FREE STATES ON CCR ALGEBRAS

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Contents

Abstract. Free states on CCR algebras are reviewed with emphasis on transition probabilities among them.

1. Introduction

Let $\mathcal{A}$ be the *-algebra of bounded $\mathbb{C}$-valued Borel functions on a Borel space Ω and $\mathfrak{P}$ be the set of probability measures on Ω.

Consider the free $\mathcal{A}$-module over the set of formal symbols $\{\varphi^{1/2}; \varphi \in \mathfrak{P}\}$ on which we introduce a positive sesquilinear form by

$$\left(\sum_{\varphi \in \mathfrak{P}} a_\varphi \varphi^{1/2} \middle| \sum_{\varphi \in \mathfrak{P}} b_\varphi \varphi^{1/2} \right) = \sum_{\varphi, \psi \in \mathfrak{P}} \int_\Omega \overline{a_\varphi(\omega)} b_\psi(\omega) \sqrt{\varphi(d\omega)} \sqrt{\psi(d\omega)}.$$

Here the Hellinger integral in the right hand side is defined by

$$\int_\Omega f(\omega) \sqrt{\varphi(d\omega)} \sqrt{\psi(d\omega)} = \int_\Omega f(\omega) \sqrt{\frac{d\varphi}{d\mu}(\omega) \frac{d\psi}{d\mu}(\omega)}\, \mu(d\omega)$$

for $f \in \mathcal{A}$, where μ is any measure majorizing φ and ψ.

The associated Hilbert space is denoted by $L^2(\Omega)$, which contains $\varphi^{1/2}$ as a special vector so that $(\varphi^{1/2}|\psi^{1/2}) \geq 0$. By the way of definition, $\varphi^{1/2} \perp \psi^{1/2}$ if and only if φ and ψ are disjoint[1].

Now let Ω be the product Borel space of a sequence of Borel spaces $\{\Omega_n\}_{n\geq 1}$ and $\varphi = \prod \varphi_n$ and $\psi = \prod \psi_n$ be product probability measures on Ω.

Theorem 1.1 (von Neumann-Kakutani)**.**

$$(\varphi^{1/2}|\psi^{1/2}) = \prod_{n\geq 1}(\varphi_n^{1/2}|\psi_n^{1/2}).$$

Theorem 1.2 (Kakutani's Dichotomy)**.** Assume that φ_n and ψ_n are equivalent[2] for every $n \geq 1$. Then φ and ψ are either equivalent or disjoint according to $(\varphi^{1/2}|\psi^{1/2}) > 0$ or $(\varphi^{1/2}|\psi^{1/2}) = 0$.

We shall apply these results to so-called gaussian measures. Let V be a finite-dimesional real vector space. A gaussian measure φ on V^* is characterized by its Fourier transform (the characteristic function of φ) by

$$\int_{V^*} e^{i\omega(x)}\, \varphi(d\omega) = e^{-S(x)/2+i\alpha(x)}, \quad x \in V,$$

where S is a positive quadratic form[3] on V (the covariance form) and $\alpha : V \to \mathbb{R}$ is a linear functional (the mean functional). We write $\varphi = \varphi_{\alpha,S}$.

Theorem 1.3.

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = \sqrt{\det\left(\frac{2\sqrt{ST}}{S+T}\right)}\, e^{-\frac{1}{4}(S+T)^{-1}(\alpha-\beta)},$$

where, given a positive quadratic form Q on V, Q^{-1} is a quadratic form on V^* defined by

$$Q^{-1}(f) = \begin{cases} Q(v) & \text{if } f(\cdot) = Q(v,\cdot), \\ +\infty & \text{otherwise.} \end{cases}$$

One may expect similar results for an infinite-dimensional V as well, but subtleties come into here. To see these, we shall be more specific.

Let $\mathbb{R}^\infty$ be the set of sequences of real numbers with the product Borel structure. Given a sequence $S = (s_n)_{n\geq 1}$ of positive reals, let φ_S

[1] $\varphi(E) = 1 = \psi(\Omega \setminus E)$ for some Borel $E \subset \Omega$.
[2] $\varphi_n(E) = 0$ if and only if $\psi_n(E) = 0$ for any Borel $E \subset \omega$.
[3] $S(x) = S(x,x)$ with $S(x,y)$ the associated positive sesquilinear form on V.

be the infinite product of gaussian measures of variances s_j ($j \geq 1$): If we denote by ω_j the j-the component of $\omega \in \mathbb{R}^\infty$, then

$$\int_{\mathbb{R}^\infty} e^{i\sum_{j=1}^n x_j\omega_j}\,\varphi_S(d\omega) = e^{-\sum_{j=1}^n s_j x_j^2/2}.$$

Proposition 1.4. For a sequence $T = \{t_j\} \in \mathbb{R}_+^\infty$ of positive reals and $\beta \in \mathbb{R}^\infty$, set

$$\mathbb{R}^\infty_{\beta,T} = \{\omega = (\omega_j) \in \mathbb{R}^\infty; \sum_{j=1}^\infty t_j(\omega_j + \beta_j)^2 < \infty\}.$$

Then

$$\varphi_S(\mathbb{R}^\infty_{\beta,T}) = \begin{cases} 1 & \text{if } \sum_j s_j t_j < \infty \text{ and } \sum_j t_j\beta_j^2 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\sum_j t_j(\omega_j+\beta_j)^2 < \infty$ for φ_S-a.e. $\omega \in \mathbb{R}^\infty$ if $\sum_j s_j t_j < \infty$ and $\sum_j t_j\beta_j^2 < \infty$, whereas $\sum_j t_j(\omega_j + \beta_j)^2 = \infty$ for φ_S-a.e. $\omega \in \mathbb{R}^\infty$ if not.

This reveals that, if S is non-degenerate, then the measure φ_S is not supported by the topological dual of V with repsect to S. To get a supporting dual, we need to replace V with a smaller subspace V_0 and endow V_0 with a stronger topology so that $V_0^* \supset V^*$ is big enough.

A warning is in order here that there is no preferable choice of V_0. This kind of arbitrariness, however, can be avoided if one works with a formal function algebra $\mathcal{A}$ generated by e^{iv} ($v \in V$) and reformulate $\varphi_{\alpha,S}$ as a positive linear functional on $\mathcal{A}$.

