}$ is called a measurability sequence if it satisfies the conditions stated above. Given a measurability sequence, the measurability of a section is defined exactly in the same way.

Let $\{\eta_n\}_{n\geq 1}$ be another measurability sequence. We say that $\{\xi_n\}$ and $\{\eta_n\}$ are equivalent if they give rise to the same classes of measurable sections. It is immediate to see that $\{\xi_n\}$ and $\{\eta_n\}$ are equivalent if and only if functions $\omega \mapsto (\xi_m(\omega)|\eta_n(\omega))$ $(m,n \geq 1)$ are measurable.

A family $\{\mathcal{H}_\omega\}$ of separable Hilbert spaces is called a **measurable field** if it is equipped with an equivalence class of measurability sequences. The direct integral of $\{\mathcal{H}_\omega\}$ with respect to a measure μ is now an obvious analogue of the Hilbert space of square-intergrable functions: If square-integrable measurable sections are identified with respect to the positive sesquilinear form

$$(\xi|\eta) = \int_\Omega (\xi(\omega)|\eta(\omega))\,\mu(d\omega),$$

we obtain the **direct integral Hilbert space**

$$\int_\Omega^\oplus \mathcal{H}_\omega\,\mu(d\omega)$$

with a square-integrable measurable section ξ denoted by

$$\int_\Omega^\oplus \xi(\omega)\,\mu(d\omega)$$

when it is regarded as an element in the direct integral space.

Clearly $L^\infty(\Omega,\mu)$ is represented in $\int_\Omega^\oplus \mathcal{H}_\omega\,\mu(d\omega)$ by multiplication and an operator in this class is said to be **diagonal**.

Let $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ be measurable fields of Hilbert spaces over a common measure space (Ω,μ). A family of bounded linear maps $\{T_\omega : \mathcal{H}_\omega \to \mathcal{K}_\omega\}$ is called **measurable** if $(\eta(\omega)|T_\omega\,\xi(\omega))$ is measurable whenever ξ and η are measurable sections. A measurable family $\{T_\omega\}$ is defined to be essentially bounded if the essential supremum $\|T\|_\infty$ of the function $\|T_\omega\|$ is finite. If this is the case, a bounded linear map $T : \int_\Omega^\oplus \mathcal{H}_\omega\,\mu(d\omega) \to \int_\Omega^\oplus \mathcal{K}_\omega\,\mu(d\omega)$ is defined by

$$T\left(\int_\Omega^\oplus \xi(\omega)\,\mu(d\omega)\right) = \int_\Omega^\oplus T_\omega\,\xi(\omega)\,\mu(d\omega)$$

with the operator norm $\|T\|$ equal to $\|T\|_\infty$. We call a bounded linear map of this type **decomposable** and denoted by

$$T = \int_\Omega^\oplus T_\omega\,\mu(d\omega).$$

Measurable fields $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ are then said to be unitarily equivalent if we can find a decomposable unitary map between $\int_\Omega^\oplus \mathcal{H}_\omega\mu(d\omega)$ and $\int_\Omega^\oplus \mathcal{K}_\omega\mu(d\omega)$.

Here are obvious algebraic relations between measurable operator families and integrated decomposable operators.

Proposition 9.5. Let $\{S_\omega : \mathcal{K}_\omega \to \mathcal{L}_\omega\}$ be another measurable family of bounded linear maps. Then $\{S_\omega T_\omega\}$ and $\{T_\omega^*\}$ are measurable and, if $\|S\|_\infty < \infty$, the following holds.

(i)

$$\left(\int_\Omega^\oplus S_\omega\,\mu(d\omega)\right)\left(\int_\Omega^\oplus T_\omega\,\mu(d\omega)\right) = \int_\Omega^\oplus S_\omega T_\omega\,\mu(d\omega).$$

(ii)

$$\left(\int_\Omega^\oplus T_\omega\,\mu(d\omega)\right)^* = \int_\Omega^\oplus T_\omega^*\,\mu(d\omega).$$

Theorem 9.6 (multiplicity decomposition)**.** Two measurable fields $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ are unitarily equivalent if and only if their dimension functions $\dim \mathcal{H}_\omega$ and $\dim \mathcal{K}_\omega$ coincide for μ-a.e. ω.

Proof. If one applies the Gram-Schmidt orthogonalization to a measurability sequence $\{\xi_n\}$, then we obtain a sequence $\{\delta_n\}$ of measurable sections satisfying

(i) $\sum_{j=1}^n \mathbb{C}\xi_j = \sum_{j=1}^n \mathbb{C}\delta_j$ for $n \geq 1$,

(ii) (semi-orthonormality) $(\delta_j(\omega)|\delta_k(\omega)) = 0$ for $j \neq k$ and $\|\delta_n(\omega)\| \in \{0, 1\}$ for $n \geq 1$.

Let $\Omega_n = \{\omega \in \Omega; \delta_n(\omega) \neq 0\}$ and rearrange δ_n by cutting and pasting in the following way: Set

$$\delta_1'(\omega) = \begin{cases} \delta_1(\omega) & \text{if } \omega \in \Omega_1, \\ \delta_2(\omega) & \text{if } \omega \in \Omega_2 \setminus \Omega_1, \\ \dots & \\ \delta_n(\omega) & \text{if } \omega \in \Omega_n \setminus (\Omega_1 \cup \dots \Omega_{n-1}) \\ \dots & \end{cases}$$

and

$$\delta_n'(\omega) = \begin{cases} \delta_n(\omega) & \text{if } \omega \in \Omega_n \cap (\Omega_1 \cup \dots \Omega_{n-1}) \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 2$. Then we have the equality of algebraic sums $\sum_{n\geq 1} \mathbb{C}\delta_n(\omega) = \sum_{n\geq 1} \mathbb{C}\delta_n'(\omega)$ for any $\omega \in \Omega$ and $\{\delta_n'\}_{n\geq 1}$ is semi-orthonormal.

Repeat the rearrangement to the semi-orthonormal system $\{\delta_n'|_{\Omega'}\}_{n\geq 2}$ ($\Omega' = \cup_{n\geq 1}\Omega_n$) to get $\{\delta_n''\}_{n\geq 2}$, $\Omega'' = \bigcup_{n\geq 2} \Omega_n' \subset \Omega'$ and so on. Now the diagonal choice

$$\epsilon_n(\omega) = \begin{cases} \delta_n^{(n)}(\omega) & \text{if } \omega \in \Omega^{(n)}, \\ 0 & \text{if } \omega \in \Omega \setminus \Omega^{(n)} \end{cases}$$

satisfies $\sum_{n\geq 1} \mathbb{C}\delta_n(\omega) = \sum_{n\geq 1} \mathbb{C}\epsilon_n(\omega)$ at each $\omega \in \Omega$ and we observe that, if $\omega \in \Omega^{(n)} \setminus \Omega^{(n+1)}$ for $n \geq 0$ or $\omega \in \Omega^{(\infty)}$ for $n = \infty$, $\{\epsilon_j(\omega)\}_{1\leq j\leq n}$ is an orthonormal basis for $\mathcal{H}(\omega)$ with $\epsilon_j(\omega) = 0$ for $j > n$. Note here that

$$\Omega^{(n)} = \{\omega \in \Omega; \dim \mathcal{H}_\omega \geq n\}$$

with $\Omega^{(0)} = \Omega$ and $\Omega^{(\infty)} = \bigcap_{n\geq 1} \Omega^{(n)}$. □

Corollary 9.7. Let $\Omega_n = \{\omega \in \Omega; \dim \mathcal{H}_\omega = n\}$ and $\ell^2(n)$ be the standard Hilbert space of dimension n for $n = 1, 2, \cdots, \infty$. Then $\{\mathcal{H}_\omega\}_{\omega\in\Omega_n}$ is unitarily equivalent to the constant field $\{\ell^2(n)\}_{\omega\in\Omega_n}$.

Theorem 9.8. A bounded linear map T between direct integral Hilbert spaces $\int_\Omega^\oplus \mathcal{H}_\omega\,\mu(d\omega)$ and $\int_\Omega^\oplus \mathcal{K}_\omega\,\mu(d\omega)$ is decomposable if and only if T intertwines the diagonal representations of $L^\infty(\Omega,\mu)$.

Proof. This follows from the multiplicity decomposition of measurable fields of Hilbert spaces and results on tensor products. $\square$

Lemma 9.9. Assume that a uniformly bounded sequence $\{a_n\}_{n\geq 1}$ of decomposable operators converges to a in the strong operator topology. Then we can find a subsequence $\{n_k\}_{k\geq 1}$ so that

$$\lim_{k\to\infty} a_{n_k}(\omega) = a(\omega)$$

in the strong operator topology of $\mathcal{B}(\mathcal{H}_\omega)$ at almost every $\omega \in \Omega$.

Proof. By replacing a_n with $a_n - a$, we may assume that $a = 0$. For each $\xi = \int^\oplus \xi(\omega)\,\mu(d\omega)$, choose a subsequence $\{n'\}_{n\geq 1}$ so that $\|a_{(n+1)'}\xi - a_{n'}\xi\| \leq 1/2^n$ for $n \geq 1$. By the subadditivity of L^2-norm, we have

$$\left(\int_\Omega \left(\sum_{k=1}^n \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\|\right)^2 \mu(dx)\right)^{1/2} \leq \sum_{k=1}^n \|a_{(k+1)'}\xi - a_{k'}\xi\| \leq 1$$

and then, by taking the limit $n \to \infty$,

$$\int_\Omega \left(\sum_{k=1}^\infty \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\|\right)^2 \mu(dx) \leq 1.$$

Thus

$$\sum_{k=1}^\infty \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\| < \infty \quad \text{for } \mu\text{-a.e. } \omega.$$

and hence

$$\lim_{n\to\infty} a_{n'}(\omega)\xi(\omega) = a_{1'}(\omega)\xi(\omega) + \sum_{k=1}^\infty (a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega))$$

is norm-convergent at almost every $\omega \in \Omega$.

In view of $\|a_n(\omega)\xi(\omega)\| \leq \sup\{\|a_n\|\}\|\xi(\omega)\|$ at almost every ω and $\int \|\xi(\omega)\|^2\,\mu(d\omega) = \|\xi\|^2 < \infty$, the dominated convergence theorem is here applied to get

$$\int_\Omega \lim_{n\to\infty} \|a_{n'}(\omega)\xi(\omega)\|^2\,\mu(d\omega) = \lim_{n\to\infty} \|a_{n'}\xi\|^2 = 0,$$

i.e., $\lim_{n\to\infty} \|a_{n'}(\omega)\xi(\omega)\| = 0$ at almost every $\omega \in \Omega$.

Let $\{\xi_j\}_{j\geq 1}$ be a measurability sequence. If one applies a Cantor's diagonal argument to choosing subsequences for the convergence $\lim_{n\to\infty} \|a_n(\omega)\xi_j(\omega)\| = 0$, we can find a subsequence $\{n_k\}_{k\geq 1}$ so that

$$\lim_{k\to\infty} \|a_{n_k}(\omega)\xi_j(\omega)\| = 0$$

at almost every $\omega \in \Omega$ for $j \geq 1$. By the totality of $\{\xi_j(\omega)\}_{j\geq 1}$ in $\mathcal{H}_\omega$ and the uniform boundedness $\sup\{\|a_n\|; n \geq 1\} < \infty$, this implies

$$\lim_{k\to\infty} a_{n_k}(\omega) = 0$$

in the strong operator topology at almost every $\omega \in \Omega$. □

Let $\{\mathcal{H}_\omega\}$ be a measurable field of separable Hilbert spaces and $\{M_\omega \subset \mathcal{B}(\mathcal{H}_\omega)\}_{\omega\in\Omega}$ be a family of W*-algebras on $\{\mathcal{H}_\omega\}$. A measurable operator family $\{a(\omega)\}$ is said to be **adapted** to $\{M_\omega\}$ if $a(\omega) \in M_\omega$ at almost every $\omega \in \Omega$. Thanks to the previous lemma, we see that the set of decomposable operators associated to adapted operator families is a W*-algebra on $\int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ and is denoted by

$$\int_\Omega^\oplus M_\omega \, \mu(d\omega).$$

Definition 9.10. A family of W*-algebras $\{M_\omega\}_{\omega\in\Omega}$ on $\{\mathcal{H}_\omega\}$ is called **measurable** if we can find a sequence of adapted families $\{a_n(\omega)\}$ $(n = 1, 2, \cdots)$ such that M_ω is generated by $\{a_n(\omega)\}_{n\geq 1}$ at μ-a.e. $\omega \in \Omega$. Such a sequence $\{a_n(\omega)\}$ is referred to as a generating sequence of measurability.

Example 9.11. Let $C \subset M$ be a central W*-subalgebra of a W*-algebra M in a seprable Hilbert space $\mathcal{H}$ and realize C as $C = L^\infty(\Omega, \mu)$ with $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ the associated direct integral decomposition. The W*-algebra $M \subset C'$ then consists of decomposable operators.

Since M_* is separable, we can find a sequence $\{a_n(\omega)\}_{n\geq 1}$ of measurable families of operators such that the integrated decomposable operators $\{a_n\}_{n\geq 1}$ are dense in M with repsect to the σ-strong* topology and, if M_ω denotes the W*-algebra on $\mathcal{H}_\omega$ generated by $\{a_n(\omega); n \geq 1\}$, the family $\{M_\omega\}$ of W*-algebras is measurable.

Proposition 9.12. Let $\{M_\omega\}$ be a measurable family of W*-algebras and e (resp. e') be a decomposable projection satisfying $e(\omega) \in M_\omega$ (resp. $e'(\omega) \in M_\omega$) at almost every $\omega \in \Omega$. Then the reduced family $\{e(\omega)M_\omega e(\omega)\}$ (resp. the induced family $\{e'(\omega)M_\omega\}$) is measurable.

Theorem 9.13. Let $\{M_\omega\}$ be a measurable family of W*-algebras. Then $\int_\Omega^\oplus M_\omega\,\mu(d\omega)$ is equal to the W*-algebra M generated by any generating sequence $\{a_n\}_{n\geq 1}$ of measurability together with the diagonal algebra $L^\infty(\Omega,\mu)$.

Proof. Since $M' = \{a_n, a_n^*; n \geq 1\}' \cap L^\infty(\Omega,\mu)'$ is realized on a separable Hilbert space, it is generated by a sequence $\{a'_n\}_{n\geq 1}$ of decomposable operators. From $a_j a'_k = a'_k a_j$ for $j,k \geq 1$, we see that $a_j(\omega)a'_k(\omega) = a'_k(\omega)a_j(\omega)$ at almost every $\omega \in \Omega$. Thus, $a'_k(\omega) \in M'_\omega$ for all $k \geq 1$ at almost every $\omega \in \Omega$. □

Example 9.14. The W*-algebra M in Example 9.11 is recovered from the measurable family $\{M_\omega\}$ as the integrated W*-algebra.

Moreover, the family $\{M_\omega\}$ does not depend on the choice of countable generators up to μ-negligible sets of Ω.

In fact, let $\{b_n\}$ be another sequence of generators of M and set $N_\omega = \{b_n(\omega)\}''$. From $b_n \in \int_\Omega^\oplus M_\omega\,\mu(d\omega)$, $b_n(\omega) \in M_\omega$ at almost all ω, which implies $N_\omega \subset M_\omega$ at almost all ω. By symmetry, we also have $M_\omega \subset N_\omega$ at almost all ω. Thus $M_\omega = N_\omega$ at almost all $\omega \in \Omega$.

Theorem 9.15. Let $\{M_\omega\}$ be a measurable family of W*-algebras on a measurable field $\{\mathcal{H}_\omega\}$ of Hilbert spaces. Then the family $\{M'_\omega\}$ of commutants is measurable and the integrated W*-algebra $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$ is the commutant of $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$ on $\int_\Omega^\oplus \mathcal{H}_\omega\,\mu(d\omega)$.

Proof. Although the measurability of $\{M'_\omega\}$ seems to be very reasonable in appearance, its proof is not so obvious as can be witnessed in Dixmier §II.3.3, Pedersen §4.11.7 or Takesaki §IV.8. We shall see this with the help of standard spaces below.

