A Fine Path to Measure

Yamagami Shigeru

Graduate School of Mathematics
Nagoya University
Japan

2021/2/22, QAS, Nagoya

Based on six lectures delivered during 2020/6/10–10/14 in Nagoya.

Route Map of Mt. Lebesgue

Twin peaks mountain

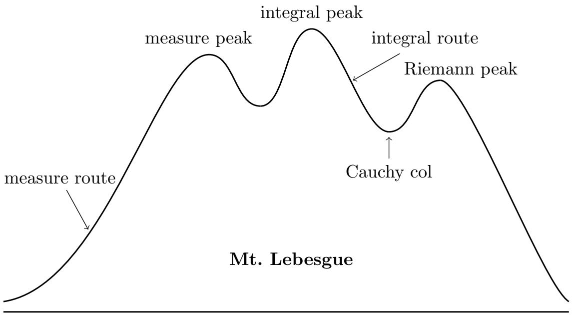

Figure: Route Map

Historical Developments

- Lebesgue (1902)
- Carathéodory (1918)
- Daniell (1918)
- Kolmogorov (1933)
- Saks (1937)
- Nakano (1937, 1940)
- Weil (1938)
- Stone (1948)
- Halmos (1950)
- Bourbaki (1952)
- Riesz-Nagy (1952)
- Loomis (1953)
- Mikusiński (1978)

Lattice Notation

Lattice notation in $\mathbb{R}$

$$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\}$$

satisfying

$$a + b = a \vee b + a \wedge b, \quad |a - b| = a \vee b - a \wedge b$$

is extended to real-valued functions by

$$\begin{aligned}(f \vee g)(x) &= f(x) \vee g(x) = \max\{f(x), g(x)\}, \\ (f \wedge g)(x) &= f(x) \wedge g(x) = \min\{f(x), g(x)\}\end{aligned}$$

with relations

$$f + g = f \vee g + f \wedge g, \quad |f - g| = f \vee g - f \wedge g.$$

Integration System

An integration system (X, L, I):

L is a complex vector space of functions on a set X such that

- $f \in L \implies \overline{f} \in L$, i.e., $L = \mathrm{Re}\, L + i\mathrm{Re}\, L$,
- (lattice condition) $f, g \in \mathrm{Re}\, L \implies f \vee g, f \wedge g \in \mathrm{Re}\, L$

($\mathrm{Re}\, L =$ the real part of L).

$\mathrm{Re}\, L$ is an ordered vector space with a positive part $L^+ = \{0 \leq f \in \mathrm{Re}\, L\}$ and the lattice condition is equivalent to

- $f \in \mathrm{Re}\, L \implies |f| \in L^+$.

Example

Given a locally compact space X, $L = C_c(X)$ satisfies a stronger condition $|f| \in L^+$ ($f \in L$).

$I : L \to \mathbb{C}$ (called an integral) is a linear functional satisfying

- (positivity) $f \in L^+ \implies I(f) \geq 0$,
- (continuity) $f_n \downarrow 0$ $(f_n \in L^+) \implies I(f_n) \downarrow 0$.

Lemma (Dini's theorem)

A positive linear functional on $C_c(X)$ is always continuous.

Example

The Cauchy-Riemann integral on $C_c(\mathbb{R}^d)$.
The Riemann-Stieltjes integral on $C([a,b])$

$$\int_a^b f(x)\, d\Phi(x).$$

A weighted sum on the space $\ell(X)$ of functions of finite support

$$I(f) = \sum_{x \in X} \rho(x) f(x).$$

Dominated Series Expression

A function $f : X \to \mathbb{C}$ is said to be I-**integrable** if it has a **dominated series expression**:
$\exists\ (f_n)_{n\geq 1} \in L$ and $(\varphi_n)_{n\geq 1} \in L^+$ such that

1. $|f_n| \leq \varphi_n$ ($|f_n| \in L$ being NOT assumed),
2. $\displaystyle\sum_{n=1}^{\infty} I(\varphi_n) < \infty$,
3. $\displaystyle\sum_{n=1}^{\infty} \varphi_n(x) < \infty \implies f(x) = \sum_{n=1}^{\infty} f_n(x)$.