In this setting, we can still construct the Hilbert space $L^2(\mathcal{A})$ without referring to measure spaces so that the square root of $\varphi_{\alpha,S}$ lives there and exactly the same formula holds for $(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2})$. Now the Kakutani dichotomy takes the following form for gaussian measures.

Theorem 1.5. Two gaussian measures $\varphi_{\alpha,S}$ and $\varphi_{\beta,T}$ are equivalent or disjoint according to non-vanishing or vanishing of $(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2})$.

The main purpose of this series of lectures is to generalize these results in such a way that it allows quantum effects at its most basic level.

2. Algebras and Representations

An algebra $\mathcal{A}$ over $\mathbb{C}$ is called a ***-algebra** if it is furnished with a conjugate linear involution $* : \mathcal{A} \to \mathcal{A}$ (called a *-operation) satisfying

$$(ab)^* = b^*a^*, \quad a, b \in \mathcal{A}.$$

An element a in a *-algebra $\mathcal{A}$ is said to be **hermitian** if $a^* = a$ and a hermitian element p is called a **projection** if $p^2 = p$. When $\mathcal{A}$ has a unit 1, a is said to be **unitary** if $aa^* = a^*a = 1$.

A *-algebra is said to be **unitary**[4] if it is generated by unitaries.

Example 2.1. Given a *-algebra $\mathcal{A}$, the $n \times n$ matrix algebra $M_n(\mathcal{A})$ with entries in $\mathcal{A}$ is a *-algebra.

Example 2.2. Let $\mathbb{C}[X]$ be the polynomial algebra of indeterminate X and make it into a *-algebra by $(\sum_{n\geq 0} a_n X^n)^* = \sum_{n\geq 0} \overline{a_n} X^n$. Then 0 and 1 are all the projections and constant polynomials of modulus 1 are all the unitaries.

Example 2.3. Given a group G, the free vector space $\mathbb{C}G$ generated by elements in G is a *-algebra (the group algebra) by extending the group product to the algebra multipliction and defining the *-operation so that elements in G are unitary. The group algebra $\mathbb{C}G = \sum_{g\in G} \mathbb{C}g$ is unitary.

Exercise 1. Let $\mathcal{A}$ be the vector space of functions on a group G of finite support and make it into a *-algebra (the convolution algebra) by

$$(ab)(g) = \sum_{g'g''=g} a(g')b(g''), \quad a^*(g) = \overline{a(g^{-1})}.$$

The convolution algebra $\mathcal{A}$ of G is naturally isomorphic to the group algebra $\mathbb{C}G$.

Given *-algebras $\mathcal{A}$ and $\mathcal{B}$, their direct sum $\mathcal{A}\oplus\mathcal{B}$ and tensor product $\mathcal{A}\otimes\mathcal{B}$ are again *-algebras in an obvious manner.

Exercise 2. The matrix algebra $M_n(\mathcal{A})$ is naturally identified with the tensor product $M_n(\mathbb{C}) \otimes \mathcal{A}$.

Let $\mathcal{H}$ be a pre-Hilbert space; $\mathcal{H}$ is a complex vector space with a positive definite inner product $(\,|\,)$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is called the **adjoint** of a linear operator $S : \mathcal{H} \to \mathcal{H}$ (and denoted by S^*) if it satisfies $(\xi|S\eta) = (T\xi|\eta)$ (for $\xi, \eta \in \mathcal{H}$). A linear operator S on $\mathcal{H}$ is sadi to be bounded (on the unit ball) if $\|S\| = \sup\{\|S\xi\|; \xi \in \mathcal{H}, \|\xi\| \leq 1\}$ is finite. Let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$ having adjoints, which is a unital *-algebra in an obvious way. The subset $\mathcal{B}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ consisting of bounded operators is a *-subalgebra. When $\mathcal{H}$ is complete, a linear operator on $\mathcal{H}$ has an adjoint if and only if it is bounded thanks to the closed graph theorem and the Riesz lemma, whence $\mathcal{L}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$.

[4] This is not a common usage of terminology.

Exercise 3. A positive semidefinite sesquilinear form $(\ |\ )$ on a complex vector space K produces a Hilbert space by taking completion after quotient of K.

By a ***-representation** of a *-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$, we shall mean an algebra-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^* = \pi(a^*)$ for $a \in \mathcal{A}$. When $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$, the *-representation is said to be **bounded**. If $\mathcal{A}$ is unitary, any *-representation is automatically bounded.

If a *-representation $(\pi, \mathcal{H})$ is bounded and $\mathcal{H}$ is complete, we can associate several operator algebras to it:

(i) A norm-closed operator algebra (C*-algebra) $\overline{\pi(\mathcal{A})}$ as a norm-closure of $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$.
(ii) A weakly closed operator algebra (W*-algebra) $\pi(\mathcal{A})'$ as the commutant $\{b \in \mathcal{B}(\mathcal{H}); \pi(a)b = b\pi(a), \forall a \in \mathcal{A}\}$ of $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$.
(iii) Another W*-algebra $\overline{\pi(\mathcal{A})}^w$ as a weak closure of $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$.

Theorem 2.4 (von Neumann, [15, Theorem 4.15])**.** Let $\mathcal{H}$ be a Hilbert space. For any *-subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ satisfying $\overline{\mathcal{B}\mathcal{H}} = \mathcal{H}$, we have $\overline{\mathcal{B}}^w = (\mathcal{B}')'$.

Exercise 4. Let $E \in \mathcal{B}(\mathcal{H})$ be a projection to the closed subspace $\mathcal{K} \subset \mathcal{H}$. Then $\mathcal{B}\mathcal{K} \subset \mathcal{K}$ if and only if $E \in \mathcal{B}'$.

It is often convenient to regard the representation space $\mathcal{H}$ as a left $\mathcal{A}$-module by $a\xi = \pi(a)\xi$. Thus a right $\mathcal{A}$-module structure corresponds to a *-antirepresentation, i.e., an algebra-antihomomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^* = \pi(a^*)$, by the relation $\xi a = \pi(a)\xi$. A pre-Hilbert space $\mathcal{H}$ is called an $\mathcal{A}$-$\mathcal{B}$ bimodule ($\mathcal{B}$ being another *-algebra) if we are given a *-representation $\lambda : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a *-antirepresentation $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ satisfying $\lambda(a)\rho(b) = \rho(b)\lambda(a)$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, i.e., $(a\xi)b = a(\xi b)$ in the module notation. An $\mathcal{A}$-$\mathcal{A}$ bimodule $\mathcal{H}$ is called a ***-bimodule** if we are given an antiunitary [5] involution ξ^* on $\mathcal{H}$ satisfying $(a\xi b)^* = b^*\xi^*a^*$ for $a, b \in \mathcal{A}$ and $\xi \in \mathcal{H}$.