Once the measurability of commutants is established, the commutant relation between integrated W*-algebras is immediate: Let $\{a'_n(\omega\}$ be a generating sequence for $\{M'_\omega\}$ and assume that a decomposable operator $a = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ is in the commutant of $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$. Since $\int_\Omega^\oplus a'_n(\omega)\,\mu(d\omega)$ belongs to $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$, we have $a(\omega)a'_n(\omega) = a'_n(\omega)a(\omega)$ for $n \geq 1$ at almost every $\omega \in \Omega$, whence $a(\omega) \in M'_\omega$ at almost every $\omega \in \Omega$. □

Given a measurable family of W*-algebras $\{M_\omega\}$ on $\{\mathcal{H}_\omega\}$, denote by $\mathcal{M}$ the *-algebra of adapted operator families. A family $\{\phi_\omega : M_\omega \to \mathbb{C}\}$ of normal functionals is said to be **measurable** if $\phi_\omega(a(\omega))$ is a measurable function of ω for every $\{a(\omega)\} \in \mathcal{M}$.

Example 9.16. There are plenty of measurable families of normal functionals: Let $\{\xi(\omega)\}$ and $\{\eta(\omega)\}$ be measurable sections of $\{\mathcal{H}_\omega\}$.

Then $\{\phi_\omega(\cdot) = (\xi(\omega)|(\cdot)\eta(\omega))\}$ is a mesurable families of normal functionals of $\{M_\omega\}$.

Moreover, if $\{\xi_n\}_{n\geq 1}$ is a sequence of measurability of $\{\mathcal{H}_\omega\}$, then

$$\varphi_\omega(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{(\xi_n(\omega)|x\xi_n(\omega))}{(\xi_n(\omega)|\xi_n(\omega))}, \quad x \in M_\omega$$

defines a measurable family of faithful normal states on $\{M_\omega\}$.

Proof. Given measurable families $\{\varphi_\omega\}$, $\{\psi_\omega\}$ of positive normal functionals, we have two measurable families of positive sesquilinear forms $\{(\varphi_\omega)_L\}$ and $\{(\varphi_\omega)_R\}$ on $\{M_\omega\}$ by

$$(\varphi_\omega)_L(a, b) = \varphi_\omega(a^*b), \quad (\psi_\omega)_R(a, b) = \psi_\omega(ba^*)$$

for $a, b \in M_\omega$ and then a measurable family of sesquilinear forms as their geometric means. Thus functions of $\omega \in \Omega$

$$(a(\omega)\varphi_\omega^{1/2}|\psi_\omega^{1/2}b(\omega)) = \sqrt{(\varphi_\omega)_L(\psi_\omega)_R}(a(\omega), b(\omega)),$$

for $\{a(\omega)\}$, $\{b(\omega)\} \in \mathcal{M}$ are measurable, which makes $\{L^2(M_\omega)\}$ into a measurable field of separable Hilbert spaces.

Let $J_\omega : L^2(M_\omega) \to L^2(M_\omega)$ be the canonical conjugation. Then the family $\{J_\omega\}$ is measurable[13] as the phase part of the polar decomposition of the measurable family $\{S_\omega : a(\omega)\varphi_\omega^{1/2} \mapsto a(\omega)^*\varphi_\omega^{1/2}\}$ of closable operators, where $\{\varphi_\omega\}$ is a measurable family of faithful normal states on $\{M_\omega\}$.

Thus $\{M'_\omega = J_\omega M_\omega J_\omega\}$ is a measurable family on $\{L^2(M_\omega)\}$ because it is generated by the sequence $\{J_\omega a_n(\omega) J_\omega\}$ of measurable operator families if $\{a_n(\omega)\}$ is a generating sequence of $\{M_\omega\}$. Now the measurable version of Dixmier's theorem on normal homomorphisms in Appendix L.3 shows the measurabilty of commutants in the general situation. □

We shall now identify the standard space $L^2(M)$ for $M = \int_\Omega^\oplus M_\omega\, \mu(d\omega)$ with $\int_\Omega^\oplus L^2(M_\omega)\, \mu(d\omega)$. Given a measurable family $\{\varphi_\omega\}$ of normal positive functionals on $\{M_\omega\}$ satisfying

$$\int_\Omega \|\varphi_\omega^{1/2}\|^2\, \mu(d\omega) = \int_\Omega \varphi_\omega(1)\, \mu(d\omega) < \infty,$$

[13] See Appendix I.1 for the measurable version of polar decomposition.

we define a positive normal functional φ on M by

$$\begin{aligned}\varphi(a) &= \Big(\int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)\Big|\Big(\int_\Omega^\oplus a(\omega)\,\mu(d\omega)\Big)\int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)\Big)\\ &= \int_\Omega \varphi_\omega(a(\omega))\,\mu(d\omega).\end{aligned}$$

Since the sesquilinear forms φ_L is realized in the form of integration as

$$\varphi(a^*b) = \int_\Omega \varphi_\omega(a(\omega)^*b(\omega))\,\mu(d\omega)$$

and similarly for ψ_R with $\{\psi_\omega\}$ another measurable family satisfying $\psi(1) < \infty$, we have

$$\begin{aligned}(a\varphi^{1/2}|\psi^{1/2}b) = \sqrt{\varphi_L\psi_R}(a,b) &= \int_\Omega \sqrt{(\varphi_\omega)_L(\psi_\omega)_R}(a(\omega),b(\omega))\,\mu(d\omega)\\ &= \int_\Omega (a(\omega)\varphi_\omega^{1/2}|\psi_\omega^{1/2}b(\omega))\,\mu(d\omega)\end{aligned}$$

for $a = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ and $b = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ in M. Thus the correspondence $\varphi^{1/2} \mapsto \int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)$ is extended to a unitary map from $L^2(M)$ onto $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$ so that it intertwines the bimodule actions of M.

Moreover, as the antiunitary part in the polar decomposition of

$$S = \int_\Omega^\oplus S_\omega\mu(d\omega)$$

with respect to a faithful normal state $\varphi = \int_\Omega^\oplus \varphi_\omega\mu(d\omega)$, we see that the canonical conjugation in $L^2(M)$ is identified with $\int_\Omega^\oplus J_\omega\,\mu(d\omega)$.

Conversely, given $\varphi \in M_*^+$, let

$$\varphi^{1/2} = \int_\Omega^\oplus \varphi^{1/2}(\omega)\,\mu(d\omega)$$

be the decomposition in $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$. Then vector functionals $\varphi_\omega(\cdot) = (\varphi^{1/2}(\omega)|(\cdot)\varphi^{1/2}(\omega))$ constitute a measurable family of positive normal functionals on $\{M_\omega\}$ and $\varphi^{1/2}$ is realized by the vector $\int_\Omega^\oplus \varphi_\omega^{1/2}\,\mu(d\omega)$. Thus $\varphi^{1/2}(\omega) = \varphi_\omega^{1/2}$ at almost every $\omega \in \Omega$.

Theorem 9.17. Given a measurable family $\{M_\omega\}$ of W*-algebras, the standard space $L^2(M)$ of the integrated W*-algebra $M = \int_\Omega^\oplus M_\omega\mu(d\omega)$ is naturally identified with $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$ as *-bimodules of M in such a way that $\xi = \int_\Omega^\oplus \xi(\omega)\,\mu(d\omega) \in L^2(M)$ belongs to $L^2(M)_+$ if and only if $\xi(\omega) \in L^2(M_\omega)_+$ at almost every $\omega \in \Omega$.

Theorem 9.18. Let (Ω, μ) be a σ-finite measure space and suppose that, for each $n \geq 1$, we are given a measurable family $\{M_n, \omega\}$ of W*-algebras on a measurable field $\{\mathcal{H}_\omega\}$ of Hilbert spaces. Then $\{\vee_{n\geq 1} M_{n,\omega}\}$, $\{\cap_{n\geq 1} M_{n,\omega}\}$ are measurable families and

$$\bigvee_{n\geq 1} \int_\Omega^\oplus M_{n,\omega}\, \mu(d\omega) = \int_\Omega^\oplus \bigvee_{n\geq 1} M_{n,\omega}\, \mu(d\omega),$$

$$\bigcap_{n\geq 1} \int_\Omega^\oplus M_{n,\omega}\, \mu(d\omega) = \int_\Omega^\oplus \bigcap_{n\geq 1} M_{n,\omega}\, \mu(d\omega).$$

Proof. Let $\{a_{n,k}(\omega)\}_{k\geq 1}$ be a generating sequence for $\{M_{n,\omega}\}$. Then $\{\vee_{n\geq 1} M_{n,\omega}\}$ is generated by $\{a_{n,k}(\omega)\}_{n,k\geq 1}$. Now Theorem 9.13 is applied to get

$$\bigvee_{n\geq 1} M_n = \{a_{n,k}, f; f \in L^\infty(\Omega), n, k \geq 1\}'' = \int_\Omega^\oplus \bigvee_{n\geq 1} M_{n,\omega}\, \mu(d\omega).$$

The relation for intersections is then obtained by taking commutants. □

Appendix A. Analytic Elements

Let X be a Banach space which is the dual space of a Banach space X_*, i.e., X_* is a predual of X. An X-valued function f on a topological space Ω is said to be weak*-continuous if $\phi \circ f$ is continuous for each $\phi \in X_*$.

Exercise 44. An X-valued function f is weak*-continuous if and only if $f : \Omega \to X$ is continuous when X is furnished with the weak* topology.

Let Ω be locally compact and μ be a complex Radon measure on Ω. If the function f is norm-bounded, i.e.,

$$\|f\|_\infty = \sup\{\|f(\omega)\|; \omega \in \Omega\} < \infty,$$

then we can define an element

$$\int_\Omega f(\omega)\, \mu(d\omega) \in X$$

by the relation

$$\left\langle \int_\Omega f(\omega)\, \mu(d\omega), \phi \right\rangle = \int_\Omega \langle f(\omega), \phi \rangle\, \mu(d\omega)$$

in view of the estimate

$$\left| \int_\Omega \langle f(\omega), \phi \rangle\, \mu(d\omega) \right| \leq |\mu|(\Omega) \|f\|_\infty\, \|\phi\|.$$

Note that, when Ω is compact, the weak*-continuity of f is enough to have the norm-boundedness $\|f\|_\infty < \infty$ thanks to the principle of uniform boundedness.

Proposition A.1. Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \to X$ be a weak*-continuous function. Then the follwoing conditions are equivalent.

(i) The function $\|f(w)\|$ of $w \in \Omega$ is locally boounded and $\langle f(w), \phi\rangle$ is holomorphic function of $w \in \Omega$ for ϕ in a dense subset of X_*.
(ii) For each $\phi \in X_*$, $\langle f(w), \phi\rangle$ is a holomorphic function of $w \in \Omega$.
(iii) If $z \in \Omega$ and $r > 0$ satisfies $\{w \in \mathbb{C}; |w - z| < r\} \subset \Omega$, then we can find a sequence $\{f_n\}_{n\geq 0}$ in X such that

$$f(w) = \sum_{n=0}^{\infty} (w - z)^n f_n$$

holds in an absolutely norm-convergent manner for $|w-z| < r$.

If f satisfies these equivalent conditions, we say that f is holomorphic on Ω.

Proof. By Cauchy's integral formula,

$$\oint_{|\zeta - z| = r-\epsilon} \frac{\langle f(\zeta), \phi\rangle}{(\zeta - z)^{n+1}}\, d\zeta \in X,$$

with $\phi \in X_*$ is independent of the choice of $0 < \epsilon < r$ and we can define $f_n \in X$ by

$$f_n = \frac{1}{2\pi i} \oint_{|\zeta - z| = r-\epsilon} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \in X.$$

Then the Cauchy's estimate $\|f_n\| \leq \|f\|_\infty/(r-\epsilon)^n$ shows that the series in question is absolutely norm-convergent if $|w - z| < r$. □

Definition A.2. An X-valued weak*-continuous function f defined on a subset D of C is said to be **analytic** if it is holomorphic when restricted to the inner part $D \setminus \partial D$.

Let $I_t : X \to X$ be a one-parameter group of isometries in X and suppose that

(i) $\mathbb{R} \ni t \mapsto I_t(x) \in X$ is weak*-continuous for each $x \in X$.
(ii) For $\phi \in X_*$ and $t \in \mathbb{R}$, the functional $\langle I_t(\cdot), \phi\rangle$ belongs to X_*.

Definition A.3. An element $x \in X$ is said to be **analytic** for $\{I_t\}$ if we can find $r > 0$ and an analytic function on the strip domain $f : (-r, r) + i\mathbb{R} \to X$ such that $f(t) = \langle I_t(x), \phi\rangle$ for $\phi \in X_*$ and $t \in \mathbb{R}$.

An analytic element $x \in X$ is said to be entirely analytic if we can find an analytic function $f : \mathbb{C} \to X$ such that $f(t) = \langle I_t(x), \phi\rangle$ for $\phi \in X_*$ and $t \in \mathbb{R}$.

We have plenty of entirely analytic elements: Let

$$x_n = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-nt^2} I_t(x)\, dt.$$

Since $\sqrt{n/\pi}e^{-nt^2}$ gives an approsimate delta function, $x_n \to I_0(x) = x$ as $n \to \infty$ in the weak*-topology. Moreover x_n is entirely analytic because

$$I_t(x_n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} I_{s+t}(x)\, ds = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-t)^2} I_s(x)\, ds.$$

indicates to set

$$f(z) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} I_s(x)\, ds$$

for $z \in \mathbb{C}$, which is analytic.

Proposition A.4. Let Δ be a positive self-adjoint operator in a Hilbert space $\mathcal{H}$ with a trivial kernel and $r \in \mathbb{R}$. Then the following conditions on $\xi \in \mathcal{H}$ are equivalent.

(i) $\xi \in D(\Delta^r)$.

(ii) The continuous function $\Delta^{it}\xi$ of $t \in \mathbb{R}$ is analytically continued to the strip region $\mathbb{R} - ir[0, 1]$.

Proof. We may assume that $r > 0$. Let $\Delta = \int_0^\infty \lambda E(d\lambda)$ be the spectral decomposition.

(i) $\Longrightarrow$ (ii): For $0 \leq s \leq r$,

$$\begin{aligned}\|\Delta^s\xi\|^2 &= \int_0^\infty \lambda^{2s}(\xi|E(d\lambda)\xi) \leq \int_{(0,1)} (\xi|E(d\lambda)\xi) + \int_{[1,\infty)} \Delta^{2r}(\xi|E(d\lambda)\xi)\\ &\leq (\xi|\xi) + (\Delta^r\xi|\Delta^r\xi) < \infty\end{aligned}$$

shows that $\xi \in D(\Delta^r)$ and then the dominated convergence theorem ensures that the function

$$\Delta^{iz}\xi = \int_0^\infty \lambda^{iz} E(d\lambda)$$

is norm-continuous on $\mathbb{R} - ir[0, 1]$ and analytic in $\mathbb{R} - ir(0, 1)$.

(ii) $\Longrightarrow$ (i): Assume that the function $\Delta^{it}\xi$ is analytically continued to $f(z)$ ($z \in \mathbb{R} - ir[0, 1]$). If $\eta \in D(\Delta^r)$, then $(\eta|\Delta^{it}\xi) = (\Delta^{-it}\eta|\xi)$ is analytically continued to the relation $(\eta|f(z)) = (\Delta^{-i\overline{z}}\eta|\xi)$. Thus $(\eta|f(-ir)) = (\Delta^r\eta|\xi)$ for $\eta \in D(\Delta^r)$, which means $\xi \in D((\Delta^r)^*) = D(\Delta^r)$ and $\Delta^r\xi = f(-ir)$. $\square$

Example A.5. Let $\mathcal{A}$ be the set of entirely analytic elements for $\sigma_t = \mathrm{Ad}(\Delta^{it})$ on $\mathcal{B}(\mathcal{H})$. Then, for $\xi \in D(\Delta^r)$ ($r \in \mathbb{R}$) and $a \in \mathcal{A}$, the relation $\Delta^{it}(a\xi) = \sigma_t(a)\Delta^{it}\xi$ is analytically continued to $\Delta^r(a\xi) = \sigma_{-ir}(a)\Delta^r\xi$ with $a\xi \in D(\Delta^r)$.