Let L^1 be the set of I-integrable functions. Then L^1 is a vector lattice extension of L and $I^1 : L^1 \to \mathbb{C}$ is well-defined by

$$I^1(f) = \sum_{n=1}^{\infty} I(f_n).$$

Lebesgue Integrals as Daniell Extensions

Starting with the above definition, more or less standard estimation arguments lead us to the following.

Theorem

The triplet (X, L^1, I^1), the **Daniell extension** *of (X, L, I), is again an integration system.*

Example

The Daniell extension of $(\mathbb{R}^d, C_c(\mathbb{R}^d), I)$ is nothing but the Lebesgue integral on the space $L^1(\mathbb{R}^d)$ of Lebesgue-integrable functions.

Theorem (Transfer Principle)

Let (X, L, I), (Y, M, J) be integration systems ρ.
If a map $\phi : X \to Y$, together with a function $\rho : X \to [0, \infty)$, satisfies $\rho(M \circ \phi) \subset L$ and $I(\rho(g \circ \phi)) = J(g)$ $(g \in M)$, then

$$\rho(M^1 \circ \phi) \subset L^1, \quad I^1(\rho \circ \phi)) = J^1(g) \ (g \in M^1).$$

Example

1. Restriction $L^1(U) \subset L^1(\mathbb{R}^d)$ for an open $U \subset \mathbb{R}^d$.
2. Weighted integrals.
3. Change-of-variables: For $\phi : U \to V$ in $\mathbb{R}^d$,

$$\int_V f(y)\, dy = \int_U f(\phi(x))|\det \phi'(x)|\, dx.$$

Completeness and Convergence Theorems

The Daniell extension is complete or maximal, which is the whole source of convergence theorems.

Theorem

The Daniell extension of (X, L^1, I^1) *is identical to* (X, L^1, I^1).

From the difference-summation relation, we have

Corollary (Monotone Convergence Theorem)

Let $f_n \uparrow$ *in Re* L^1 *and* $f = \lim f_n$ *is a real-valued function on* X, f *is* I*-integrable iff* $\lim I^1(f_n) < \infty$.
Moreover $I^1(f) = \lim I^1(f_n)$ *if this is the case.*

Once the monotone convergence theorem is established, it is a routine work to get other convergence theorems.

Lemma (Fatou)

If $f_n \in \text{Re}\, L^1$ and $g \in L^1$ satisfy $|f_n| \le g$ $(n \ge 1)$, then

$$\inf_{n\ge 1} f_n, \quad \sup_{n\ge 1} f_n, \quad \liminf_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n$$

are all integrable and

$$I^1(\liminf f_n) \le \liminf I(f_n) \le \limsup I(f_n) \le I^1(\limsup f_n).$$

Theorem (Dominated convergence theorem)

If $f_n \in L^1$, $g \in L^1$ satisfy $|f_n| \le g$ $(n \ge 1)$ and $f = \lim_{n\to\infty} f_n$, then f is integrable and

$$I(f) = \lim_{n\to\infty} I(f_n).$$

Null functions and sets

$f \in L^1$ is a **null function** $\iff I^1(|\mathrm{Re}\, f|) = I^1(|\mathrm{Im}\, f|) = 0$.
$A \subset X$ is a **null set** $\iff$ its indicator 1_A is a null function.

Example

If $\sum I(\varphi_n) < \infty$, then $[\sum \varphi_n = \infty]$ is a null set.

Lemma

f is a null function $\iff [f \neq 0]$ is a null set.

Corollary

- *Subsets of a null set are null sets.*
- *Countable unions of null sets are null sets.*

Historical comments

F. Riesz revealed the functional meaning of null sets in his treatise on functional analysis as

Lemma A: For a decreasing sequence (h_n) of step functions which converges to 0 almost everywhere, $I(h_n) \downarrow 0$.

Lemma B: For an increasing sequence (h_n) of step functions, if $(I(h_n))$ is bounded, then h_n converges almost everywhere.