Given another bounded *-representation ${}_A\mathcal{K}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$, a bounded linear map $T : \mathcal{H} \to \mathcal{K}$ is called an **intertwiner** if it satisfies $T(a\xi) = aT(\xi)$ for $a \in A$ and $\xi \in \mathcal{H}$. We denote the space of intertwiners by $\mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{K})$, which is a closed subspace of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. When ${}_A\mathcal{H} = {}_A\mathcal{K}$, $\mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{K})$, which is also denoted by $\mathrm{End}({}_A\mathcal{H})$, is equal to the commutant $\pi(\mathcal{A})'$ of $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$.

[5] A conjugate-linear operator J on a pre-Hilbert space $\mathcal{H}$ is called an antiunitary if $(J\xi|J\eta) = (\eta|\xi)$ and $J\mathcal{H} = \mathcal{H}$.

According to the obvious block representation of linear operators, we have

$$\mathrm{End}({}_{\mathcal{A}}(\mathcal{H} \oplus \mathcal{K})) = \begin{pmatrix} \mathrm{End}(\mathcal{H}) & \mathrm{Hom}(\mathcal{K}, \mathcal{H}) \\ \mathrm{Hom}(\mathcal{H}, \mathcal{K}) & \mathrm{End}(\mathcal{K}) \end{pmatrix}$$

and the information of intertwiners is encoded in the commutant (of a suitable representation).

An $\mathcal{A}$-submodule ${}_{\mathcal{A}}\mathcal{K}$ is called a **subrepresentation** of ${}_{\mathcal{A}}\mathcal{H}$. When $\mathcal{H}$ is complete and $\mathcal{K}$ is closed, let e be the projection to $\mathcal{K} \subset \mathcal{H}$. Then $e \in \pi(\mathcal{A})'$ and there is a one-to-one correspondence between (closed) subrepresentations of ${}_{\mathcal{A}}\mathcal{H}$ and projections in $\pi(\mathcal{A})'$.

In what follows, the completeness of $\mathcal{H}$ is assumed when one talks about bounded representations.

Two bounded *-representations $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ are said to be **unitarily equivalent** (resp. **quasi-equivalent**) if we can find a unitary intertwiner $T : \mathcal{H}_1 \to \mathcal{H}_2$ (resp. a *-isomorphism $\phi : \pi_1(\mathcal{A})'' \to \pi_2(\mathcal{A})''$ satisfying $\pi_2(a) = \phi(\pi_1(a))$). Quasi-equivalence is an equivalence up to multiplicities:

Theorem 2.5 (Dixmier, [15, Theorem 5.8])**.** Two bounded *-representations $(\pi_j, \mathcal{H}_j)$ $(j = 1, 2)$ are quasi-equivalent if and only if we can find Hilbert spaces $\mathcal{K}_j$ so that ${}_{\mathcal{A}}\mathcal{H}_1 \otimes \mathcal{K}_1$ and ${}_{\mathcal{A}}\mathcal{H}_2 \otimes \mathcal{K}_2$ are unitarily equivalent.

A linear functional φ on a *-algebra $\mathcal{A}$ is defined to be **positive** if $\varphi(a^*a) \geq 0$ for $a \in \mathcal{A}$. A positive linear functional φ on a unital *-algebra $\mathcal{A}$ is called a **state** if $\varphi(1_{\mathcal{A}}) = 1$ ($1_{\mathcal{A}}$ being the unit element of $\mathcal{A}$). A linear functional τ on an algebra $\mathcal{A}$ is called a **trace** or said to be **tracial** if $\tau(ab) = \tau(ba)$ for $a, b \in \mathcal{A}$.

Example 2.6. Let $\mathcal{C}_0(\mathcal{H})$ be the set of finite rank operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{C}_0(\mathcal{H})$ is a *-ideal of $\mathcal{B}(\mathcal{H})$ and the ordinary trace defines a positive tracial functional tr on $\mathcal{C}_0(\mathcal{H})$.

Example 2.7. Every probability measure μ on the real line of finite moments defines a state on the polynomial algebra $\mathbb{C}[X]$ by

$$\varphi(\sum_n a_n X^n) = \sum_n a_n \int_{\mathbb{R}} t^n \mu(dt).$$

Conversely, any state arises in this way (the existence part of the Hamburger moment problem). See §X.1 in Reed-Simon for more information.

Example 2.8. In the group algebra $\mathbb{C}G$, positive linear functionals φ are one-to-one correspondence with positive definite functions on G by restriction and linear extension. The state associated to the positive definition function

$$\delta(g) = \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{otherwise} \end{cases}$$

is called the standard trace.

Exercise 5. The standard trace δ has the trace property: $\delta(ab) = \delta(ba)$ for $a, b \in \mathbb{C}G$.

Given a positive linear functional φ on a *-algebra $\mathcal{A}$, we define a *-representation as follows: The inner product $(a|b) = \varphi(a^*b)$ on $\mathcal{A}$ is positive semidefinite and the representation space is given by the associated pre-Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}$ is the quotient vector space relative to the kernel of $(\ |\ )$. The non-degenerate inner product on the quotient space is also denoted by $(\ |\ )$, whereas the quotient vector of $x \in \mathcal{A}$ in $\mathcal{H}$ is denoted by $x\varphi^{1/2}$. The inner product then looks like $(x\varphi^{1/2}|y\varphi^{1/2}) = \varphi(x^*y)$ and we introduce a representation π by $\pi(a)(x\varphi^{1/2}) = (ax)\varphi^{1/2}$.

Exercise 6. Check that the representation π is well-defined.

The representation obtained in this way is referred to as the **GNS-representation**[6] or its process as the GNS-construction. When $\mathcal{A}$ is unital, we have a distinguished vector $\varphi^{1/2} = 1_{\mathcal{A}}\varphi^{1/2}$ in the representation space, which is **cyclic** with respect to π in the sense that $\mathcal{H} = \pi(\mathcal{A})\varphi^{1/2}$.