Definition A.6. Let h be a densely defined hermitian operator in a Hilbert space $\mathcal{H}$. An element $\xi \in \mathcal{H}$ is called an **analytic vector** for h if $\xi \in D(h^n)$ for $n = 1, 2, \cdots$ and

$$\sum_{n=0}^\infty \frac{1}{n!}\|h^n\xi\|\, r^n < \infty \quad \text{for some } r > 0.$$

Example A.7. Let h be a self-adjoint operator and $\xi \in D(e^{rh}) \cap D(e^{-rh})$ for some $r > 0$. Then ξ is an analytic vector for h:

$$e^{zh}\xi = \sum_{n=0}^\infty \frac{z^n}{n!} h^n\xi \quad \text{for any } z \in \mathbb{C} \text{ satisfying } |z| < r.$$

In fact, by the assumption $(\xi|e^{\pm 2rh}\xi) < \infty$, we have

$$\begin{aligned} \sum_{n=0}^\infty \frac{(2r)^n}{n!} \| |h|^{n/2}\xi\|^2 &= \sum_{n=0}^\infty \frac{(2r)^n}{n!} \int_{\mathbb{R}} |\lambda|^n (\xi|E(d\lambda)\xi) \\ &\leq (\xi|e^{2rh}\xi) + (\xi|e^{-2rh}\xi) < \infty \end{aligned}$$

and hence

$$\|h^n\xi\| \leq \| |h|^n\xi\| \leq \ell^2\left(\frac{\sqrt{(2n)!}}{(2r)^n}\right) = \ell^2\left(\frac{n!}{r^n}\right).$$

Thus $\sum_n |z|^n\|h^n\xi\|/n! < \infty$ for $|z| < r$. The Taylor expansion identity follows from the dominated convergence theorem in view of inequalities

$$\begin{aligned} \left\| e^{zh}\xi - \sum_{k=0}^n \frac{z^k}{k!} h^k\xi \right\|^2 &= \int_{\mathbb{R}} \left| e^{z\lambda} - \sum_{k=0}^n \frac{(z\lambda)^k}{k!} \right|^2 (\xi|E(d\lambda)\xi) \\ &\leq \int_{\mathbb{R}} \left(\sum_{k>n} \frac{|z\lambda|^k}{k!} \right)^2 (\xi|E(d\lambda)\xi) \\ &\leq \int_{\mathbb{R}} e^{2r|\lambda|} (\xi|E(d\lambda)\xi) \leq \|e^{rh}\xi\|^2 + \|e^{-rh}\xi\|^2 \end{aligned}$$

for $|z| \leq r$.

A linear combination of analytic vectors is again analytic by taking the common domain of convergence and, if ξ is an analytic vector for h, $h^n\xi$ ($n = 1, 2, \cdots$) is analytic with the same radius of convergence by the Cauchy-Hadamard formula.

Theorem A.8 (E. Nelson)**.** If a densely defined hermitian operator h has a total set of analytic vectors, then it is essentially self-adjoint, i.e., $h^* = \overline{h}$.

Proof. By the previous observation, we may assume that the domain D of h consists of anaytic vectors and satisfies $hD \subset D$. By the von Neumann's criterion of self-adjointness, it suffices to prove the density of $(h \pm i)D$ in $\mathcal{H}$. To see this, we shall show that $\eta \in \mathcal{H}$ orthogonal to $(h-i)D$ or $(h+i)D$ satisfies $(\eta|D) = 0$.

For $\xi \in D$ with a radius $r > 0$ of convergence, let p be the projection to $\overline{\mathbb{C}[h]\xi} \subset \mathcal{H}$ and let $h_\xi = h|_{\mathbb{C}[h]\xi}$ be a restriction of h. Then the conjugation Γ in $p\mathcal{H}$ defined by

$$\Gamma\left(\sum_{k=0}^n \lambda_k h^k \xi\right) = \sum_{k=0}^n \overline{\lambda_k} h^k \xi$$

commutes with h_ξ. Consequently, $\Gamma(\ker(h_\xi^* + i)) = \ker(h_\xi^* - i)$ and h_ξ has a self-adjoint extension H. Let $H = \int_{\mathbb{R}} \lambda E(d\lambda)$ be the spectral decomposition. Then, for $0 < s \leq r/2$ and $m = 0, 1, 2, \dots$,

$$\begin{aligned}
\|e^{\pm sH}\xi\|^2 &= \int_{\mathbb{R}} e^{\pm 2s\lambda}(\xi|E(d\lambda)\xi) \leq \sum_{n=0}^\infty \frac{(2s)^n}{n!} \int_{\mathbb{R}} |\lambda|^n (\xi|E(d\lambda)\xi) \\
&= \sum_{n=0}^\infty \frac{(2s)^n}{n!}(\xi||H|^n\xi) \leq \sum_{n=0}^\infty \frac{(2s)^n}{n!}\|\xi\|\,\|H^n\xi\| \\
&= \sum_{n=0}^\infty \frac{(2s)^n}{n!}\|\xi\|\,\|h^n\xi\| \leq \sum_{n=0}^\infty \frac{r^n}{n!}\|\xi\|\,\|h^n\xi\| < \infty
\end{aligned}$$

shows that $\xi \in D(e^{zH})$ for $z \in [-r/2, r/2] + i\mathbb{R}$ and

$$e^{zH}\xi = \int_{\mathbb{R}} e^{\lambda z} E(d\lambda)\xi$$

is an $\mathcal{H}$-valued analytic function of $z \in [-r/2, r/2] + i\mathbb{R}$ with the Taylor expansion at $z = 0$ given by

$$e^{zH}\xi = \sum_{n=0}^\infty \frac{z^n}{n!} h^n \xi, \quad |z| \leq \frac{r}{2}$$

because of

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|z|=r/2} dz\, \frac{1}{z^{n+1}} e^{zH}\xi &= \frac{1}{2\pi i}\int_{\mathbb{R}}\left(\oint_{|z|=r/2} \frac{e^{\lambda z}}{z^{n+1}}\, dz\right) E(d\lambda)\xi \\
&= \frac{1}{n!}\int_{\mathbb{R}} \lambda^n E(d\lambda)\xi = \frac{1}{n!}H^n\xi = \frac{1}{n!}h^n\xi.
\end{aligned}$$

Now assume that $(\eta|(h-i)D) = 0$. Then η satisfies $(\eta|h^{n+1}\xi) = i(\eta|h^n\xi)$ for $n = 0, 1, \cdots$, whence $(\eta|h^n\xi) = i^n(\eta|\xi)$. Thus,

$$(\eta|e^{zH}\xi) = \sum_{n=0}^{\infty} \frac{z^n}{n!}(\eta|h^n\xi) = (\eta|\xi)e^{iz}$$

first for $|z| \leq r/2$ and then for $z \in [-r/2, r/2] + i\mathbb{R}$ by analytic continuation. Since the left hand side is bounded for $z \in i\mathbb{R}$, we conclude that $(\eta|\xi) = 0$. $\square$

Example A.9. Let $\{\Delta^{it}\}$ be a one-parameter group of unitaries on a Hilbert space $\mathcal{H}$, M be a W*-algebra on $\mathcal{H}$ such that $\Delta^{it}M\Delta^{-it} = M$ for $t \in \mathbb{R}$, and $\mathcal{M}$ be the set of entirely analytic elements of M with respect to $\{\sigma_t(\cdot) = \Delta^{it}(\cdot)\Delta^{-it}\}$. Let $\xi \in \mathcal{H}$ be cyclic for M and satisfy $\Delta^{it}\xi = \xi$ for $t \in \mathbb{R}$. Then $\mathcal{M}\xi$ is a core of the positive self-adjoint Δ^r for any $r \in \mathbb{R}$.

In fact, for $f \in C_c^2(\mathbb{R})$, its inverse Fourier transform $\widehat{f}$ is an entirely analytic integrable function and

$$\sigma_{\widehat{f}}(a) = \int_{\mathbb{R}} \widehat{f}(t)\sigma_t(a)\,dt$$

belongs to $\mathcal{M}$ for any $a \in M$. Since Δ^r operates on

$$\sigma_{\widehat{f}}(a)\xi = \int_{\mathbb{R}} \widehat{f}(t)\Delta^{it}(a\xi)\,dt = 2\pi \int_{\mathbb{R}} f(\lambda)E(d\lambda)(a\xi)$$

boundedly, this is an entirely analytic vector for Δ^r and, if we denote by $\mathcal{M}_0$ the set of all such vectors, $\overline{\Delta^r|_{\mathcal{M}_0\xi}}$ is self-adjoint by the Nelson's theorem. As a self-adjoint extension of $\overline{\Delta^r|_{\mathcal{M}_0\xi}}$, Δ^r coincides with this and $\overline{\Delta^r|_{\mathcal{M}\xi}}$ is equal to Δ^r as an intermediate extension.

Appendix B. Haar Measure

V.S. Varadarajan, Geometry of Quantum Theory.

Appendix C. Pontryagin Duality

For a locally compact abelian group G, the set

$$\widehat{G} = \{\chi : G \to \mathbb{T}; \chi \text{ is continuous and satisfies } \chi(ab) = \chi(a)\chi(b)\}$$

is a subgroup of the product group $\prod_{g\in G} \mathbb{T}$ and, if we furnish it with the topology of uniform convergence on compact subsets of G, $\widehat{G}$ is a locally compact group, called the Pontryagin dual of G.

Theorem C.1 (Pontryagin)**.** The second dual $\widehat{\widehat{G}}$ is naturally identified with G as a locally compact abelian group.

Given Haar measures dg on G and a function f in $L^1(G)$,

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)\,dg$$

defines a function in $C_0(\widehat{G})$, which belong to $L^2(\widehat{G})$ for $f \in L^1(G) \cap L^2(G)$. The correspondence $L^1(G) \cap L^2(G) \ni f \mapsto \widehat{f} \in L^2(\widehat{G})$ gives rise to a unitary map between $L^2(G)$ and $L^2(\widehat{G})$ when the Haar measure of $\widehat{G}$ is appropriately normalized relative to dg.

Appendix D. Group Representations

By a unitary representation of a locally compact group G on a Hilbert space, we shall mean a group homomorphism $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ such that $G \ni g \mapsto U_g\xi \in \mathcal{H}$ is continuous for any $\xi \in \mathcal{H}$. Here the weak continuity is enough to have the norm-continuity of $U_g\xi$ in view of $\|U_g\xi - \xi\|^2 = 2(\xi|\xi) - (\xi|U_g\xi) - (U_g\xi|\xi)$.

Let dg be a left Haar measure. The inequality

$$\int_G |f(g)(\xi|U_g\eta)|\,dg \leq \|\xi\|\,\|\eta\| \int_G |f(g)|\,dg$$

for $f \in L^1(G)$ implies the existence of a bounded operator U_f satisfying

$$(\xi|U_f\eta) = \int_G f(g)(\xi|U_g\eta)\,dg.$$

The Banach space $L^1(G)$ is then made into a Banach *-algebra so that $f \mapsto U_f$ is a *-homomorpshim:

$$(f_1f_2)(g) = \int_G f_1(h)f_2(h^{-1}g)\,dh, \quad f^*(g) = \frac{dg^{-1}}{dg}\overline{f(g^{-1})}.$$

Conversely, given a *-representation π of $L^1(G)$ on a Hilbert space $\mathcal{H}$, a unitary representation U_g is recovered by

$$U_g(\pi(f)\xi) = \pi(gf)\xi,$$

where $(gf)(h) = f(g^{-1}h)$ $(g, h \in G)$. Thus there exits a one-to-one correspondence between unitary representations of G on $\mathcal{H}$ and *-represenations of $L^1(G)$ on $\mathcal{H}$. By the way of construction, the one-to-one correspondence is valid for a dense *-subalgebra $\mathcal{A}$ of $L^1(G)$ satisfying $ga \in \mathcal{A}$ for $g \in G$ and $a \in \mathcal{H}$. Example: $\mathcal{A} = C_c(G)$ or $\mathcal{A} = C_c^\infty(G)$ for a Lie group G.

The universal C*-algebra $C^*(G)$ of $L^1(G)$ (or any smaller $\mathcal{A}$ which is dense in $L^1(G)$) is referred to as the group C*-algebra. In this way, we have established a canonical correspondence between unitary representations of G and *-representations of $C^*(G)$.

Note here that the correspondence between group representations and algebra representations is also valid if we weaken the continuity condition on group representations to the measurability one with respect to the Haar measure class:

Theorem D.1 (von Neumann)**.** If a measurable family $\{T_g\}$ of unitary operators on a Hilbert space $\mathcal{H}$ satisfies $T_g T_h = T_{gh}$ for almost all $(g, h) \in G \times G$, then we can find a unitary representation U_g on G such that $T_g = U_g$ for almost all $g \in G$.

The correspondence of representations also enables us to embed G into $C^*(G)^{**}$ in such a way that

$$\int_G f(g) g \, dg \in C^*(G)^{**}$$

belongs to $C^*(G) \subset C^*(G)^{**}$ for $f \in L^1(G)$. In fact, it is equal to the image of $f \in L^1(G)$ in $C^*(G)$.

A continuous function $\varphi(g)$ of $g \in G$ is said to be **positive definite** if

$$\sum_{1 \leq j,k \leq n} \varphi(g_j^{-1} g_k) \overline{z_j} z_k \geq 0$$

for any $\{z_j\}_{j=1}^n \in \mathbb{C}^n$.

Proposition D.2. There is a one-to-one correspondence between positive definition functions and positive functionals on $C^*(G)$.

Now restrict ourselves to the case of abelian groups. Then a character $\chi : C^*(G) \to \mathbb{C}$ is nothing but a *-representation on a one-dimensional Hilbert space $\mathbb{C}$ and it is rephrased as a unitary representation of G on $\mathbb{C}$, i.e., a continuous group homomorphism $G \to \mathbb{T}$. In this way, the Gelfand spectrum $\sigma_{C^*(G)}$ of $C^*(G)$ is identified with the dual group $\widehat{G}$ and the Gelfand transform of $f \in L^1(G)$ with the function

$$\widehat{G} \ni \chi \mapsto \int_G f(g) \chi(g) \, dg,$$

which is nothing but the Fourier transform of f.

Example D.3 (unitary spectral decomposition)**.** Any unitary representation of the additive group $\mathbb{Z}$ on $\mathcal{H}$ corresponds to a single unitary U on $\mathcal{H}$ and we can find a projection-valued measure E on $\widehat{\mathbb{Z}} = \mathbb{T}$ so that

$$U = \int_{\mathbb{T}} z \, E(dz).$$

For the vector group $\mathbb{R}^n$, its dual group is identified with $\mathbb{R}^n$ itself by

$$\langle s, t \rangle = e^{is \cdot t}.$$

Example D.4 (Stone)**.** Given a unitary representation U of the vector group $\mathbb{R}^n$, we can find a projection-valued measure E so that

$$U_t = \int_{\mathbb{R}^n} e^{is\cdot t}\, E(ds).$$

Example D.5 (Bochner)**.** Given a continuous positive definite function φ of a locally compact abelian group G, we can find a finite Radon measure μ on the dual group $\widehat{G}$ so that

$$\varphi(g) = \int_{\widehat{G}} \chi(g)\, \mu(d\chi).$$

Appendix E. Projective Representations

A.A. Kirillov, Elements of the Theory of Representations, Springer, 1976.

A. Kleppner, Multipliers on Abelian Groups, Math. Annalen 158(1965), 11–34.

Baggett-Kleppner, Multiplier representations of abelian groups, JFA, 14(1973), 299-324.

Let a locally compact group G be represented by a measurable family of unitaries U_g $(g \in G)$ in a projective way:

$$U_g U_{g'} = \gamma(g, g')U_{gg'}$$

with $\gamma(g, g') \in \mathbb{T}$ a measurable function. Here the function $\gamma(g, g')$ is referred to as a Schur multiplier or simply a cocyle of G. From associativity, we see that γ satisfies the cocycle condition

$$\gamma(g, g')\gamma(gg', g'') = \gamma(g', g'')\gamma(g, g'g'') \quad \text{for all } g, g', g'' \in G$$

and two cocycles γ and γ' belong to the same projective representation if and only if they are equivalent in the sense that we can find a measurable function $\beta : G \to \mathbb{T}$ so that $\gamma'(g, g') = \gamma(g, g')\beta(g)\beta(g')\beta(gg')^{-1}$.

In a reverse way, given a cocycle γ of G, a γ-representation of G is an assignment of unitaries U_g satisfying $U_g U_{g'} = \gamma(g, g')U_{gg'}$.

A cocycle γ is said to be normalized if $\gamma(g, e) = \gamma(e, g) = 1$ for $g \in G$. Any cocycle is equivalent to a normalized one: If we put $g' = e$ in the cocycle condition, the relation $\gamma(g, e) = \gamma(e, g'')$ for $g, g'' \in G$ implies $\gamma(g, e) = \gamma(e, e) = \gamma(e, g)$ for $g \in G$ and therefore γ is equivalent to $\gamma(g, g')/\gamma(e, e)$ $(\beta(g) = \gamma(e, e)^{-1})$. Problems related to Schur multipliers are consequently redeuced to normalized ones.