Later J. Mikusinski reinterpreted Riesz' two lemmas and generalized in his book on Bochner integral as follows:

A real-valued function f is integrable if there exist elementary functions $f_n \in L$ satisying

1. $\sum I(|f_n|) < \infty$ and
2. $f(x) = \sum_n f_n(x)$ whenever $\sum |f_n(x)| < \infty$.

From integral to measure

A subset $A \subset X$ is **I-integrable** $\iff 1_A \in L^1$.
$A \subset X$ is **σ-integrable** $\iff A = \bigcup A_n$ with A_n I-integrable.
I is **σ-finite** $\iff X$ is σ-integrable.
The measurability is less obvious but $A \subset X$ is **I-measurable** $\iff A \cap Q$ is integrable for any integrable Q.
Let $\mathcal{L}(I)$ be the totality of I-measurable sets.

Proposition

1. *Integrable sets are closed under sum, difference and countable intesection.*
2. *σ-integrable sets are closed under difference and countable sum.*
3. *$\mathcal{L}(I)$ is a σ-algebra $\supset$ σ-integrable sets.*

Theorem

Among measures on $\mathcal{L}(I)$ satisfying $|Q| = I^1(1_Q)$ for any integrable Q, there exist minimal and maximal ones specified by

$$|A|_I = \sup\{I^1(1_Q); Q \subset A \text{ is integrable}\},$$

$$|A|^I = \begin{cases} I^1(1_A) & \text{if } A \text{ is integrable,} \\ \infty & \text{otherwise.} \end{cases}$$

In other words, $|A|_I \leq |A| \leq |A|^I$ $(A \in \mathcal{L}(I))$ and $|A|$ is determined by I on σ-integrable sets.

In particular, if I is σ-finite, then $|A|_I = |A|^I$ $(A \in \mathcal{L}(I))$.

Example

Let $I_\rho(f) = \sum_{x \in X} \rho(x) f(x)$ be a weighted sum on $\ell(X)$ with $\rho : X \to [0, \infty)$. Then,

$$A \text{ is } I_\rho\text{-integrable} \iff A \text{ is countable and } \sum_{x \in A} \rho(x) < \infty,$$

$$A \text{ is } \sigma\text{-integrable} \iff A \text{ is countable.}$$

Any A is I_ρ-measurable and $|A|_I = \sum_{x \in A} \rho(x)$, whereas $|A|^I = \infty$ if A is uncountable even for $\rho \equiv 0$.

From premeasure to integral

A set-function $\mu : \mathcal{A} \to [0, \infty]$ on a Boolean ring is a premeasure if it is σ-additive: $\mu(\emptyset) = 0$ and

$$A = \bigsqcup A_n \text{ (in } \mathcal{A}) \Longrightarrow \mu(A) = \sum \mu(A_n).$$

Given a premeasure μ, $L(X, \mu)$ is the totality of functions of the form

$$f = \sum_{j=1}^{n} \lambda_j \mathbf{1}_{A_j}, \quad \mu(A_j) < \infty$$

is a vector lattice and $I_\mu(f) = \sum \lambda_j \mu(A_j)$ gives an integral on $L(X, \mu)$.
Let $(L^1(X, \mu), I^1_\mu)$ be the Daniell extension of $(L(X, \mu), I_\mu)$ with $\mathcal{L}_\mu$ the σ-algebra of I_μ-measurable functions.
The maximal I_μ-measure $|A|^\mu$ then extends μ to $\mathcal{L}_\mu \supset \mathcal{A}$.

Integral-Measure-Integral

Given an integration system (L, I), let μ be the minimal I-measure. Then $L^1(X, \mu) \subset L^1(I)$.
Stone condition on L: $1_X \wedge h \in L$ $(h \in L^+)$.

Lemma

$L^1(I)$ satisfies the Stone condition if so does L or I is σ-finite.

Theorem

Under the Stone condition on $L^1(I)$,
$L^1(X, \mu) = L^1(I) + \{I_\mu\text{-null functions}\}$.
When I is σ-finite, $L^1(X, \mu) = L^1(I)$.