Conversely, if we are given a *-representation $(\pi, \mathcal{H})$ of a *-algebra $\mathcal{A}$ and a cyclic vector $\xi \in \mathcal{H}$ for π, the formula $\varphi(a) = (\xi|\pi(a)\xi)$ defines a positive linear functional and the associated GNS-representation is unitarily equivalent to the initial one by the unitary map $a\varphi^{1/2} \mapsto \pi(a)\xi$ ($a \in \mathcal{A}$).

A positive functional φ is said to be **bounded** if the associated GNS-representation is bounded.

Exercise 7. A positive functional is bounded if and oly if, given $a \in \mathcal{A}$, we can find $M > 0$ such that $\varphi(x^*a^*ax) \leq \varphi(x^*x)$ for any $x \in \mathcal{A}$.

Exercise 8. Formulate the GNS-construction for right $\mathcal{A}$-modules.

Example 2.9. The GNS-representation associated to the state on $\mathbb{C}[X]$ realized by a probability measure μ on $\mathbb{R}$ is identified with the multiplication operator by polynomial functions on the Hilbert space $L^2(\mathbb{R}, \mu)$.

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[6]Named after I.M. Gelfand, M.A. Naimark and I.E. Segal.

Example 2.10. Given a positive trace τ on a *-algebra $\mathcal{A}$, the associated GNS-representation space $\mathcal{A}\tau^{1/2}$ is made into a *-bimodule by $(a\tau^{1/2})^* = a^*\tau^{1/2}$ $(a \in \mathcal{A})$.

Example 2.11. The GNS-representation of the standard trace of a group algebra $\mathbb{C}G$ is identified with the regular representation of G:

$$(a\delta^{1/2}|b\delta^{1/2}) = \delta(a^*b) = \sum_{g\in G} \overline{a_g} b_g \quad \text{for } a = \sum_{g\in G} a_g g,\ b = \sum_{g\in G} b_g g.$$

By the trace property of δ, the representation space $\ell^2(G)$ is a *-bimodule of $\mathbb{C}G$.

When G is commutative, $\ell^2(G)$ is unitarily mapped onto $L^2(\widehat{G})$ ($\widehat{G}$ being the Pontryagin dual of G) with the representation of $\mathbb{C}G$ unitarily transformed into the multiplication operator on $L^2(\widehat{G})$ given by the function

$$\widehat{G} \ni \omega \mapsto \sum_{g\in G} a_g \langle g, \omega\rangle \quad \text{for } a = \sum_{g\in G} a_g g \in \mathbb{C}G.$$

Exercise 9. For $G = \mathbb{Z}$, identify $\widehat{\mathbb{Z}}$ with $\mathbb{T}$ and the unitary map $\ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ with the Fourier expansion.

Definition 2.12. Given a vector η in a Hilbert space $\mathcal{H}$, the linear functional $\eta^* : \mathcal{H} \to \mathbb{C}$ is defined by $\eta^*(\xi) = (\eta|\xi)$ for $\xi \in \mathcal{H}$. By Riesz lemma, the dual space $\mathcal{H}^*$ of $\mathcal{H}$ is of the form $\mathcal{H}^* = \{\eta^*; \eta \in \mathcal{H}\}$ and it is a Hilbert space by the inner product $(\xi^*|\eta^*) = (\eta|\xi)$. The *-algebra $\mathcal{B}(\mathcal{H})$ then naturally acts on $\mathcal{H}^*$ from the right by $\eta^* a = (a^*\eta)^*$. For $\xi, \eta \in \mathcal{H}$, define a rank one operator $\xi\eta^* \in \mathcal{C}_0(\mathcal{H})$ by[7]

$$(\xi\eta^*)\zeta = (\eta|\zeta)\xi, \quad \zeta \in \mathcal{H}.$$

The notation is compatible with the multiplications by elements in $\mathcal{B}(\mathcal{H})$: $a(\xi\eta^*)b = (a\xi)(\eta^* b)$.

Example 2.13. Let tr be the ordinary trace on the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$. Then the correspondence $\xi\eta^*\mathrm{tr}^{1/2} \mapsto \xi \otimes \eta^*$ gives rise to a unitary map from the GNS-representation space $\overline{\mathcal{C}_0(\mathcal{H})\mathrm{tr}^{1/2}}$ onto $\mathcal{H} \otimes \mathcal{H}^*$.

On a *-algebra $\mathcal{A}$, we introduce a seminorm $\|\cdot\|_{C^*}$ by

$$\|a\|_{C^*} = \sup\{\|\pi(a)\|; \pi \text{ is a bounded *-representation}\},$$

which satisfies

$$\|ab\|_{C^*} \le \|a\|_{C^*}\, \|b\|_{C^*}, \quad \|a^*a\|_{C^*} = \|a\|_{C^*}^2.$$

[7]According to Dirac, $\xi\eta^*$ is often denoted by $|\xi)(\eta|$.

The completion of the quotient *-algebra $\mathcal{A}/\mathcal{I}$ relative to $\|\cdot\|_{C^*}$ ($\mathcal{I} = \{a \in \mathcal{A}; \|a\|_{C^*}\}$) is a C*-algebra, which is universal in the sense that any bounded *-representation of $\mathcal{A}$ splits through A in a unique way. Thus, instead of bounded *-representations of $\mathcal{A}$, we can work with *-representations of A.

Exercise 10. Check the following: $\|a^*\|_{C^*} = \|a\|_{C^*}$ for $a \in \mathcal{A}$ and $\{a \in \mathcal{A}; \|a\|_{C^*} = 0\}$ is a *-ideal of $\mathcal{A}$.

Example 2.14. The closure of the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$ in the operator topology on $\mathcal{B}(\mathcal{H})$ is a C*-algebra as a norm-closed *-ideal of $\mathcal{B}(\mathcal{H})$, which is referred to as the **compact operator** algebra and denoted by $\mathcal{C}(\mathcal{H})$.

The norm $\|a\|_2 = \|a\mathrm{tr}^{1/2}\| = \sqrt{\mathrm{tr}(a^*a)}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **Hilbert-Schmidt norm** and satisfies

$$\|ab\|_2 \leq \|a\|\|b\|_2, \quad \|b^*\|_2 = \|b\|_2 \geq \|b\|, \quad a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H}).$$

Thus the completion $\mathcal{C}_2(\mathcal{H})$ of $\mathcal{C}_0(\mathcal{H})$ relative to the Hilbert-Schmidt norm, which is included in $\mathcal{C}(\mathcal{H})$ as a *-ideal of $\mathcal{B}(\mathcal{H})$ and is, at the same time, isomorphic to $\mathcal{H} \otimes \mathcal{H}^*$. In other words, $\mathcal{C}_2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$ is a *-bimodule of $\mathcal{B}(\mathcal{H})$.