Given a normalized cocycle γ of G, the product set $G \times \mathbb{T}$ is made into a group (denoted by $G \times_\gamma \mathbb{T}$) so that $(g, z) \mapsto zU_g$ is a unitary representation for any γ-representation U_g of G:

$$(g, z)(g', z') = (gg', zz'\gamma(g, g')).$$

Note that $(e,1)$ is a unit element in the group $G \times_\gamma \mathbb{T}$ and $(zU_g)^{-1} = z^{-1}U_g^{-1} = z^{-1}\gamma(g,g^{-1})^{-1}\gamma(e,e)^{-1}U_{g^{-1}}$ implies

$$(g,z)^{-1} = (g^{-1}, z^{-1}\gamma(g,g^{-1})^{-1}).$$

Note also that $\mathbb{T}$ is identified with the central subgroup of $G \times_\gamma \mathbb{T}$ by the embedding $\mathbb{T} \ni z \mapsto (e,z) \in G \times \mathbb{T}$.

Let $\xi \in L^2(G)$ be identified with a measurable function ξ on $G \times \mathbb{T}$ satisfying $\xi(g,z) = z\xi(g,1)$ for $(g,z) \in G \times \mathbb{T}$. Then a γ-representation $\{R_g^\gamma\}$ of G on $L^2(G)$ is obtained via the right translation:

$$(R_g^\gamma \xi)(h,1) = \xi((h,1)(g,1)) = \xi(hg, \gamma(h,g)) = \gamma(h,g)\xi(hg,1).$$

Exercise 45. Check the relation $R_a^\gamma R_b^\gamma = \gamma(a,b)R_{ab}^\gamma$ for $a, b \in G$.

$$(R_a^\gamma R_b^\gamma \xi)(g) = \gamma(g,a)\gamma(ga,b)\xi(gba) = \gamma(a,b)(R_{ab}^\gamma \xi)(g).$$

In what follows we focus on locally compact second countable abelian groups G. Given a measurable cocycle γ of G, let $[\gamma]$ be another cocycle defined by $[\gamma](a,b) = \gamma(a,b)/\gamma(b,a)$. By the cocycle condition on γ, together with the commutativity in G, we see that $[\gamma]$ is a bicharacter of G, whence it is continuous. Note here that cocycles equivalent to γ give the same bicharacter $[\gamma]$.

Conversely, if $[\gamma] \equiv 1$, then γ is coboundary; we can find a measurable function $\beta : G \to \mathbb{T}$ such that $\gamma(g,g') = \beta(g)\beta(g')\beta(gg')^{-1}$ for all $(g,g') \in G \times G$. In fact, if $[\gamma](g,g') \equiv 1$, then $G \times_\gamma \mathbb{T}$ is commutative and, as an irreducible component of a measurable representation of $G \times_\gamma \mathbb{T}$, we can find a measurable homomorphism $\alpha : G \times_\gamma \mathbb{T} \to \mathbb{T}$ satisfying $\alpha(1,z) = z$.

$$\alpha(g,1)\alpha(g',1) = \alpha((g,1)(g',1)) = \alpha((gg',1)(1,c(g,g'))) = \alpha(gg',1)c(g,g').$$

Exercise 46. By using the averaging method, show that any separately continuous bicharacter $\langle g, g' \rangle$ is jointly continuous in $(g,g') \in G \times G$.

The point is the continuity at (e,e); $\langle a_n, b_n \rangle \to 1$ if $(a_n, b_n) \to (e,e)$. Since $\langle a, b \rangle$ is separately continuous, the integrals in

$$\int \langle a_n, g \rangle f(b_n^{-1}g)\, dg = \langle a_n, b_n \rangle \int \langle a_n, g \rangle f(g)\, dg$$

converge as $n \to \infty$, which implies $\langle a_n, b_n \rangle \to 1$.

Theorem E.1. The map $\gamma \mapsto [\gamma]$ induces a group-isomorphism from the second cohomology group of G into the group of alternating bicharacters of G.

Let $H = \{h \in G; \gamma(g,h) = \gamma(h,g) \forall g \in G\}$ be the kernel of $[\gamma]$, which is a closed subgroup of G. Since γ is symmetric when restricted to H, we can find a measurable function $\alpha : H \to \mathbb{T}$ so that $\gamma(h,h') = \alpha(h)\alpha(h')\alpha(hh')^{-1}$ for $h,h' \in H$. By choosing a measurable extension $\beta : G \to \mathbb{T}$ of α and replacing γ with $\gamma(g,g')d\beta(g,g')$, we may assume that $\gamma(h,h') = 1$ for $h,h' \in H$ from the outset.

Given a γ-representstion U_g of G on a separable Hilbert space $\mathcal{H}$, consider $U_g \otimes R_g$ ($R_g = R^\gamma_g$ for $\gamma \equiv 1$). Define a unitary operator S on $L^2(G) \otimes \mathcal{H}$ by $(S\xi)(g) = U_g\xi(g)$. Then

$$(S(R_g \otimes U_g)\xi)(h) = U_h U_g \xi(hg) = \gamma(h,g)U_{hg}\xi(hg) = \gamma(h,g)(S\xi)(hg)$$

shows that $S(R_g \otimes U_g)S^* = R^\gamma_g \otimes 1$ for $g \in G$.

If one applies the Fourier transform on the $L^2(G)$ part, the representation $\{R_g \otimes U_g\}$ takes the form $\{T_g\}$ with

$$(T_g\widehat{\xi})(\chi) = \chi(g)U_g\widehat{\xi}(\chi)$$

If the bicharacter $[\gamma]$ gives an isomorphism $G \to \widehat{G}$ of abelian groups, the unitary operator Φ defined by

$$(\Phi\xi)(g) = \int_G [\gamma](g,h)\xi(h)\,dh$$

satisfies

$$[\gamma](h,g)U_g(\Phi\xi)(h) = U_h U_g U_h^*(\Phi\xi)(h) = S(1 \otimes U_g)S^*(\Phi\xi)(h),$$

whence $\{R_g \otimes U_g\}$ is unitarily equivalent to $\{1 \otimes U_g\}$. Thus, $1_{L^2(G)} \otimes U$ is unitarily equivalent to $R^\gamma \otimes 1_{\mathcal{H}}$ for any γ-representation U on $\mathcal{H}$.

Theorem E.2 (Stone-von Neumann)**.** Under the condition that $G \cong \widehat{G}$ via $[\gamma]$, all the γ-representations are quasi-equivalent. In particular, G has a unique (up to unitary equivalence) irreducible γ-representation.

Example E.3. Let $G = H \times \widehat{H}$ with H a locally compact abelian group and $\gamma((a,\chi),(a',\chi')) = \chi(a')$. Then the bicharacter

$$[\gamma]((a,\chi),(a',\chi')) = \frac{\chi(a')}{\chi'(a)}$$

satisfies the condition $G \cong \widehat{G}$.

Let G be as above and define a γ-representation of G on $L^2(H)$ by

$$(U_{(a,\chi)}\xi)(b) = \chi(ba)^{-1}\xi(ba).$$

Then it is irreducible. In fact, $U_{(1,\chi)}$ generates $L^\infty(H)$ on $L^2(H)$, whence the commutant $\{U_g\}'$ is in the fixed point algebra of $L^\infty(H)' = L^\infty(H)$ under the adjoint action of $\{U_{a,1}\}$.

Appendix F. Tensor Products

Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, their tensor product is a Hilbert space $\mathcal{H} \otimes \mathcal{K}$ together with a bilinear map $\mathcal{H} \times \mathcal{K} \ni (\xi, \eta) \mapsto \xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ satisfying the following properties.

(i) For $\xi, \xi' \in \mathcal{H}$ and $\eta, \eta' \in \mathcal{K}$, $(\xi \otimes \eta | \xi' \otimes \eta') = (\xi|\xi')\,(\eta|\eta')$.
(ii) Linear combinations of elements of the form $\xi \otimes \eta$ are dense in $\mathcal{H} \otimes \mathcal{K}$.

If $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$ are orthonormal bases in $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{\xi_i \otimes \eta_j\}_{(i,j) \in I \times J}$ is an orthonormal basis in $\mathcal{H} \otimes \mathcal{K}$.

Proposition F.1. Tensor product exists and is unique.

Clearly, given $\xi \in \mathcal{H}$, the linear map $\mathcal{K} \ni \eta \mapsto \xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ is a scalar multiplication of an isometry and its adjoint, denoted by $\langle\cdot\rangle_{\xi\otimes 1}$, is specified by $\xi' \otimes \eta \mapsto (\xi|\xi')\eta \in \mathcal{K}$ and referred to as a partial evaluation by $\xi \in \mathcal{H}$.

*-representations $\mathcal{B}(\mathcal{H}) \ni a \mapsto a \otimes 1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and $\mathcal{B}(\mathcal{K}) \ni b \mapsto \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ are defined by

$$(a \otimes 1)(\xi \otimes \eta) = a\xi \otimes \eta, \quad (1 \otimes b)(\xi \otimes \eta) = \xi \otimes b\eta,$$

and set $a \otimes b = (a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1)$.

Given W*-algebras M on $\mathcal{H}$ and N on $\mathcal{K}$, their tensor product $M \otimes N$ is a W*-algebra on $\mathcal{H} \otimes \mathcal{K}$ obtained as the σ-weak closure of the *-subalgebra generated by $a \otimes b$ ($a \in M$, $b \in N$). Note that $M \otimes \mathbb{C}1_{\mathcal{K}} = \{a \otimes 1; a \in M\}$ and similarly for $\mathbb{C}1_{\mathcal{H}} \otimes N$.

Appendix G. Infinite Tensor Products

J. von Neumann, On infinite direct products, Composi. Math., 6(1939), 1–77.

A. Guichardet, Produits tensoriels infinis et représentations des relations d'anticommutation, Annales scientifiques de l'É.N.S., 83(1966), 1–52.

An algebraically sophisticated way to introduce finite tensor products $\bigotimes V_i$ is to define them as subspaces of multilinear functionals on $\prod V_i^*$: Given a finite family $\{v_i\}_{i \in I}$ of vectors, let $\otimes_{i \in I} v_i$ be a multilinear functional defined by

$$\otimes_{i\in I} v_i : (v_i^*) \mapsto \prod_{i \in I} \langle v_i, v_i^* \rangle.$$

Then $\bigotimes_{i \in I} V_i$ is the linear span of $\{\otimes_{i \in I} v_i\}$. From the associativity of direct products, the associativity for tensor products follows: If $I =$

$\bigsqcup_{j\in J} I_j$, there is a natural isomorphism

$$\bigotimes_{j\in J}\left(\bigotimes_{i\in I_j} V_i\right) \cong \bigotimes_{i\in I} V_i.$$

In particular, given a decomposition $I = I'\sqcup I''$ and a family $\{v_{i''}\}_{i''\in I''}$,

$$\bigotimes_{i'\in I'} V_{i'} \ni \bigotimes_{i'\in I'} v_{i'} \mapsto \bigotimes_{i\in I} v_i \in \bigotimes_{i\in I} V_i \quad \text{with } \{v_i\} = \{v_{i'}\}\cup\{v_{i''}\}$$

is extended to a linear map.

When the index set is linearly ordered, it is customary to notationally reflect it in a geometric position. For example, if $I = \{1, 2, \dots, n\}$, we shall write $\otimes_{i\in I} v_i = v_1 \otimes \cdots \otimes v_n$.

Now consider a general family $\{\mathcal{A}_i\}$ of unital *-algebras. Then the family of unital *-algebras $\{\bigotimes_{i\in F}\mathcal{A}_i\}$, where F runs through finite subsets of I, is directed for inclusions of F by the map

$$\bigotimes_{i\in F} a_i \mapsto \bigotimes_{i\in F'} a_i',$$

where

$$a_i' = \begin{cases} a_i & \text{if } i \in F, \\ 1_i & \text{otherwise.}\end{cases}$$

The inductive limit $\lim\limits_{F\to I} \bigotimes_{i\in F}\mathcal{A}_i$ is denoted by $\bigotimes_{i\in I}\mathcal{A}_i$ and called the algebraic tensor product of $\{\mathcal{A}_i\}_{i\in I}$. Note that $\bigotimes \mathcal{A}_i$ is generated by $\otimes a_i$ ($a_i = 1$ except for finite indices) and $\bigotimes\mathcal{A}_i$ is unitary if so is each $\mathcal{A}_i$. Given a family $\{\varphi_i\}$ of states, a state $\varphi = \otimes\varphi_i$ of $\bigotimes\mathcal{A}_i$ is defined by $\varphi(\otimes a_i) = \prod \varphi_i(a_i)$. The positivity of φ is a consequence of

$$\sum_{j,k} \overline{z_j} z_k \varphi(\otimes_{i\in I} a_{i,j}^* a_{i,k}) = \sum_{j,k} \overline{z_j} z_k \prod_i \varphi_i(a_{i,j}^* a_{i,k}) \geq 0.$$

For a family of positive functionals $\{\varphi_i\}$, the positive functional $\otimes\varphi_i$ is defined by $\prod\varphi_i(1) \otimes (\varphi_i/\varphi_i(1))$ if $\prod\varphi_i(1) < \infty$.

The following formula looks quite reasonable but is in the heart of the celebrated Kakutani dichotomy on infinite product measures.

Theorem G.1. For $x = \otimes x_i$ and $y = \otimes y_i$ in $\bigotimes \mathcal{A}_i$, we have

$$(x(\otimes\varphi_i)^{1/2} | (\otimes\psi_i)^{1/2} y) = \prod_{i\in I} (x_i\varphi_i^{1/2} | \psi_i^{1/2} y_i).$$

Corollary G.2. If $\prod_{i\in I}(\varphi_i^{1/2}|\psi_i^{1/2}) = 0$, $\otimes\varphi_i$ and $\otimes\psi_i$ are disjoint.

Given a family $\{z_\alpha\}_{\alpha\in I}$ of complex numbers, let $z_F = \prod_{\alpha\in F} z_\alpha$ for a finite subset $F \subset I$. If $\{z_F\}$ is a coonvergent net of complex numbers, its limit is denoted by $\prod_{\alpha\in I} z_\alpha$ and we see $\prod_{\alpha\in I} |z_\alpha| = \left|\prod_{\alpha\in I} z_\alpha\right|$.

If $z_\alpha = 0$ for some $\alpha \in I$, then $\prod_{\alpha\in I} z_\alpha = 0$. If not, $\{|z_G|\}$ is bounded for any finite $G \subset I$ satisfying $|z_\alpha| > 1$ $(\alpha \in G)$. In fact, if it is not bounded, given any finite $F \subset I$, we can choose $G \subset I \setminus F$ so that $|z_G|$ is arbitrarily large while $|z_F| \neq 0$, i.e., $|z_{F\sqcup G}|$ can be arbitrarily large, which contradicts with the convergence of $\prod_{\alpha\in I} |z_\alpha|$.

Thus the condition $\prod z_\alpha = 0$ is equivalent to (i) $z_\alpha = 0$ for some $\alpha \in I$ or (ii) $\prod_{\alpha:|z_\alpha|>1} |z_\alpha| < \infty$ and $\prod_{\alpha:|z_\alpha|\leq 1} |z_\alpha| = 0$.

Now assume that $\prod z_\alpha \neq 0$. Then $(\prod z_\alpha)^{-1} = \prod z_\alpha^{-1} \neq 0$, implies that

$$\prod_{\alpha:|z_\alpha|>1} |z_\alpha| < \infty, \qquad \prod_{\alpha:|z_\alpha|\leq 1} |z_\alpha| > 0$$

and $\prod e^{i\theta_\alpha}$ is convergent to a complex number of modulus one, where $z_\alpha = |z_\alpha| e^{i\theta_\alpha}$ with $-\pi < \theta_\alpha \leq \pi$. From the convergence of $\prod e^{i\theta_\alpha}$, we see that, given $\epsilon > 0$, we can find a finite $F \subset I$ such that $|\theta_\alpha| \leq \epsilon$ for any $\alpha \in I \setminus F$. Thus the convergence of $\prod e^{i\theta_\alpha}$ is equivalent to the convergence of $\sum \theta_\alpha$, which is combined with the convergence of $\sum_{\alpha\in I_\pm} \log|z_\alpha|$ $(I_\pm = \{\alpha \in I; \pm\log|z_\alpha| > 0\})$ to get the convergence of $\sum \log z_\alpha$, i.e., $\sum |z_\alpha - 1| < \infty$. Conversely, this condition implies the absolute convergence of $\sum \log z_\alpha$ and we see that

$$\prod_{\alpha\in I} z_\alpha = e^{\sum \log z_\alpha}$$

is convergent.