Measure-Integral-Measure

Starting with a measure space $(X, \mathcal{B}, \mu)$,
construct an integration system $(L(X,\mu), I_\mu)$
and then I_μ- measure on I_μ-measurable sets.

Theorem

$$L^1(X,\mu) = \{\mathcal{B}\text{-measurable and } I_\mu\text{-integrable functions}\} + \{I_\mu\text{-null functions}\}.$$

When μ is σ-finite, I_μ-measure is the completion of μ and $L^1(X,\mu) = L^1(I_\mu)$.

Markov map and repeated integrals

Let L (resp. M) be a vector lattice on S (resp. T).

Definition

A linear map $\Lambda : L \to M$ is called a Markov map if it satisfies

Positivity : $f \in L^+ \implies \Lambda(f) \in M^+$.

Continuity : $f_n \downarrow 0$ in $L \implies \Lambda(f_n) \downarrow 0$ in M.

Λ is equivalently described by a family (λ_t) of integrals on L.

Example

Given measure spaces $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y, \mu_Y)$, let $L = L(X, \mu_X) \otimes L(Y, \mu_Y)$ on $S = X \times Y$ and $M = L(X, \mu_X)$ on $T = X$. Then λ_x is given by

$$\lambda_x(f \otimes g) = f(x) \int_Y g(y)\, \mu_Y(dy).$$

Assume that each λ_t is σ-finite and we are given a σ-finite integral J on M with the asoosicated σ-finite measure μ.

An integra I is then defined on L by

$$I(f) = \int_T \lambda_t(f)\,\mu(dt).$$

We assume that I is σ-finite with λ the associated σ-finite measure on $\mathcal{L}(I)$.

Practically, these assumptions are satisfied under a mild condition of σ-finiteness on L.

Theorem (repeated integral formulas)

Fubini *For an integrable function $f \in L^1$, we can find a μ-null set $N \subset T$ so that f is λ_t-integrable for $t \in T \setminus N$, $\lambda_t(f)$ is μ-integral as a function of t and*

$$I^1(f) = \int_T \lambda_t(f)\,\mu(dt).$$

Tonelli *For an I-measurable function $f : S \to [0, \infty]$, f is λ_t-measurable for μ-a.e. $t \in T$, $\int_S f(s)\,\lambda_t(ds) \in [0, \infty]$ is μ-measurable as a function of t and*

$$\int_S f(s)\,\lambda(ds) = \int_T \int_S f(s)\,\lambda_t(ds)\,\mu(dt).$$

Projective limits

Let $(X_\alpha, \mathcal{B}_\alpha, \mu_\alpha)$ be a family of probability spaces indexed by a directed set $\mathcal{D}$ and consider a projective system $\pi_\alpha^\beta : X_\beta \to X_\alpha$ of surjective measurable maps satisfying $(\pi_\alpha^\beta)_* \mu_\beta = \mu_\alpha$ (consistency condition).
Assume that the projective limit $\varprojlim X_\alpha$ exists as a set and we seek for projective limits of probability measures.

Theorem (Daniell-Kolmogorov)

If X_α are locally compact and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ are continuous, then the projective limit exists as a probability space.

Another situation: Suppose that there exist measures ${}_x\lambda_\beta$ on $(X_\beta, \mathcal{B}_\beta)$ for $\alpha \prec \beta$ and $x \in X_\alpha$ such that

1. ${}_x\lambda_\beta$ is supported by $(\pi_\alpha^\beta)^{-1}(x) \subset X_\beta$.
2. For each $B \in \mathcal{B}_\beta$, ${}_x\lambda_\beta(B)$ is $\mathcal{B}_\alpha$-measurable in $x \in X_\alpha$ and

$$\mu_\beta(B) = \int_{X_\alpha} {}_x\lambda_\beta(B)\, \mu_\alpha(dx).$$

3. For $\alpha \prec \beta \prec \gamma$, $x \in X_\alpha$ and $B \in \mathcal{B}_\gamma$,

$$ {}_x\lambda_\gamma(B) = \int_{X_\beta} {}_y\lambda_\gamma(B)\, {}_x\lambda_\beta(dy).$$

Theorem (Kakutani)

The projective limit exists as a probability space.