The norm $\|a\|_1 = \sup\{|\mathrm{tr}(ab)|; b \in \mathcal{C}_0(\mathcal{H}), \|b\| \leq 1\}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **trace norm** and satisfies

$$\|ab\|_1 \leq \|a\|\|b\|_1, \quad \|b^*\|_1 = \|b\|_1 \geq \|b\|_2, \quad |\mathrm{tr}(b)| \leq \|b\|_1$$

for $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H})$. Thus the completion of $\mathcal{C}_0(\mathcal{H})$ relative to the trace norm, which is included in $\mathcal{C}_2(\mathcal{H})$ and denoted by $\mathcal{C}_1(\mathcal{H})$, is a Banach *-algebra and realized as a *-ideal of $\mathcal{B}(\mathcal{H})$ with the trace functional extended to $\mathcal{C}_1(\mathcal{H})$ by continuity.

$$\mathcal{C}_0(\mathcal{H}) \subset \mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}_2(\mathcal{H}) \subset \mathcal{C}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}).$$

Exercise 11. Check the inequalities for the Hilbert-Schmidt and the trace norms.

Exercise 12. Show that, for a positive operator $a \in \mathcal{B}(\mathcal{H})$,

$$\mathrm{tr}(a) = \sum_j (\xi_j | a\xi_j)$$

does not depend on the choice of an orthonormal basis $\{\xi_j\}$ in $\mathcal{H}$.

Exercise 13. Show that $a \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{C}_1(\mathcal{H})$ if and only if $\mathrm{tr}(|a|) < \infty$. Here $|a| = \sqrt{a^*a}$. If this is the case, $\|a\|_1 = \mathrm{tr}(|a|)$.

Exercise 14. Show that $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_2(\mathcal{H})\mathcal{C}_2(\mathcal{H})$ and deduce the inequality $\|ab\|_1 \leq \|a\|_2\|b\|_2$ from $|\mathrm{tr}(ab)| \leq \|a\|_2\|b\|_2$ (the Cauchy-Schwarz inequality).

Proposition 2.15. Given a bounded positive linear functional φ on $\mathcal{C}(\mathcal{H})$, we can find a positive operator $\rho \in \mathcal{C}_1(\mathcal{H})$ such that $\varphi(x) = \mathrm{tr}(\rho x)$ for $x \in \mathcal{C}(\mathcal{H})$. Moreover, $\rho^{1/2} \in \mathcal{C}_2(\mathcal{H})$ is identified with $\varphi^{1/2}$ by the left multiplication of $\mathcal{C}(\mathcal{H})$.

Proof. Define a positive sesquilinear form Φ on $\mathcal{H}$ by $\Phi(\xi, \eta) = \varphi(\eta\xi^*)$. Then, from $\|\xi\xi^*\| = \|\xi\|^2$, we have $\Phi(\xi, \xi) \leq \|\varphi\|(\xi|\xi)$ and therefore a positive operator ρ satisfying $\varphi(\eta\xi^*) = (\xi|\rho\eta)$ for $\xi, \eta \in \mathcal{H}$. If $\{\xi_j\}$ is an orthonormal basis, $\sum_{j=1}^n \xi_j\xi_j^*$ is a projection in $\mathcal{C}_0(\mathcal{H})$ and

$$\sum_{j=1}^n (\xi_j|\rho\xi_j) = \varphi(\sum_{j=1}^n \xi_j\xi_j^*) \leq \|\varphi\|$$

shows that $\mathrm{tr}(\rho) = \sum_{j=1}^\infty (\xi_j|\rho\xi_j)$ is finite, i.e., ρ is in the trace class. Now $\varphi(\eta\xi^*) = \mathrm{tr}(\rho(\eta\xi^*))$ is extended to $x \in \mathcal{C}(\mathcal{H})$ by linearity and then by continuity. □

Exercise 15. Through the identification $\mathcal{C}_2(\mathcal{H}) = \mathcal{H}\otimes\mathcal{H}^*$, $\overline{\mathcal{C}(\mathcal{H})\rho^{1/2}} = \mathcal{H} \otimes \mathcal{H}^*[\rho]$, where $[\rho]$ denotes the support projection of ρ.

A bounded *-representation ${}_{\mathcal{A}}\mathcal{H}$ is said to be **irreducible** if $\mathrm{End}({}_{\mathcal{A}}\mathcal{H}) = \mathbb{C}1_{\mathcal{H}}$. A positive functional is said to be **pure** if the associated GNS-representation is irreducible. A family $\{{}_{\mathcal{A}}\mathcal{H}_j\}$ of bounded *-representations is said to be **disjoint** if $\mathrm{Hom}({}_{\mathcal{A}}\mathcal{H}_j, {}_{\mathcal{A}}\mathcal{H}_k) = \{0\}$ for $j \neq k$. Two bounded positive functionals φ and ψ of A are said to be **disjoint** (resp. **quasi-equivalent**) if the associated GNS representations are disjoint (resp. quasi-equivalent).

Lemma 2.16. Let ω be a positive functional on a unitary algebra $\mathcal{A}$ with $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ the associated GNS-representation. Then the following formula gives a one-to-one correspondence between positive functionls ω_T on $\mathcal{A}$ majorized by ω and positive operators T in the commutant $\pi(\mathcal{A})' = \{T \in \mathcal{B}(\mathcal{H}); T\pi(a) = \pi(a)T,\ \forall a \in \mathcal{A}\}$ majorized by the identity operator 1_H.

$$\omega_T(a) = (T\omega^{1/2}|\pi(a)\omega^{1/2}), \quad a \in \mathcal{A}.$$

Proof. Let φ be majorized by ω, i.e., $\varphi(a^*a) \leq \omega(a^*a)$ for $a \in \mathcal{A}$. Then by Schwarz inequality

$$|\varphi(x^*y)| \leq \varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} \leq \omega(x^*x)^{1/2}\omega(y^*y)^{1/2} = \|x\omega^{1/2}\|\,\|y\omega^{1/2}\|,$$

we see that $x\omega^{1/2} \times y\omega^{1/2} \mapsto \varphi(x^*y)$ gives a bounded sesquilinear form on the completed Hilbert space $\mathcal{H}$, whence we can find a bounded linear operator T on $\mathcal{H}$ satisfying

$$\varphi(x^*y) = (x\omega^{1/2}|T(y\omega^{1/2})) \quad x, y \in \mathcal{A}.$$

By equating $\varphi(x^*(ay))$ and $\varphi((a^*x)^*y)$, we have $T \in \pi(\mathcal{A})'$. Furthermore, the condition $0 \leq \varphi(a^*a) \leq \omega(a^*a)$ means the operator inequality $0 \leq T \leq 1_{\mathcal{H}}$.