Let $\{\mathcal{H}_i\}_{i\in I}$ be an infinite family of Hilbert spaces and $\{\iota_i \in \mathcal{H}_i\}$ be a family of unit vectors. Given a finite subset F of I, let

$$\mathcal{H}_F = \left(\bigoplus_\phi \mathcal{H}_{\phi(1)} \otimes \mathcal{H}_{\phi(2)} \otimes \cdots \otimes \mathcal{H}_{\phi(n)} \right)^{S_n}$$

be the symmetrized tensor product of $\{\mathcal{H}_i\}_{i\in F}$, where $\phi : \{1, \dots, n\} \to F$ $(n = |F|)$ runs through bijections and $(\ )^{S_n}$ denotes the fixed-point subspace under the obvious action of the symmetric group S_n. Thus, for each ϕ, $\mathcal{H}_F$ can be identified with the Hilbert space $\mathcal{H}_{\phi(1)} \otimes \cdots \otimes \mathcal{H}_{\phi(n)}$.

Define an imbedding of $\mathcal{H}_F$ into $\mathcal{H}_{F'}$ for $F \subset F'$ by

$$\xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)} \mapsto \xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)} \otimes \iota_{\phi'(n+1)} \otimes \cdots \otimes \iota_{\phi'(n')},$$

where $n' = |F'|$ and $\phi' : \{1, \dots, n'\} \to F'$ is any extension of ϕ.

The inductive limit Hilbert space $\lim_{F\nearrow I} \mathcal{H}_F$ is called the infinite tensor product of $\{\mathcal{H}_i\}_{i\in I}$ with respect to the reference vector $\{\iota_i\}$ and denoted by $\bigotimes_{i\in I}\mathcal{H}_i$. The image of $\xi_F \in \mathcal{H}_F$ in $\bigotimes_{i\in I}\mathcal{H}_i$ is denoted by

$$\xi_F \otimes \bigotimes_{i\in I\setminus F} \iota_i.$$

Lemma G.3. Given a family of vectors $\{0 \neq \xi_i \in \mathcal{H}_i\}_{i\in I}$, let $\xi_F \in \mathcal{H}_F$ be defined by $\xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)}$. Then the limit

$$\otimes\xi_i = \lim_{F\nearrow I} \xi_F \otimes \bigotimes_{i\in I\setminus F} \iota_i$$

exists in $\bigotimes_{i\in I}\mathcal{H}_i$ and $\otimes\xi_i \neq 0$ if and only if

$$\sum_{i\in I} |\log(\xi_i|\xi_i)| < \infty \quad \text{and} \quad \sum_{i\in I} |1 - (\iota_i|\xi_i)| < \infty.$$

Moreover, given another such family $\{\eta_i\}$, we have

$$(\otimes\xi_i|\otimes\eta_i) = \prod(\xi_i|\eta_i),$$

where the infinite product converges absolutely;

$$\sum_{i\in I} |1 - (\xi_i|\eta_i)| < \infty.$$

Appendix H. Polarity in Banach Spaces

Let X, Y be complex vector spaces and suppose that they are coupled by a non-degenerate bilinear form $\langle x, y\rangle$. If they are furnished with weak topologies, then continuous linear functionals are given by the coupling becuase the continuity of a linear functional $f : X \to \mathbb{C}$ relative to the seminorm $|\langle x, y_1\rangle| + \cdots + |\langle x, y_n\rangle|$ implies that f passes through the linear map $X \ni x \mapsto (\langle x, y_1\rangle, \ldots, \langle x, y_n\rangle) \in \mathbb{C}^n$. For a subspece E of X or Y, let $E^\perp$ be the polar of E with respect to the pairing $\langle\ ,\ \rangle$. Clearly $E \subset F$ implies $F^\perp \subset E^\perp$ and $E \subset E^{\perp\perp}$, which are combined to see that $E^{\perp\perp\perp} = E^\perp$. $E^{\perp\perp}$ is the weak closure of E by Hahn-Banach theorem. Thare is a one-to-one correspondence between weakly closed subspaces of X and weakly closed subspaces of Y by taking polars.

Now let X be a Banach space and $Y = X^*$ the dual Banach space of X.

Again, by Hahn-Banach theorem, norm closure and weak closure coincide for convex subsets of X. The weak topology on X^* via the natural coupling $\langle x, f\rangle = f(x)$ ($x \in X$, $f \in X^*$) is referred to as the weak* topology to avoid confusion with the weak topology of the pairing between X^* and X^{**}.

Proposition H.1. Let $F \subset X^*$ be a subspace. Then the weak* closure $F^{\perp\perp}$ of F is naturally identified with the dual $(X/F^\perp)^*$ of the quotient Banach space $X/F^\perp$ and we have

$$\sup\{|\langle x, f\rangle|; f \in F^{\perp\perp}, \|f\| \leq 1\} = \inf\{\|x+e\|; e \in F^\perp\}.$$

Proof. For $e \in F^\perp$ and $f \in f^{\perp\perp}$ with $\|f\| \leq 1$, we see

$$|\langle x, f\rangle| = |\langle x+e, f\rangle| \leq \|x+e\|$$

and then, by taking inf for e and sup for f,

$$\sup\{|\langle x, f\rangle|; f \in F^{\perp\perp}, \|f\| \leq 1\} \leq \inf\{\|x+e\|; e \in F^\perp\}.$$

By Hahn-Banach theorem, we can find $\varphi : X/F^\perp \to \mathbb{C}$ such that $\|\varphi\| = 1$ and $\varphi(x+F^\perp) = \|x+F^\perp\|$. Let $f \in F^{\perp\perp}$ be the composition $f(x) = \varphi(x+F^\perp)$. Then $\|f\| \leq 1$ and $|\langle x, f\rangle| = \|x+F^\perp\|$. □

Appendix I. Radon Measures

Riesz-Radon-Banach-Markov-Kakutani theorem.

Given a commutative C*-algebra $A = C_0(\Omega)$ with Ω a locally compact space, there is a one-to-one correspondence between elements of A^* and regular complex Borel measures on Ω by the relation

$$\varphi(a) = \int_\Omega a(\omega)\, \mu(d\omega)$$

so that $\|\varphi\| = |\mu|(\Omega)$. Under this correspondence, φ is positive if and only if μ is positive.

The essence in this correspondence can be summarized as follows:

Theorem I.1. Let $B(\Omega)$ be the *-algebra of bounded Baire functions. Then the natural embedding $C_0(\Omega) \to C_0(\Omega)^{**}$ is extended to a *-isomorphism of $B(\Omega)$ onto $C_0(\Omega)^{**}$ in such a way that, if a uniformly bounded sequence $f_n \in B(\Omega)$ converges to $f \in B(\Omega)$ point-wise, then $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ for any $\varphi \in C_0(\Omega)^*$.

Now it is immediate to get the spectral decomposition theorem. Let $\pi : C_0(\Omega) \to \mathcal{B}(\mathcal{H})$ be a *-representation on a Hilbert space $\mathcal{H}$. Since π is extended to a normal *-homomorphism $\pi^{**} : B(\Omega) = C_0(\Omega)^{**} \to \pi(C_0(\Omega))''$, if we define a projection-valued Baire measure E by $E(S) = \pi^{**}(1_S)$, then

$$\pi^{**}(f) = \int_\Omega f(\omega)\, E(d\omega) \quad \text{for } f \in B(\Omega).$$

Note that the class of Borel functions coincides with that of Baire functions when Ω is second countable. To avoid measure-theoretical complexities, we assume this condition from here on.

A group G is said to be locally compact if it is furnished with a locally compact topology so that the group operations, $G \times G \ni (a, b) \mapsto ab \in G$ and $G \ni g \mapsto g^{-1} \in G$ are continuous.

A positive Radon measure μ on a locally compact group G is called a left (resp. right) Haar measure if it is invariant under the left (resp. right) translations.

Theorem I.2. A left Haar measure exists and it is unique up to scalar multiplications.

Appendix J. Sesquilinear Forms

A sesquilinear form θ on a complex vector space D is said to be hermitian (resp. positive) if $\theta(x, x) \in \mathbb{R}$ (resp. $\theta(x, x) \geq 0$) for $x \in D$. By the polarization identity, a hermitian form satisfies $\theta(y, x) = \overline{\theta(x.y)}$.

A positive form θ defined on a dense linear subspace D of a Hilbert space $\mathcal{H}$ is said to be closed if D is complete with respect to the inner product $(\xi|\eta)_\theta = (\xi|\eta) + \theta(\xi, \eta)$.

Example J.1. Let Θ be a densely defined positive operator in $\mathcal{H}$. Then $\theta(\xi, \eta) = (\xi|\Theta\eta)$ is a positive form on $D(\Theta)$.

Example J.2. A positive form associated to a positive operator Θ is closable in the following sense:

Let D be the completion of $D(\Theta)$ relative to the inner product $(\ |\ )_\theta$. Since the imbedding $D(\Theta) \subset \mathcal{H}$ is norm-decreasing, it gives rise to a contractive linear map $\phi : D \to \mathcal{H}$, which turns out to be injective: Suppose that $\xi \in D$ and $\phi(\xi) = 0$. Then we can find a sequence $\xi_n \in D(\Theta)$ such that $\|\xi_n - \xi\|_\theta \to 0$ and $\|\xi_n\| \to 0$.

$$\|\xi\|_\theta^2 = \lim_{m,n\to\infty} (\xi_m|\xi_n)_\theta = \lim_{n\to\infty} \lim_{m\to\infty} \Big((\xi_m|\xi_n) + (\xi_m|\Theta\xi_n) \Big) = 0.$$

Clearly $D(\Theta) \subset \phi(D)$ and $(\phi^{-1}\xi|\phi^{-1}\eta)_\theta - (\xi|\eta)$ is a closed positive form which extends θ.

Given a closed positive form θ on $D \subset \mathcal{H}$, we want to get a positive self-adjoint operator Θ such that $\theta(\xi, \eta) = (\xi|\Theta\eta)$. To see this, let $\phi : D \to \mathcal{H}$ denote the imbedding, which is norm-decreasing, i.e., $\|\phi(\xi)\| \leq \|\xi\|_\theta$ for $\xi \in D$, and set $R = \phi\phi^* : \mathcal{H} \to \mathcal{H}$, which is a positive contraction and satisfies the relation

$$(\xi|R\eta)_\theta = (\xi|\phi^*\eta)_\theta = (\xi|\eta), \quad \xi, \eta \in \mathcal{H}.$$

Since D is dense in $\mathcal{H}$, ϕ^* is injective and so is R. Thus R^{-1} with $D(R^{-1}) = R\mathcal{H} = \phi^*\mathcal{H} \subset D$ is a positive self-adjoint operator satisfying

$R^{-1} \geq 1$. We notice here that $\phi^*\mathcal{H}$ is dense in D. In fact, if $\eta \in D$ is orthogonal to $\phi^*\mathcal{H}$, $0 = (\eta|\phi^*\xi)_\theta = (\eta|\xi)$ for any $\xi \in \mathcal{H}$ implies $\eta = 0$.

Let $\Theta = R^{-1} - 1$ with $D(H) = D(R^{-1}) = \phi^*\mathcal{H}$ be a positive self-adjoint operator. Then

$$\begin{aligned}(\phi^*\xi|\Theta\phi^*\eta) &= (\phi\phi^*\xi|R^{-1}\phi\phi^*\eta) - (\phi^*\xi|\phi^*\eta) = (\phi\phi^*\xi|\eta) - (\phi^*\xi|\phi^*\eta)\\ &= (\phi^*\xi|\phi^*\eta)_\theta - (\phi^*\xi|\phi^*\eta) = \theta(\phi^*\xi, \phi^*\eta).\end{aligned}$$

Finally observe that $D = D(\Theta^{1/2})$ in view of the density of $D(\Theta)$ in D and θ coincides with the closure of the positive form $(\xi|\Theta\eta)$ on $D(\Theta)$.

We shall review basics on the Pusz-Woronowicz theory of functional calculus on sesquilinear forms. Let α, β be positive (sesquilinear) forms on a complex vector space H. By a **representation** of the unordered pair $\{\alpha, \beta\}$, we shall mean a linear map $i : H \to K$ of H into a Hilbert space K together with positive (self-adjoint) operators A, B in K such that A commutes with B in the spectral sense, $i(H)$ is a core for the self-adjoint operator $A + B$ and

$$\alpha(x, y) = (i(x)|Ai(y)), \quad \beta(x, y) = (i(x)|Bi(y))$$

for $x, y \in H$. Note that $i(H)$ is included in the domains of $A = \frac{A}{A+B+I}(A + B + I)$ and $B = \frac{B}{A+B+I}(A + B + I)$. When A and B are bounded, we say that the representation is **bounded**. Note that, the core condition is reduced to the density of $i(H)$ in K for a bounded representation.

If A and B are commuting self-adjoint operators with spectral measures $e_A(ds)$, $e_B(dt)$ respectively and (s, t) be a complex-valued Borel function on $\sigma(A) \times \sigma(B) \subset \mathbb{R}^2$, then the normal operator $f(A, B)$ is defined by

$$f(A, B)\xi = \int_{\sigma(A)\times\sigma(B)} f(s, t)e_A(ds)e_B(dt)\xi.$$

Lemma J.3. Any pair of positive forms α, β on H admits a bounded representation.

Proof. Let K be the Hilbert space associated to the positive form $\alpha+\beta$ and $i : H \to K$ be the natural map. By the Riesz lemma, we have bounded operators A and B on K representing α and β respectively, which commute becuase of $A + B = 1_K$. $\square$

Lemma J.4 (cf. Reed-Simon §VIII.6)**.**

(i) Let D be a core of a positive self-adjoint operator C on a Hilbert space $\mathcal{H}$. Then D is a core for $C^{1/2}$. In particular, we have the domain inclusion $D(C) \subset D(C^{1/2})$.

(ii) Let A and B be commuting positive (self-adjoint) operators on a Hilbert space $\mathcal{H}$ with $A+B$ denoting the closure of $A|_{D(A)\cap D(B)} + B|_{D(A)\cap D(B)}$. Let $D \subset \mathcal{H}$ be a core for the positive operator $A+B$, then D is a core for A and B as well.

Proof. (i) By a spectral representation of C, we may assume that $\mathcal{H} = \int_{t\geq 0} \mathcal{H}(t)\,\mu(dt)$ with C given by multiplication of t. Then any vector in $D(C^{1/2})$ is of the form $\int_{t\geq 0}^{\oplus} f(t)\,\mu(dt)$ with $\{f(t) \in \mathcal{H}(t)\}_{t\geq 0}$ a measurable field satisfying

$$\int_{t\geq 0} (f(t)|f(t))_t\,\mu(dt) < +\infty, \quad \int_{t>0} t(f(t)|f(t))_t\,\mu(dt) < +\infty.$$

Assume that $f \in \mathcal{H}$ satisfy $f(t) = 0$ for $t > M$ with $M \geq 1$. Then $f \in D(C)$ and, by assumption, we can find a sequence $\{f_n(t)\}_{n\geq 1}$ in D such that $\|f_n - f\| \to 0$ and $\|Cf_n - Cf\| \to 0$, i.e.,

$$\int_{t\leq M} \|f_n(t) - f(t)\|_t^2\,\mu(dt) + \int_{t>M} \|f_n(t)\|_t^2\,\mu(dt) \to 0$$

and

$$\int_{t\leq M} t^2\|f_n(t) - f(t)\|_t^2\,\mu(dt) + \int_{t>M} t^2\|f_n(t)\|_t^2\,\mu(dt) \to 0,$$

which imply

$$\begin{aligned}\|C^{1/2}f_n - C^{1/2}f\|^2 &= \int_{t\leq M} t\|f_n(t) - f(t)\|_t^2\,\mu(dt) + \int_{t>M} t\|f_n(t)\|_t^2\,\mu(dt)\\ &\leq M\int_{t\leq M} \|f_n(t) - f(t)\|_t^2\,\mu(dt) + \int_{t>M} t^2\|f_n(t)\|_t^2\,\mu(dt)\\ &\to 0\end{aligned}$$

Thus spectrally truncated vectors for $C^{1/2}$ are approximated by vectors in D relative to the graph norm, which in turn constitute a core for $C^{1/2}$.