The converse implication is immediate and the proof is left to the reader. □

Theorem 2.17. Let $\mathcal{A}$ be a *-algebra.

(i) A bounded positive functional φ on $\mathcal{A}$ is pure if and only if any positive functional ψ satisfying $\psi \leq \varphi$ is proportional to φ.
(ii) A bounded *-representation ${}_{\mathcal{A}}\mathcal{H}$ is irreducible if and only if $\overline{\mathcal{A}\xi} = \mathcal{H}$ for any $0 \neq \xi \in \mathcal{H}$.
(iii) Two bounded *-representations ${}_{\mathcal{A}}\mathcal{H}$ and ${}_{\mathcal{A}}\mathcal{K}$ are not disjoint if and only if we can find non-zero subrepresentations ${}_{\mathcal{A}}\mathcal{H}' \subset {}_{\mathcal{A}}\mathcal{H}$ and ${}_{\mathcal{A}}\mathcal{K}' \subset {}_{\mathcal{A}}\mathcal{K}$ such that ${}_{\mathcal{A}}\mathcal{H}'$ and ${}_{\mathcal{A}}\mathcal{K}'$ are unitarily equivalent.

Corollary 2.18. The set of pure states of a unital *-algebra $\mathcal{A}$ is invariant under *-automorphisms of $\mathcal{A}$.

Exercise 16. Prove the theorem.

2.1. **Von Neumann's Reduction Theory.** In the study of group structures, one of key strategies is to focus on commutative subgroups such as $\mathbb{Z}_n$, $\mathbb{Z}$ and $\mathbb{T}$, where Fourier analysis plays significant roles (Pontryagin duality, 1934).

Theorem 2.19 (Gelfand, [15, Theorem 2.22])**.** Any commutative C*-algebra C is naturally isomorphic to $C_0(\Omega)$ (the C*-algebra of continuous functions vanishing at infinity), where a locally compact space Ω is captured as the Gelfand spectrum $\sigma_C = \{\omega : C \to \mathbb{C}; \chi$ is a one-dimensional *-representation$\}$ of C.

Example 2.20. Let V be a finite-dimensional real vector space and let $C_c(V)$ be the vector space of $\mathbb{C}$-valued continuous functions of compact support, which is a *-algebra by the convolution product

$$(f \star g)(v) = \int_V dv' f(v')g(v - v'), \quad f^*(v) = \overline{f(-v)}.$$

Here dv' is a preassigned Lebesgue measure on V.

Proposition 2.21. There exists a one-to-one correspondence between a continuous unitary representation U of the vector group V and a bounded *-representation π of $C_c(V)$.

$$\pi(f) = \int_V f(v)U(v)\, dv, \quad U(v)(\pi(f)\xi) = \pi(v.f)\xi.$$

Here $(v.f)(v') = f(v' - v)$.

Exercise 17. Prove the proposition.

Example 2.22. Let C be the commutative C*-algebra associated to $C_c(V)$. Then $\sigma_C = V^*$ by

$$\chi : \int_V f(v)e^{iv}\,dv \mapsto \int_V f(v)e^{i\omega(v)}\,dv = \widehat{f}(\omega).$$

and $C_c(V) \ni f \mapsto \widehat{f} \in C_0(V^*)$ is extended to an isomorphism $C \cong C_0(V^*)$. Note that $\lim_{\omega\to\infty} \widehat{f}(\omega) = 0$ by Riemann-Lebesgue lemma.

Consider a *-representation $\imath$ of a C*-algebra A on a Hilbert space $\mathcal{H}$. Let $C \subset A$ be a central C*-subalgebra and express $C = C(\Omega)$ with Ω a compact Hausdorff space.

Theorem 2.23 (Riesz-Radon-Banach-Markov-Kakutani)**.** There is a one-to-one correspondence between states, say ϕ, on C and probability measures (Radon measures), say μ, on Ω.

$$\phi(f) = \int_\Omega f(\omega)\mu(d\omega).$$

In what follows, ϕ is identified with the associated Radon measure.

From here on, $\mathcal{H}$ is assumed to be separable. Write $\mathcal{H} = \bigoplus_{j=1}^\infty \overline{\pi(C)\xi_j}$ and set

$$\phi = \sum_{j=1}^\infty \frac{1}{2^j}\phi_j, \quad \phi_j(a) = (\xi_j|a\xi_j).$$

Then $\pi(C)''$ is *-isomorphic to $L^\infty(\Omega,\phi)$ on $L^2(\Omega,\phi)$ and $\mathcal{H}$ is identified with a closed subspace of $L^2(\Omega,\phi)\otimes\mathcal{K}$ ($\mathcal{K}$ being some Hilbert space). Thus,

$$\mathcal{H} \cong \int_\Omega^\oplus \mathcal{H}_\omega\,\phi(d\omega), \quad \xi \longleftrightarrow \int_\Omega^\oplus \xi_\omega\,\phi(d\omega), \quad \mathcal{H}_\omega \subset \mathcal{K}$$

so that

$$\mathrm{End}({}_C\mathcal{H}) \cong \int_\Omega^\oplus \mathcal{B}(\mathcal{H}_\omega)\,\phi(d\omega), \quad \pi(a) \longleftrightarrow \int_\Omega^\oplus \pi_\omega(a)\,\phi(d\omega).$$

Let ${}_A\mathcal{H}'$ be another *-representation of A ($\mathcal{H}'$ being separable) and choose a measure ϕ' so that $\mathcal{H}' \cong \int_\Omega^\oplus \mathcal{H}'_\omega\,\phi'(d\omega)$. Then

$$\mathrm{Hom}({}_C\mathcal{H}, {}_C\mathcal{H}') \cong \int_\Omega^\oplus \mathcal{B}(\mathcal{H}_\omega,\mathcal{H}'_\omega)\,\sqrt{\phi\phi'}(d\omega).$$

Recall that $\sqrt{\phi\phi'}$ is a measure on Ω defined by

$$\sqrt{\phi\phi'}(d\omega) = \sqrt{\frac{d\phi}{d\mu}(\omega)\frac{d\phi'}{d\mu}(\omega)}\,\mu(d\omega).$$

Note that $\sqrt{\phi\phi'}$ can be replaced with any measure equivalent to it.