(ii) By the trivial inclusion $\frac{A}{A+B+I}(A+B+I) \subset A$ of unbounded operators, $D \subset D(A+B) \subset D(A)$. Let $\xi \in D(A+B) = D(A+B+I)$ be a vector in the spectral subspace of condition $A+B+I \leq M$ with M a sufficiently large positive real number, then we can find a sequence $\{\xi_n\}$ in D such that $\|\xi_n - \xi\| \to 0$ and $\|(A+B+I)\xi_n - (A+B+I)\xi\| \to 0$. Then

$$\|A\xi_n - A\xi\| = \left\|\frac{A}{A+B+I}(A+B+I)\xi_n - \frac{A}{A+B+I}(A+B+I)\xi\right\| \to 0$$

implies that the domain of the closure of $A|_D$ contains a dense set of entirely analytic vectors of A, whence $A|_D$ is essentially self-adjoint. □

A complex-valued Borel function f on the closed first quadrant $[0,\infty)^2$ is called a **form function** if it is locally bounded and homogeneous of degree one; f is bounded when restricted to a compact subset of $[0,\infty)^2$ and $f(rs,rt) = rf(s,t)$ for $r,s,t \geq 0$. Clearly $f(0,0) = 0$ and there is a one-to-one correspondence between form functions and bounded Borel functions on the unit interval $[0,1]$ by the restriction $f(t,1-t)$ $(0 \leq t \leq 1)$. Let $\mathcal{F}$ be the the vector space of form functions.

Theorem J.5. For $f \in \mathcal{F}$, the sesquilinear form on H defined by

$$\gamma(x,y) = (i(x)|f(A,B)i(y)), \quad x,y \in H$$

does not depend on the choice of representations of $\{\alpha,\beta\}$, which will be reasonably denoted by $\gamma = f(\alpha,\beta)$.

Proof. Let $\mathcal{F}_0$ be the set of functions $f \in \mathcal{F}$ satisfying the property in the theorem. Clearly $\mathcal{F}_0$ is a linear subspace of $\mathcal{F}$, closed under taking pointwise limit in a locally uniformly bounded fashion and $\alpha,\beta \in \mathcal{F}_0$. By the lemma below, if $f \geq 0$ and $g \geq 0$ belong to $\mathcal{F}_0$, we have $fg/(f+g) \in \mathcal{F}_0$. Thus, for $\mu > 0$,

$$\frac{\mu st}{s+\mu t}, \quad \frac{((s+\mu t)/2)^2}{s+\mu t}$$

are functions in $\mathcal{F}_0$ and, as a linear combination of these,

$$\frac{(s+t)^2}{(1-\lambda)s+t} = \frac{s+t}{1-\lambda s/(s+t)}$$

belongs to $\mathcal{F}_0$ for $0 < \lambda < 1$. Thus, extracting asymptotics as $\lambda \to +0$, $\frac{s^{n+1}}{(s+t)^n} \in \mathcal{F}_0$ $(n = 0,1,2,\dots)$ and then by Weierstrass approximation theorem continuous functions in $\mathcal{F}$ are included in $\mathcal{F}_0$. Since $\mathcal{F}_0$ is closed under taking locally bounded sequential limits, we conclude that $\mathcal{F}_0 = \mathcal{F}$. □

Lemma J.6. Let $\mathcal{F}_0$ be the set of functions $f \in \mathcal{F}$ satisfying the property in the theorem and let $f,g \in \mathcal{F}_0$ take values in $[0,\infty)$. Then $\frac{fg}{f+g}$ belongs to $\mathcal{F}_0$.

Proof. Let (i,K,A,B) be a bounded representation. Then the harmonic mean $C = \frac{f(A,B)g(A,B)}{f(A,B)+g(A,B)}$ of positive operators $f(A,B)$ and $g(A,B)$ is characterized by

$$(\zeta|C\zeta) = \inf\{(\xi|f(A,B)\xi)+(\eta|g(A,B)\eta); \xi,\eta \in K, \xi+\eta = \zeta\}, \quad \zeta \in K.$$

Since $i(H)$ is dense in K, this implies the assertion □

For $z \in \mathbb{C}$ in the strip region $0 \leq \mathrm{Re}z \leq 1$, $\alpha^z\beta^{1-z}(x,y)$ is continuous and holomorphic in $0 < \mathrm{Re}z < 1$ for any $x, y \in H$, which is referred as a Uhlmann's interpolation between α and β. The boundary part $\alpha^{1-t}\beta^t$ $(0 \leq t \leq 1)$, which is a continuous family of positive forms and characterized by

$$\sqrt{(\alpha^{1-s}\beta^s)(\alpha^{1-t}\beta^t)} = \alpha^{1-(s+t)/2}\beta^{(s+t)/2}$$

for $0 \leq s, t \leq 1$.

Lemma J.7.

(i) If $\alpha \leq \alpha'$ and $\beta \leq \beta'$, then $\sqrt{\alpha\beta} \leq \sqrt{\alpha'\beta'}$.

(ii) If α', α'', β', β'' are positive forms and $0 \leq s \leq 1$,

$$s\sqrt{\alpha'\beta'} + (1-s)\sqrt{\alpha''\beta''} \leq \sqrt{(s\alpha' + (1-s)\alpha'')(s\beta' + (1-s)\beta'')}.$$

Theorem J.8 (Uhlmann)**.** Uhlmann's boundary interpolations satisfy the following inequalities. Under the same situations as above, we have $\alpha^{1-t}\beta^t \leq (\alpha')^{1-t}(\beta')^t$ and

$$s(\alpha')^{1-t}(\beta')^t + (1-s)(\alpha'')^{1-t}(\beta'')^t \leq (s\alpha' + (1-s)\alpha'')^{1-t}(s\beta' + (1-s)\beta'')^t$$

for $0 \leq t \leq 1$.

Proof. Let I be the set of parameters $0 \leq t \leq 1$ satisfying the inequalities. Then $0, 1 \in I$ and $t, t' \in I$ implies $(t + t')/2 \in I$. Since I is a closed subset, this means $I = [0, 1]$. □

Definition J.9. Let α and β be positive forms on a vector space H. A hermitian form γ on H is said to be **dominated** by $\{\alpha, \beta\}$ if $|\gamma(x,y)|^2 \leq \alpha(x,x)\,\beta(y,y)$ for $x, y \in H$. Note that the order of α and β is irrelevant in the domination.

Theorem J.10 (Pusz-Woronowicz)**.** Let α, β be positive forms on a complex vector space H. Then, for $x \in H$, we have the following variational expression.

$$\sqrt{\alpha\beta}(x,x) = \sup\{\gamma(x,x); \gamma \text{ is a positive form dominated by } \{\alpha,\beta\}\}.$$

Proof. Let $(i : H \to K, A, B)$ be a representation of $\{\alpha, \beta\}$. We first prove the formula for bounded representations. Assume that a positive form γ is dominated by $\{\alpha, \beta\}$. Then the inequality

$$|\gamma(x,y)|^2 \leq \alpha(x,x)\,\beta(y,y) \leq \|A\|\,\|B\|\|\,i(x)\|^2\,\|i(y)\|^2$$

enables us to find a positive bounded operator C on K such that $\gamma(x,y) = (i(x)|Ci(y))$. Since $i(H)$ is dense in K, we have

$$|(\xi|C\eta)| \leq (\xi|A\xi)\,(\eta|B\eta) \leq (\xi|(A+\epsilon)\xi)\,(\eta|(B+\epsilon)\eta)$$

for any $\epsilon > 0$. Replacing ξ and η with $(A+\epsilon)^{-1/2}\xi$ and $(B+\epsilon)^{-1/2}\eta$ respectively, we have $\|(A+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2}\| \leq 1$ and hence

$$(A+\epsilon)^{-1/2}C(B+\epsilon)^{-1}C(A+\epsilon)^{-1/2} \leq 1_K.$$

Multiplying the positive operator

$$(B+\epsilon)^{-1/2}(A+\epsilon)^{1/2} = (A+\epsilon)^{1/2}(B+\epsilon)^{-1/2}$$

from the left and right sides, we get

$$\left((B+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2}\right)^2 \leq (A+\epsilon)(B+\epsilon)^{-1}$$

and then by taking square roots (taking square roots is operator-monotone)

$$(B+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2} \leq (A+\epsilon)^{1/2}(B+\epsilon)^{-1/2}$$

and therefore $C \leq (A+\epsilon)^{1/2}(B+\epsilon)^{1/2}$. Thus $C \leq A^{1/2}B^{1/2}$.

Now let us deal with the case of unbounded A and B. Since $i(H)$ is assumed to be a core for $A+B+I$, it is a core for $(A+B+I)^{1/2}$ as well and, if we set $j(x) = (A+B+I)^{1/2}i(x)$, the linear map $j : H \to K$ has a dense range. By the identity

$$\begin{aligned}(j(x)|\frac{A}{A+B+I}j(x)) &= (\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)|\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)) \\ &= (A^{1/2}i(x)|A^{1/2}i(x)) = \alpha(x,x)\end{aligned}$$

and a similar expression for $\beta(x,x)$, we obtain a bounded representation $(j, \frac{A}{A+B+I}, \frac{B}{A+B+I})$ and then

$$\begin{aligned}\sqrt{\alpha\beta}(x,x) &= (j(x)|\left(\frac{A}{A+B+I}\right)^{1/2}\left(\frac{B}{A+B+I}\right)^{1/2}j(x)) \\ &= (\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)|\frac{B^{1/2}}{(A+B+I)^{1/2}}j(x)) \\ &= (A^{1/2}i(x)|B^{1/2}i(x)).\end{aligned}$$

□

Corollary J.11. Given positive forms α and β on H, we can find a positive form $\sqrt{\alpha\beta}$, called the **geometric mean** of α and β, satisfying

$$\sqrt{\alpha\beta}(x,x) = \sup\{\gamma(x,x); \gamma \text{ is a positive form dominated by } \{\alpha,\beta\}\}$$

for $x \in H$.

Remark 4. From the proof, we also have

$$\sqrt{\alpha\beta}(x,x) = \sup\{\gamma(x,x); \gamma \text{ is a hermitian form dominated by } \{\alpha,\beta\}\}.$$

Appendix K. Transition Probabilities

Let ω be a positive functional of a C*-algebra A. According to [Pusz-Woronowicz], we introduce two positive forms ω_L and ω_R on A defined by

$$\omega_L(x,y) = \omega(x^*y), \qquad \omega_R(x,y) = \omega(yx^*), \quad x,y \in A.$$

Lemma K.1. Let M be a W*-algebra and Let φ, ψ be positive normal functionals of a W*-algebra M. Then

$$\sqrt{\varphi_L\psi_R}(x,y) = \langle \varphi^{1/2}x^*\psi^{1/2}y\rangle \quad \text{for } x,y \in M.$$

Proof. By the positivity $\langle \varphi^{1/2}x^*\psi^{1/2}x\rangle = (x\varphi^{1/2}x^*|\psi^{1/2}) \geq 0$ and the Schwarz inequality $|\langle \varphi^{1/2}x^*\psi^{1/2}y\rangle|^2 \leq \varphi(x^*x)\psi(yy^*)$, the positive form $(x,y) \mapsto \langle \varphi^{1/2}x^*\psi^{1/2}y\rangle$ is dominated by $\{\varphi_L, \psi_R\}$.

Assume for the moment that φ and ψ are faithful and consider the embedding $i : M \ni x \mapsto x\varphi^{1/2} \in L^2(M)$. Then φ_L is represented by the identity operator, whereas

$$\psi(xx^*) = \|\psi^{1/2}x\|^2 = \|\psi^{1/2}(x\varphi^{1/2})\varphi^{-1/2}\|^2$$

shows that ψ_R is represented by the relative modular operator Δ ($\Delta(\xi) = \psi\xi\varphi^{-1}$). Note here that $M\varphi^{1/2}$ is a core for $\Delta^{1/2}$. Thus

$$\sqrt{\varphi_L\psi_R}(x,y) = (x\varphi^{1/2}|\Delta^{1/2}(y\varphi^{1/2})) = (x\varphi^{1/2}|\psi^{1/2}y) = \langle \varphi^{1/2}x^*\psi^{1/2}y\rangle.$$

Now we relax φ and ψ to have no-trivial supports. Let e be the support projection of $\varphi + \psi$. Then it is the supprot for $\varphi_n = \varphi + \frac{1}{n}\psi$ and $\psi_n = \frac{1}{n}\varphi + \psi$ as well. In particular, φ_n and ψ_n are faithful on the reduced algebra eMe.

Let γ be a positive form on M dominated by $\{(\varphi_n)_L, (\psi_n)_R\}$. Then $\varphi_n(1-e) = 0 = \psi_n(1-e)$ shows that

$$|\gamma(x(1-e),(1-e)y)|^2 \leq \varphi_n((1-e)x^*x(1-e))\psi_n((1-e)yy^*(1-e)) = 0,$$

i.e., $\gamma(x,y) = \gamma(xe,ey)$ for $x,y \in M$, whence we have

$$\gamma(x,y) = \gamma(xe,ey) = \overline{\gamma(ey,xe)} = \overline{\gamma(eye,exe)} = \gamma(exe,eye).$$

Since the restriction $\gamma|_{eMe}$ is dominated by $(\varphi_n|_{eMe})_L$ and $(\psi_n|_{eMe})_R$ with φ_n and ψ_n faithful on eMe, we have

$$\gamma(x,x) = \gamma(exe,exe) \leq \langle \varphi_n^{1/2}ex^*e\psi_n^{1/2}exe\rangle = \langle \varphi_n^{1/2}x^*\psi_n^{1/2}x\rangle.$$

Taking the limit $n \to \infty$, we obtain $\gamma(x,x) \leq \langle \varphi^{1/2}x^*\psi^{1/2}x\rangle$ in view of the Powers-Størmer inequality. $\square$

Remark 5.

(i) The case $\varphi = \psi$ is implicitly considered in [PW].

(ii) In the notation of [U,relative entropy], we have

$$QF_t(\varphi_L, \psi_R)(x, y) = \langle \varphi^{1-t} x^* \psi^t y \rangle$$

for $0 \leq t \leq 1$ and $x, y \in M$.

Given a positive functional φ of a C*-algebra A, let $\widetilde{\varphi}$ be the associated normal functional on the W*-envelope A^{**} through the canonical duality pairing.

Lemma K.2. Let φ and ψ be positive functionals on a C*-algebra A with $\widetilde{\varphi}$ and $\widetilde{\psi}$ the corresponding normal functionals on A^{**}. Then

$$\sqrt{\varphi_L \psi_R}(x, y) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle \quad \text{for } x, y \in A \subset A^{**}.$$

Proof. The positive form $A \times A \ni (x, y) \mapsto \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle$ (recall that $x^* \widetilde{\psi}^{1/2} x$ is in the positive cone to see the positivity) is dominated by $\widetilde{\varphi}_L$ and $\widetilde{\psi}_R$ because of

$$|\langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle|^2 \leq \widetilde{\varphi}(x^* x) \widetilde{\psi}(y y^*) = \varphi(x^* x) \psi(y y^*).$$

Consequently,

$$\langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \leq \sqrt{\varphi_L \psi_R}(x, x) \quad \text{for } x \in A.$$

To get the reverse inequality, let γ be a positive form on $A \times A$ dominated by φ_L and ψ_R. Then we have the domination inequality

$$|\gamma(x, y)|^2 \leq \varphi(x^* x) \psi(y y^*) = \|x \widetilde{\varphi}^{1/2}\|^2 \, \|\widetilde{\psi}^{1/2} y\|^2.$$

Since A is dense in A^{**} relative to the σ^*-topology, we see that γ is extended to a positive form $\widetilde{\gamma}$ on $A^{**} \times A^{**}$ so that

$$|\widetilde{\gamma}(x, y)|^2 \leq \|x \widetilde{\varphi}^{1/2}\|^2 \, \|\widetilde{\psi}^{1/2} y\|^2 \quad \text{for } x, y \in A^{**},$$

whence

$$\gamma(x, x) = \widetilde{\gamma}(x, x) \leq \sqrt{\widetilde{\varphi}_L \widetilde{\psi}_R}(x, x) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \quad \text{for } x \in A.$$

Maximization on γ then yields the inequality

$$\sqrt{\varphi_L \psi_R}(x, x) \leq \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \quad \text{for } x \in A$$

and we are done. □

Corollary K.3. Given a normal state φ of a W*-algebra M, let $\widetilde{\varphi}$ be the associated normal state of the second dual W*-algebra M^{**}. Then

$$L^2(M) \ni \varphi^{1/2} \mapsto \widetilde{\varphi}^{1/2} \in L^2(M^{**})$$

defines an isometry of M-M bimodules.