When $\pi(A)' = \pi(C)''$ and $\pi'(A)' = \pi'(C)''$, we have

$$\mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{H}') \cong L^\infty(\Omega, \sqrt{\phi\phi'}).$$

3. CCR-algebras and Analytic Representations

A **presymplectic vector space** is a pair (V, σ) of a real vector space V and an alternating form $\sigma : V \times V \to \mathbb{R}$. When σ is non-degenerate, it is called a **symplectic vector space**.

Exercise 18. A real vector space V is equivalently described by a complex vector space $V^{\mathbb{C}}$ with conjugation $(x+iy)^* = x-iy$ $(x, y \in V)$. There is a one-to-one correspondence between presymplectic forms σ on V and hermitian forms h on $V^{\mathbb{C}}$ satisfying $\overline{h} = -h$ by the relation $h(z, w) = i\sigma(z^*, w)$ $(z, w \in V^{\mathbb{C}})$ (given a sesquilinear form s on $V^{\mathbb{C}}$, we set $\overline{s}(z, w) = \overline{s(z^*, w^*)}$), where σ is bilinearly extnded to $V^{\mathbb{C}}$.

Example 3.1. Let L and V_0 be real vector spaces with L^* the algebraic dual space of L. Then the direct sum $V = V_0 \oplus L \oplus L^*$ is a presymplectic vector space with the presymplectic form defined by

$$\sigma(a \oplus x \oplus \xi, b \oplus y \oplus \eta) = \langle x, \eta\rangle - \langle y, \xi\rangle.$$

Note that $\ker \sigma = V_0 \oplus 0 \oplus 0 \cong V_0$.

Exercise 19. If $\dim V < \infty$, any presymplectic vector space is of this form.

Given presymplectic vector spaces (V, σ) and (V', σ'), a linear map $\phi : V \to V'$ is said to be **presymplectic** if $\sigma'(\phi(x), \phi(y)) = \sigma(x, y)$ for $x, y \in V$. When $(V', \sigma') = (V, \sigma)$ and ϕ is an isomorphism, it is called a presymplectic automorphism of (V, σ). The group of presymplectic automorphisms of (V, σ) is denoted by $\mathrm{Aut}(V, \sigma)$.

If V is endowed with a linear topology which makes σ continuous, it is reasonable to restrict ourselves to continuous presymplectic maps.

Associated to a presymplectic vector space (V, σ), we introduce several *-algebras, called **CCR-algebras**[8]. The first one, denoted by $\mathcal{A}(V, \sigma)$, is a unital *-algebra which is linearly and universally generated by elements in V subject to the relations

$$x^* = x, \quad xy - yx = i\sigma(x, y)1, \quad \text{for } x, y \in V.$$

Lemma 3.2. Given a real-linear map π of V into a unital algebra A satisfying $\pi(x)\pi(y) - \pi(y)\pi(x) = i\sigma(x, y)1_A$ for $x, y \in V$, π is extended to an algebra-homomorphism of $\mathcal{A}(V, \sigma)$ into A.

[8] CCR stands for the Canonical Commutation Relations.

Proof. This is a consequence of the fact that the commutation relations are invariant under the *-operation. □

Example 3.3. Let K be a complex Hilbert space and set $V^{\mathbb{C}} = K \oplus \overline{K}$, where $\overline{K}$ is the conjugate Hilbert space and the real structure (or the conjugation) in $V^{\mathbb{C}}$ is defined by $(\xi \oplus \overline{\eta})^* = \eta \oplus \overline{\xi}$. Thus the real subspace V is $\{\xi \oplus \overline{\xi}; \xi \in K\}$, which can be identified with K as a real Hilbert space by the isometry $K \ni \xi \mapsto (\xi \oplus \overline{\xi})/\sqrt{2} \in V$.

On the vector space V, we define a symplectic form σ by

$$\sigma(\xi \oplus \overline{\xi}, \eta \oplus \overline{\eta}) = 2\mathrm{Im}(\xi|\eta).$$

If σ is extended to $V^{\mathbb{C}}$ bilinearly, then

$$\sigma(\xi \oplus 0, \eta \oplus 0) = 0 = \sigma(0 \oplus \overline{\xi}, 0 \oplus \overline{\eta}), \quad \sigma(\xi \oplus 0, 0 \oplus \overline{\eta}) = i(\eta|\xi)$$

for $\xi, \eta \in K$ and the associated hermitian form is described by

$$i\sigma((\xi \oplus \overline{\eta})^*, \xi' \oplus \overline{\eta'}) = (\xi|\xi') - (\eta'|\eta) = \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi' \\ \overline{\eta'} \end{pmatrix}.$$

With the notation $a(\xi) = 0 \oplus \overline{\xi}$ and $a^*(\xi) = a(\xi)^* = \xi \oplus 0$ for generators in the CCR algebra $\mathcal{A}(V, \sigma)$, we can express the commutation relations in the following form:

$$[a(\xi), a(\eta)] = 0 = [a^*(\xi), a^*(\eta)], \quad [a(\xi), a^*(\eta)] = (\xi|\eta)1.$$

We call this the creation-annihilation form of the canonical commutation relations.

Exercise 20. In the above exmaple, a continuous symplectic automorphism is of the form

$$\begin{pmatrix} A & \overline{C} \\ C & \overline{A} \end{pmatrix},$$

where $A : K \to K$ and $C : K \to \overline{K}$ are bounded operators satisfying $A^*A - C^*C = 1_H$ and $A^*\overline{C} = C^*\overline{A}$.

The second one, called the Weyl form of CCR algebra, is a unitary algebra $\mathcal{C}(V, \sigma)$ universally generated by the symbols $\{e^{ix}; x \in V\}$ subject to the relations

$$(e^{ix})^* = e^{-ix}, \quad e^{ix}e^{iy} = e^{-i\sigma(x,y)/2}e^{i(x+y)}, \quad x, y \in V,$$

which are the exponentiated form of the canonical commutation relations. Note that e^{i0} (the zero in the exponential represents the zero vector in V) is the unit element in the algebra.