Proof. Combining two lemmas just proved, we have

$$\langle \varphi^{1/2} x^* \psi^{1/2} y \rangle = \sqrt{\varphi_L \psi_R}(x, y) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle$$

for $x, y \in M$. $\square$

In what follows, $\varphi^{1/2}$ is identified with $\widetilde{\varphi}^{1/2}$ via the above isometry: Given a positive normal functional φ of a W*-algebra M, $\varphi^{1/2}$ is used to stand for a vector commonly contained in the increasing sequence of Hilbert spaces

$$L^2(M) \subset L^2(M^{**}) \subset L^2(M^{****}) \subset \dots.$$

In accordance with this convention, the formula in the previous lemma then takes the form

$$(x\varphi^{1/2} | \psi^{1/2} y) = \sqrt{\varphi_L \psi_R}(x, y) \quad \text{for } x, y \in A.$$

Here the left hand side is the inner product in $L^2(A^{**})$, whereas the right hand side is the geometric mean of positive forms on the C*-algebra A. Note that, the formula is compatible with the invariance of geometric means:

$$\sqrt{\varphi_L \psi_R}(x, y) = \sqrt{\psi_L \varphi_R}(y^*, x^*) = \sqrt{\varphi_R \psi_L}(y^*, x^*).$$

Remark 6. A W*-algebra M satisfies $M_* = M^*$ if and only if $\dim M < +\infty$. In fact, if $\dim M = \infty$, we can find a sequence of non-zero projections $\{p_n\}_{n \geq 1}$ in M such that $\sum_n p_n = 1$. In other words, M contains $\ell^\infty(\mathbb{N})$ as a W*-subalgebra. Let f be a singular state of $\ell^\infty(\mathbb{N})$ and extend it to a state φ of M. If $M_* = M^*$ in addition, φ is normal, which contradicts with

$$1 = \varphi(1_M) = \sum_n \varphi(p_n) = \sum_n f(p_n) = 0.$$

Remark 7.

- (i) When φ and ψ are vector states of a full operator algebra $\mathcal{L}(\mathcal{H})$ associated to normalized vectors ξ, η in $\mathcal{H}$, our transition amplitude $(\varphi^{1/2} | \psi^{1/2})$ is reduced to the transition probability $|(\xi | \eta)|^2$.
- (ii) Let $P(\varphi, \psi)$ be the transition probability between states in the sense of Kakutani-Bures-Uhlmann. Then we have $P(\varphi, \psi) = \langle |\varphi^{1/2} \psi^{1/2}| \rangle^2$ (cf. [Ragio]) and

$$(\varphi^{1/2} | \psi^{1/2})^2 \leq P(\varphi, \psi) \leq (\varphi^{1/2} | \psi^{1/2})$$

for states φ and ψ on a C*-algebra.

In the text, the transition probability described above is utilized to analyse the universal representations of C*-algebras. We shall here show that the main construction remains valid under some positivity assumption on geometric means of positive forms.

Given a finite family $\{\omega_j\}_{1\leq j\leq n}$ of positive functionals of a *-algebra $\mathcal{A}$, let ω be a positive functional of $M_n(\mathcal{A})$ defined by

$$X = (x_{jk}) \mapsto \omega(X) = \sum_{j=1}^{n} \omega_j(x_{jj}).$$

Lemma K.4. Let $E_{jk} \in M_n(\mathbb{C})$ be the matrix unit.

(i) The decomposition $M_n(\mathcal{A}) = \sum_{j,k} \mathcal{A}E_{jk}$ is orthogonal with respect to the positive form $\sqrt{\omega_L \omega_R}$.

(ii) For $x, y \in \mathcal{A}$, $\sqrt{\omega_L\omega_R}(xE_{jk}, yE_{jk}) = \sqrt{(\omega_k)_L(\omega_j)_R}(x, y)$.

Proof. (i) is a consequence of the fact that $\sum \mathcal{A}E_{jk}$ is orthogonal relative to both of ω_L and ω_R.

(ii) If a hermitian form Γ on $M_n(\mathcal{A})$ is dominated by $\{\omega_L, \omega_R\}$, then

$$|\Gamma(xR_{jk}, yE_{jk})|^2 \leq \omega(R_{kj}x^*yE_{jk})\omega(yE_{jk}E_{kj}x^*) = \omega_k(x^*y)\omega_j(yx^*)$$

shows that the hermitian form $\gamma(x, y) = \Gamma(xR_{jk}, yE_{jk})$ on $\mathcal{A}$ is dominated by $\{(\omega_k)L, (\omega_j)_R\}$, whence

$$\sqrt{\omega_L\omega_R}(xE_{jk}, xE_{jk}) = \sup \Gamma(xE_{jk}, xE_{jk}) \leq \sqrt{(\omega_k)_L(\omega_j)_R}(x, x)$$

for $x \in \mathcal{A}$.

Conversely, given a hermitian form γ dominated by $\{(\omega_k)_L, (\omega_j)_R\}$, the hermitian form $\Gamma(X, Y) = \gamma(x_{jk}, y_{jk})$ on $M_n(\mathcal{A})$ is dominated by $\{\omega_L, \omega_R\}$:

$$|\Gamma(X, Y)|^2 \leq \omega_k(x_{jk}^* x_{jk})\, \omega_j(y_{jk}y_{jk}^*) \leq \omega(X^*X)\, \omega(YY^*).$$

Thus,

$$\sqrt{(\omega_k)_L(\omega_j)_R}(x, x) = \sup \gamma(x, x) \leq \sqrt{\omega_L\omega_R}(xE_{jk}, xE_{jk}).$$

□

Definition K.5. A set $\mathcal{P}$ of positive functional on a *-algebra $\mathcal{A}$ is said to be **positive** if ω is a positive functional on $M_n(\mathcal{A})$ associated to a finite family $\{\omega_j\}$ in $\mathcal{P}$, then

$$\sqrt{\omega_L\omega_R}(XX^*, YY^*) \geq 0$$

for any $X, Y \in M_n(\mathcal{A})$.

Example K.6. The set of positive functionals on a C*-algebra is positive.

We now imitate the construction of standard Hilber spaces. Let $\mathcal{P}$ be a positive set of positive functionals. On the free algebraic sum

$$\sum_{\varphi\in\mathcal{P}} \mathcal{A}\otimes\varphi^{1/2}\otimes\mathcal{A},$$

introduce a sesquilinear for by

$$\begin{aligned}\left(\sum_j x_j\otimes\omega_j^{1/2}\otimes y_j\,\middle|\,\sum_k x'_k\otimes\omega_k^{1/2}\otimes y'_k\right)&\\ =\sum_{j,k}\sqrt{(\omega_j)_L(\omega_k)_R}&((x'_k)^*x_j, y'_k y_j^*),\end{aligned}$$

which is positive because

$$\begin{aligned}\sum_{j,k}\sqrt{(\omega_j)_L(\omega_k)_R}(x_k^*x_j, y_k y_j^*) &= \sum_{j,k}\sqrt{\omega_L\omega_R}(x_k^*x_jE_{kj}, y_k y_j^*E_{kj})\\ &= \sqrt{\omega_L\omega_R}(X^*X, YY^*)\geq 0,\end{aligned}$$

where

$$X=\begin{pmatrix} x_1 & \cdots & x_n\\ 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{pmatrix},\quad Y=\begin{pmatrix} y_1 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ y_n & 0 & \dots & 0\end{pmatrix}.$$

The quotient inner product space is denoted by $\mathcal{L}^2(\mathcal{A},\mathcal{P})$ with the quotient vector of $\sum_j x_j\otimes\omega_j^{1/2}\otimes y_j$ with respect to this positive form denoted by

$$\sum_j x_j\omega_j^{1/2}y_j.$$

From

$$\begin{aligned}\|\sum_j y_j^*\omega_j^{1/2}x_j^*\|^2 &= \sum_{j,k}\sqrt{(\omega_j)_L(\omega_k)_R}(y_k y_j^*, x_k^*x_j)\\ &= \overline{\sum_{j,k}\sqrt{(\omega_j)_L(\omega_k)_R}(y_k y_j^*, x_k^*x_j)}\\ &= \sum_{j,k}\sqrt{(\omega_j)_L(\omega_k)_R}(x_k^*x_j, y_k y_j^*)\\ &= \|\sum_j x_j\omega_j^{1/2}y_j\|^2,\end{aligned}$$

the conjugation is well-defined by

$$\left(\sum_j x_j\omega_j^{1/2}y_j\right)^* = \sum_j y_j^*\omega_j^{1/2}x_j^*.$$

From the formula in the definition of pre-iiner product, we have

$$\left(\sum_j (ax_j)\otimes\omega_j^{1/2}\otimes y_j \middle| \sum_k x'_k\otimes\omega_k^{1/2}\otimes y'_k\right)$$
$$= \left(\sum_j x_j\otimes\omega_j^{1/2}\otimes y_j \middle| \sum_k (a^*x'_k)\otimes\omega_k^{1/2}\otimes y'_k\right)$$

for $a \in \mathcal{A}$, whence the left multiplication of $a \in \mathcal{A}$ on the quotient inner product space is well-defined. Similarly for the right multiplication.

In this way, we have constructed a *-bimodule $\mathcal{L}^2(\mathcal{A}, \mathcal{P})$ of $\mathcal{A}$.

A linear map $\Phi : A \to B$ between C*-algebras is said to be a Schwartz map if it satisfies the operator inequality $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ for $a \in A$.

Theorem K.7 (Uhlmann, relative entropy, Proposition 17)**.** Let $\Phi : A \to B$ be a unital Schwarz map between unital C*-algebras. Then, for $\varphi, \psi \in B_+^*$,

$$(\varphi^{1/2}|\psi^{1/2}) \leq ((\varphi\circ\Phi)^{1/2}|(\psi\circ\Phi)^{1/2}).$$

Proof. Let $\gamma : B \times B \to \mathbb{C}$ be a positive form dominated by $\{\varphi_L, \psi_R\}$. Then

$$|\gamma(\Phi(x), \Phi(y))|^2 \leq \varphi(\Phi(x)^*\Phi(x))\psi(\Phi(y)\Phi(y)^*) \leq \varphi(\Phi(x^*x))\psi(\Phi(yy^*))$$

shows that the positive form $A \times A \ni (x, y) \mapsto \gamma(\Phi(x), \Phi(y))$ is dominated by $\{(\varphi\circ\Phi)_L, (\psi\circ\Phi)_R\}$. Thus

$$\gamma(1,1) = \gamma(\Phi(1), \Phi(1)) \leq \sqrt{(\varphi\circ\Phi)_L(\psi\circ\Phi)_R}(1,1) = ((\varphi\circ\Phi)^{1/2}|(\psi\circ\Phi)^{1/2}).$$

Maximizing $\gamma(1, 1)$ with respect to γ, we obtain the inequality. □

Example K.8. Consider an inclusion of matrix algebras $\pi : M_n(\mathbb{C}) \ni x \mapsto x \otimes 1 \in M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$. By the isomorphism $M_n(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_{2n}(\mathbb{C})$, π takes the form

$$\pi(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Let a_j, b_j $(j = 1, 2)$ be hermitian matrices in $M_n(\mathbb{C})$ and set

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}.$$

Consider positive linear functionals on $M_{2n}(\mathbb{C})$ defined by $\varphi(y) = \mathrm{trace}(a^2y)$ and $\psi(y) = \mathrm{trace}(b^2y)$ for $y \in M_{2n}(\mathbb{C})$. Then we see

$$(\varphi \circ \pi)(x) = 2\mathrm{trace}((a_1^2 + a_2^2)x), \quad (\psi \circ \pi)(x) = 2\mathrm{trace}((b_1^2 + b_2^2)x)$$

for $x \in M_n(\mathbb{C})$ and the inequality $(\varphi^{1/2}|\psi^{1/2}) \leq ((\varphi \circ \pi)^{1/2}|(\psi \circ \pi)^{1/2})$ takes the form

$$\mathrm{trace}(|a|\,|b|) \leq \mathrm{trace}((a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}).$$

In view of the Jordan decompositions $a = a_+ - a_-$, $b = b_+ - b_-$ with $|a| = a_+ + a_-$, $|b| = b_+ + b_-$, we see $\mathrm{trace}(ab) \leq \mathrm{trace}(|a|\,|b|)$, which is combined with above inequality to get $\mathrm{trace}(a_1b_1 + a_2b_2) \leq \mathrm{trace}((a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2})$.

Now, by an obvious induction on m, we conclude the following: Given hermitian matrices $a_1, \dots, a_m$ and $b_1, \dots, b_m$ in $M_n(\mathbb{C})$, we have the inequality

$$\mathrm{trace}(a_1b_1 + \cdots + a_mb_m) \leq \mathrm{trace}((a_1^2 + \cdots + a_m^2)^{1/2}(b_1^2 + \cdots + b_m^2)^{1/2}).$$

Theorem K.9. Let φ and ψ be positive functional on a C*-algebra A with unit 1_A. Let $\{A_n\}_{n\in\mathcal{N}}$ be an increasing net of C*-subalgebras of A containing 1_A in common and assume that, given any $a \in A$, we can find a net $\{a_n \in A_n\}_{n\in\mathcal{N}}$ in A satisfying

$$\lim_{n\to\infty} a_n\varphi^{1/2} = a\varphi^{1/2}, \quad \lim_{n\to\infty} \psi^{1/2}a_n = \psi^{1/2}a$$

in the norm topology of $L^2(A)$. Set $\varphi_n = \varphi|_{A_n}, \psi_n = \psi|_{A_n} \in A_n^*$. Then the net $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}_{n\in\mathcal{N}}$ is decreasing and converges to $(\varphi^{1/2}|\psi^{1/2})$.

Proof. The net $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}$ is decreasing with $(\varphi^{1/2}|\psi^{1/2})$ a lower bound by the coarse-graining inequality.

Let e_n and f_n be projections in $\mathcal{B}(L^2(A))$ defined by

$$e_nL^2(A) = \overline{A_n\varphi^{1/2}}, \quad f_nL^2(A) = \overline{\psi^{1/2}A_n}.$$

By the variational expression of geometric mean, we can find positive forms $\gamma_n : A_n \times A_n \to \mathbb{C}$ for $n \in \mathcal{N}$ so that each γ_n is dominated by $\{(\varphi_n)_L, (\psi_n)_R\}$ and satisfies

$$\gamma_n(1, 1) \geq (\varphi_n^{1/2}|\psi_n^{1/2}) - \epsilon_n,$$

where $\{\epsilon_n\}$ is a net of positive reals converging to 0. From the domination inequality of γ_n, we can find a linear map $C_n' : \overline{\psi^{1/2}A_n} \to \overline{A_n\varphi^{1/2}}$ satisfying

$$\gamma_n(x, y) = (x\varphi^{1/2}|C_n'(\psi^{1/2}y)) \quad \text{for } x, y \in A_n$$

and $\|C'_n\| \leq 1$. Let $C_n = e_n C'_n f_n : \overline{\psi^{1/2}A} \to \overline{A\varphi^{1/2}}$. Since $\|C_n\| \leq 1$, we may assume that $C_n \to C$ in weak operator topology by passing to a subnet if necessary. Now set

$$\gamma(x, y) = (x\varphi^{1/2}|C(\psi^{1/2}y)),$$

which is a sesquilinear form on A satisfying $|\gamma(x, y)| \leq \|x\varphi^{1/2}\| \, \|\psi^{1/2}y\|$. Moreover, if $x \in A_m$ for some $m \in \mathcal{N}$,

$$\gamma(x, x) = \lim_{n\to\infty} (x\varphi^{1/2}|C_n(\psi^{1/2}x)) = \lim_{n\to\infty} \gamma_n(x, x) \geq 0,$$

which shows that γ is positive on $\bigcup_{m\in\mathcal{N}} A_m$ and then on A by the approximation assumption. Thus, γ is a positive form dominated by $\{\varphi_L, \psi_R\}$ and the variational estimate is used again to get

$$\begin{aligned}(\varphi^{1/2}|\psi^{1/2}) \geq \gamma(1, 1) &= \lim_{n\to\infty} (\varphi^{1/2}|C_n\psi^{1/2}) = \lim_{n\to\infty} \gamma_n(1, 1) \\ &\geq \lim_{n\to\infty} \big((\varphi_n^{1/2}|\psi_n^{1/2}) - \epsilon_n\big) = \lim_{n\to\infty} (\varphi_n^{1/2}|\psi_n^{1/2}).\end{aligned}$$

□

Appendix L. Random Operators

Random linear operators / A.V. Skorohod

A random operator is a family of operators parametrized by elements in a Borel space in such a way that its dependence is considered to be measurable in some sense.

When a measure is not specified, the measurability means that for Borel structures.

L.1. **Polar Decomposition.** Let $\{\mathcal{H}_\omega\}$ be a measurable field of separable Hilbert spaces and $\{T_\omega : D_\omega \to \mathcal{H}_\omega\}$ be a family of densely defined closed operators which is measurable in the sense that we can find a sequence of measurable sections $\{\xi_n\}_{n\geq 1}$ so that $\sum_{n\geq 1} \mathbb{C}\xi_n(\omega)$ is a core for T_ω at almost every $\omega \in \Omega$ and sections $\{T_\omega \xi_n(\omega)\}_{\omega\in\Omega}$ $(n \geq 1)$ are measurable. Let

$$\mathcal{H}_\omega \oplus \mathcal{H}_\omega = \{\xi \oplus T_\omega\xi; \xi \in D(T_\omega)\} + \{T_\omega^*\eta \oplus -\eta; \eta \in D(T_\omega^*)\}$$

be an orthogonal decomposition associated with the graphs of T_ω and T_ω^*. Let $E_\omega \in \mathcal{B}(\mathcal{H}_\omega \oplus \mathcal{H}_\omega)$ be the projection to the graph of T_ω. By the measurability assumption on $\{T_\omega\}$, $\{E_\omega\}$ is a measurable field of projections and hence so is $\{1_{\mathcal{H}_\omega} - E_\omega\}$. Thus $\{T_\omega^*\}$ is measurable because the second component of $\{(1 - E_\Omega)(\zeta_j(\omega) \oplus \zeta_k(\omega)); j, k \geq 1\}$ is a core for T_ω^*, where $\{\zeta_n\}_{n\geq 1}$ is any sequence of measurable sections such that $\{\zeta_n(\omega); n \geq 1\}$ is dense in $\mathcal{H}_\omega$ at almost every $\omega \in \Omega$.

Ginve a measurable section $\zeta(\omega)$ of $\{\mathcal{H}_\omega\}$, the orthogonal decomposition

$$\zeta(\omega) \oplus 0 = (\xi(\omega) \oplus T_\omega \xi(\omega)) + (T_\omega^* \eta(\omega) \oplus -\eta(\omega))$$

with $\xi(\omega)$ and $\{\eta(\omega)\}$ measurable sections of $\{\mathcal{H}_\omega\}$ and belonging to $D(T_\omega)$ and $D(T_\omega^*)$ respectively. The relation $\xi(\omega) = (1 + T_\omega^* T_\omega)^{-1} \zeta(\omega)$ reveals that $\{(1 + T_\omega^* T_\omega)^{-1}\}$ and then

$$\frac{T_\omega^* T_\omega}{1 + T_\omega^* T_\omega} = 1 - \frac{1}{1 + T_\omega^* T_\omega}$$

are measurable. Since the square roots of a bounded positive operator is realized as a uniform limit of polynomials,

$$\sqrt{\frac{T_\omega^* T_\omega}{1 + T_\omega^* T_\omega}}$$

is measurable as well. Now replace T_ω with tT_Ω $(t > 0)$ and then divide the result by t to get a measurable family of positive operators

$$\sqrt{\frac{T_\omega^* T_\omega}{1 + t^2 T_\omega^* T_\omega}}.$$

Thanks to the spectral calculus, we then see that

$$|T_\omega| \xi_n(\omega) = \lim_{t \to +0} \sqrt{\frac{T_\omega^* T_\omega}{1 + t^2 T_\omega^* T_\omega}} \xi_n(\omega)$$

is a measurable section for $n \geq 1$. Since the partial isometry part V_ω in the polar decomposition of T_ω, is given by $|T_\omega|\xi \mapsto T_\omega \xi$ $(\xi \in D(T_\omega) = D(|T_\omega|))$, $\{V_\omega\}$ maps measurable sections $|T_\omega|\xi_n(\omega)$ into measurable sections $T_\omega \xi_n(\omega)$ for $n \geq 1$. Thus $\{V_\omega\}$ is measurable.

Now let T be a densely defined operator in $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ defined by

$$T\xi = \int_\Omega^\oplus T_\omega \xi(\omega) \, \mu(d\omega)$$

for $\xi = \int_\Omega^\oplus \xi(\omega) \, \mu(d\omega)$ satisfying

$$\int_\Omega \|T_\omega \xi(\omega)\|^2 \, \mu(d\omega) < \infty.$$

Since the graph of T is equal to $E(\mathcal{H} \oplus \mathcal{H})$ with the projectio E defined by

$$E = \int_\Omega^\oplus E_\omega \, \mu(d\omega)$$

T is a closed operator. Furthermore,

$$|T| = \int_\Omega^\oplus |T_\omega| \, \mu(d\omega)$$

and

$$V = \int_\Omega^\oplus V_\omega \, \mu(d\omega)$$

is the partial isometry part in the polar decompositive of T.

L.2. **Sesquilinear Forms.** Let Ω be a Borel space with $\mathfrak{B}(\Omega)$ denoting the complex vector space of Borel functions on Ω. Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of complex vector spaces parametrized by elements in Ω and let $\mathfrak{H}$ be a vector space consisting of sections of $\{H_\omega\}$ fulfilling the conditions:

(i) $\xi = \{\xi(\omega)\} \in \mathfrak{H}$ and $f \in \mathfrak{B}(\Omega)$ imply $f\xi = \{f(\omega)\xi(\omega)\} \in \mathfrak{H}$.

(ii) $\mathfrak{H}$ is closed under taking point-wise sequential limits.

A family $\{\alpha_\omega\}$ of sesquilinear forms is said to be measurable if $\alpha(\xi,\eta) = \{\alpha_\omega(\xi(\omega),\eta(\omega))\} \in \mathfrak{B}(\Omega)$ for any $\xi,\eta \in \mathfrak{H}$.

Let $\phi_\omega(s,t)$ be a family of form functions which is measurable as a function of $(\omega,s,t) \in \Omega \times [0,\infty)^2$. Then, for mearurable families $\{\alpha_\omega\}$, $\{\beta_\omega\}$ of positive sesquilinear forms, the family $\{\phi_\omega(\alpha_\omega,\beta_\omega)\}$ of sesquilinear forms is measurable.

To this this, let $\mathcal{H}_\omega$ be the Hilbert space associated to the positive sesquilinear form $\alpha_\omega+\beta_\omega$ on H_ω and furnish $\{\mathcal{H}_\omega\}$ with the measurable field structure induced from $\mathfrak{H}$. Then the operator representation a_ω, b_ω of α_ω, β_ω gives measurable families $\{a_\omega\}$, $\{b_\omega\}$.

Let Φ be the set of measurable functions ϕ on $\Omega \times [0,1]$ such that $r_\omega = \sup\{|\phi(\omega,s)|; 0 \leq s \leq 1\} < \infty$ for each $\omega \in \Omega$ and Φ_0 be the subset consisting of functions ϕ for which $\{\phi(\omega,a_\omega)\}$ is a measurable family of operators on $\mathcal{H}_\omega$. Note that

$$\phi_\omega(\alpha_\omega,\beta_\omega)(\xi(\omega),\eta(\omega)) = (i(\xi(\omega))|\phi(\omega,a_\omega)i(\eta(\omega))),$$

where $\phi(\omega,s) = \phi_\omega(s,1-s)$, and the measurability of $\{\phi_\omega(\alpha_\omega,\beta_\omega)\}$ follows from that of the operator family $\{\phi(\omega,a_\omega)\}$.

Clearly $\mathfrak{B}_b(\Omega\times[0,1]) \subset \Phi$ [14] and each ϕ is pointwise limit of the sequence $\phi_n \in \mathfrak{B}_b(\Omega\times[0,1])$ defined by

$$\phi_n(\omega,s) = \begin{cases} \phi(\omega,s) & \text{if } r_\omega \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, to see $\Phi = \Phi_0$, it suffices to check $\mathfrak{B}_b(\Omega\times[0,1]) \subset \Phi_0$. In fact, Φ_0 contains $\mathfrak{B}_b(\Omega)\otimes\mathbb{C}[s]$ and is closed under taking uniformly bounded

[14] $\mathfrak{B}_b$ indicates the bounded Borel functions.

pointwise sequential limits. Thus it contains $\mathfrak{B}_b(\Omega) \otimes C[0,1]$ by the Weierstrass approximation theorem and then the whole $\mathfrak{B}_b(\Omega \times [0,1])$ and we are done.

L.3. **Normal Homomorphisms.** Let $\{\phi_\omega : M_\omega \to N_\omega\}$ be a family of normal *-homomorphisms and suppose that it is measurable: Given an adapted operator family $\{a(\omega) \in M_\omega\}$, the family $\{\phi_\omega(a(\omega))\}$ is adapted to $\{N_\omega\}$. Then we can find a measurable family $\{e_\omega \in \mathcal{B}(\mathcal{H}_\omega \otimes \ell^2)\}$ of projections belongin to the commutant of M_ω on $\mathcal{H}_\omega \otimes \ell^2$ and a measurable family of isometries $\{U_\omega^* : \mathcal{K}_\omega \to e_\omega(\mathcal{H}_\omega \otimes \ell^2)\}$ such that

$$\phi_\omega(a) = U_\omega(a \otimes 1)U_\omega^* \quad \text{for } \omega \in \Omega \text{ and } a \in M_\omega.$$

Appendix M. Geometric Approach

To avoid unbounded operators as much as possible, we adopt a sophisticated approach due to Rieffel and Van Daele after the Pedersen's book.

Let $\mathcal{H}$ be a complex Hilbert space and $\Re\mathcal{H}$ be the associated real Hilbert space. Let $H \subset \mathcal{H}$ be a cloased real subspace of $\mathcal{H}$ and assume that

$$H \cap iH = \{0\}, \quad (H + iH)^\perp = \{0\}.$$

Let E and F be real-linear projections to the real-subspaces H and iH repectively. Let $E - F = J|E - F|$ be the polar decomposition of $E - F$.

Lemma M.1.

(i) $E + F$ and $|E - F|$ are complex-linear, while $E - F$ and J are conjugate-linear. $|E - F|$ commutes with E and F.
(ii) $E + F$ and $|E - F|$ are injective with dense ranges.
(iii) J is an involution satisfying $(J\xi|J\eta) = (\eta|\xi)$ for $\xi, \eta \in \mathcal{H}$ and $JEJ = 1 - F$.

Example M.2. After unitary rotations, a standard form of real two-dimensional subspace of $\mathbb{C}^2$ is

$$H = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} i\sin\theta \\ \cos\theta \end{pmatrix},$$

where $0 \leq \theta \leq \pi/2$ and the degeneracy condition is satisfied if and only if $0 \leq \theta < \pi/2$. If we use the following real-orthonormal basis of $\mathbb{C}^2$ as a reference frame

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i\sin\theta \\ \cos\theta \end{pmatrix}, \begin{pmatrix} i\cos\theta \\ -\sin\theta \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix},$$

the relevant projections are expressed in the block matrix form of $M_2(M_2(\mathbb{C}))$ by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \sin^2\theta & I\cos\theta\sin\theta \\ -I\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} \text{ with } I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and we see that

$$J = \begin{pmatrix} \cos\theta & -I\sin\theta \\ I\sin\theta & -\cos\theta \end{pmatrix}, \quad |E-F| = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}.$$

Define a positive self-adjoint operator Δ by $\Delta = \frac{2-E-F}{E+F}$.

Lemma M.3. $\Delta^{-1} = J\Delta J$, $H + iH \subset D(\Delta^{1/2})$ and

$$J\Delta^{1/2}(\xi + i\eta) = \xi - i\eta \quad \text{for } \xi, \eta \in H.$$

The unitaries Δ^{it} commute with J and make H invariant. $\Delta^{it}H = H$ for $t \in \mathbb{R}$.

Returning to the situation of constructing the *-bimodule of a W*-algebra M, Let H be the closure of $\{h\varphi^{1/2}; h = h^* \in M\}$. The assumptions on H are satisfied by the faithfulness of φ. Since $\varphi^{1/2}$ is cyclic and seperating for M', we can use the closure H' of $\{h'\varphi^{1/2}; h' = (h')^* \in M'\}$ as well. Let E', F', J' and Δ' be the associated operators.

Although the obvious part $\mathrm{Re}(H'|iH) = 0$ is only needed in what follows, we shall make the symmetric roles of M and M' clear in the construction.

Lemma M.4. We have $E' = 1-F$, $F' = 1-E$, whence $E'-F' = E-F$, $J' = J$ and $\Delta' = \Delta^{-1}$. Thus $JH = H'$, which suggests the relation $JMJ = M'$.

Proof. The orthogonality $\mathrm{Re}(\xi|\xi') = 0$ for $\xi \in iH$ and $\xi' \in H'$ is clear. We next prove the density of $H' + iH$ in $\mathcal{H}$; let $\eta \in \mathcal{H}$ be real-orthogonal to $H' + iH$ and we shall show that $\eta = 0$.

From the orthogonality $\mathrm{Re}(\eta|H') = 0$, we see

$$(\eta|a'\varphi^{1/2}) = -(a'\varphi^{1/2}|\eta) = -(\varphi^{1/2}|a'\eta)$$

for a hermitian $a' \in M'$ and then the equality $(\eta|a'\varphi^{1/2}) = -(\varphi^{1/2}|a'\eta)$ holds for any $a' \in M'$ by the complex linearity in a'. In other words, the vector $\eta \oplus \varphi^{1/2} \in \mathcal{H} \oplus \mathcal{H}$ is orthogonal to the closed subspace $\mathcal{K} = \overline{\{a'\varphi^{1/2} \oplus a'\eta; a' \in M'\}}$ of $\mathcal{H} \oplus \mathcal{H}$. Likewise, the orthogonality $\mathrm{Re}(\eta|iH) = 0$ implies $(\eta|b\varphi^{1/2}) = (\varphi^{1/2}|b\eta)$ for any $b \in M$. Let

$$P = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

be the projection to $\mathcal{K} \subset \mathcal{H} \oplus \mathcal{H}$, where a, b and c in $\mathcal{B}(\mathcal{H})$ belong to M because $\mathcal{K}$ is invariant under the diagonal action of M'.

Since $\eta \oplus \varphi^{1/2} \in \mathcal{K}^\perp$ and $\varphi^{1/2} \oplus \eta \in \mathcal{K}$, as a part of $P \begin{pmatrix} \eta \\ \varphi^{1/2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $P \begin{pmatrix} \varphi^{1/2} \\ \eta \end{pmatrix} = \begin{pmatrix} \varphi^{1/2} \\ \eta \end{pmatrix}$, we have

$$a\eta + b\varphi^{1/2} = 0, \quad a\varphi^{1/2} + b\eta = \varphi^{1/2}.$$

All these are combined to get

$$(\eta|a\eta) = -(\eta|b\varphi^{1/2}) = -(\varphi^{1/2}|b\eta) = -(\varphi^{1/2}|(1-a)\varphi^{1/2})$$

The operator inequlity $0 \leq a \leq 1$ as a reduction of P now leads us to $(\eta|a\eta) = 0 = (\varphi^{1/2}|(1-a)\varphi^{1/2})$. Since $\varphi^{1/2}$ is separating for M and $1 - a \in M_+$, we conclude that $a = 1$ and then $(\eta|\eta) = 0$. □

Lemma M.5. The operator $J\Delta^{1/2}$ is the closure of $M\varphi^{1/2} \ni a\varphi^{1/2} \mapsto a^*\varphi^{1/2} \in M\varphi^{1/2}$.

Appendix N. Stone-Čech Compactification

J.B. Conway, A Course in Functional Analysis, Springer, Recall that a topological space X is said to be completely regular if

For a topological space X, let $C_b(X)$ be the C*-algebra of bounded continuous functions on X with Ω the Gelfand spectrum of $C_b(X)$. Let $\delta : X \ni x \mapsto \delta_x \in \Omega$ be defined by $\delta_x(f) = f(x)$ for $f \in C_b(X)$, which is continuous.

Proposition N.1. The continuous map $\delta : X \to \Omega$ gives a homeomorphism of X onto the image $\delta_X \subset \Omega$ if and only if X is completely regular.

Theorem N.2 (Stone-Čech)**.** Let X be a completely regular topological space. Then $\delta_X \subset \Omega$ is dense in Ω and each $f \in C_b(X)$ has an extension to a continuous function on Ω.