Since $\mathcal{C}(V, \sigma)$ is generated by unitaries $\{e^{ix}\}$, any *-representation is automatically bounded.

A *-representation $\pi : \mathcal{C}(V,\sigma) \to \mathcal{B}(\mathcal{H})$ is said to be **continuous** if for any $\xi, \eta \in \mathcal{H}$ and for any finite-dimensional subspace $W \subset V$, $W \ni x \mapsto (\xi|\pi(e^{ix})\eta)$ is continuous.

A positive functional on $\mathcal{C}(V,\sigma)$ is defined to be continuous if the associated GNS-representation is continuous.

The operator-norm completion with repsect to all *-representations is then a C*-algebra $C(V,\sigma)$, which is referred to as the **CCR C*-algebra**. From the very definition, there is a one-to-one correspondance between *-representations of $\mathcal{C}(V,\sigma)$ on a Hilbert space $\mathcal{H}$ and *-representations of $C(V,\sigma)$ on $\mathcal{H}$. There is also a one-to-one correspondance between states on $\mathcal{C}(V,\sigma)$ and states on $C(V,\sigma)$.

Lemma 3.4. A *-representation π of $C(V,\sigma)$ is continuous if and only if $\mathbb{R} \ni t \mapsto \pi(e^{itx})$ is weakly continuous for any $x \in V$.

Proof. Use the realtion

$$e^{i(t_1x_1+\cdots+t_nx_n)} = e^{i\sum_{j<k} t_jt_k\sigma(x_j,x_k)/2} e^{it_1x_1} \cdots e^{it_nx_n}$$

for $x_1,\dots,x_n \in V$ and $(t_1,\dots,t_n) \in\in \mathbb{R}^n$. □

Exercise 21. Use the algebraic representation

$$(\pi(e^{ix})f)(y) = e^{-i\sigma(x,y)/2} f(y-x), \quad x,y \in V, f \in F(V)$$

of $\mathcal{C}(V,\sigma)$ and its differential, where $F(V)$ is the vector space of complex-valued (finite-dimensionally) differentiable functions on the set V, to show that $V \to \mathcal{A}(V,\sigma)$ is injective.

If we are given a presymplectic map $\phi : V \to V'$, it induces a *-homomorphism $C(V,\sigma) \to C(V',\sigma')$ by universality and similarly for other CCR algebras. In particular, $\mathrm{Aut}(V,\sigma)$ acts on $C(V,\sigma)$ as a *-automorphism group.

Lemma 3.5. If $V = V_1 \oplus V_2$ and $\sigma = \sigma_1 \oplus \sigma_2$, then $\mathcal{C}(V,\sigma) = \mathcal{C}(V_1,\sigma_1) \otimes \mathcal{C}(V_2,\sigma_2)$ and similarly for other CCR algebras.

In particular, if $V' \subset V$ is a complementary subspace of $\ker\sigma$ with σ' the restriction of σ to V', then $\mathcal{C}(V,\sigma) = \mathcal{C}(\ker\sigma, 0) \otimes \mathcal{C}(V',\sigma')$.

Let $f : V \to \mathbb{R}$ be a linear functional. Then $e^{ix} \mapsto e^{if(x)}e^{ix}$ gives a *-automorphism of $C(V,\sigma)$, which is referred to as a **gauge automorphism**.

Proposition 3.6. Given a continuous positive functional $\varphi : C(V,\sigma) \to \mathbb{C}$, its characteristic function $\widehat{\varphi}(x) = \varphi(e^{ix})$ $(x \in V)$ is characterized by the (finite-dimensional) continuity and the positivity condition that

$$\sum_{1\le j,k\le n} \overline{z_j} z_k \widehat{\varphi}(x_k - x_j) e^{i\sigma(x_j,x_k)/2} \ge 0$$

for any finite sequences $\{x_j\}_{1\leq j\leq n}$ in V and $\{z_j\}_{1\leq j\leq n}$ in $\mathbb{C}$.

Exercise 22. Check this tautological claim.

Let $(\pi, \mathcal{H})$ be a continuous *-representation of $C(V,\sigma)$ with $\mathcal{H}$ a Hilbert space. A vector $\xi \in \mathcal{H}$ is said to be **entirely analytic** if $W \ni x \mapsto \pi(e^{ix})\xi \in \mathcal{H}$ is analytically continued to $W^{\mathbb{C}}$ for any finite-dimensional subspace W of V.

Let $\mathcal{D}$ be the set of entirely analytic vectors in $\mathcal{H}$, which is a subspace of $\mathcal{H}$. For $\xi \in \mathcal{D}$ and $v \in V^{\mathbb{C}}$, the vector $\pi(e^v)\xi$ is well-defined as an analytic continuation and it belongs to $\mathcal{D}$ and satisfies

$$\pi(e^v)(\pi(e^w)\xi) = e^{i\sigma(v,w)/2}\pi(e^{v+w})\xi, \quad \xi \in \mathcal{D},\ v,w \in V^{\mathbb{C}}$$

in view of the analytic extension of the identity $\pi(e^{ix})\pi(e^{iy})\xi = e^{-i\sigma(x,y)/2}\pi(e^{i(x+y)}\xi$ for $x, y \in V$.

Thus,if we set $\pi(v)\xi = \frac{d}{dt}\pi(e^{tv})\xi|_{t=0}$, it again belongs to $\mathcal{D}$. Define linear operators $\pi_{\mathcal{D}}(e^v)$ and $\pi_{\mathcal{D}}(v)$ on $\mathcal{D}$ by $\pi_{\mathcal{D}}(e^v)\xi = \pi(e^v)\xi$ and $\pi_{\mathcal{D}}(v)\xi = \pi(v)\xi$.

Lemma 3.7. These operators belong to $\mathcal{L}(\mathcal{D})$ with $\pi_{\mathcal{D}}(e^v)^* = \pi_{\mathcal{D}}(e^{v^*})$, $\pi_{\mathcal{D}}(v)^* = \pi_{\mathcal{D}}(v^*)$ and

$$\pi_{\mathcal{D}}(e^v)\pi_{\mathcal{D}}(e^w) = e^{i\sigma(v,w)/2}\pi_{\mathcal{D}}(e^{v+w}), \quad \pi_{\mathcal{D}}(v)\pi_{\mathcal{D}}(w) - \pi_{\mathcal{D}}(w)\pi_{\mathcal{D}}(v) = i\sigma(v,w)1.$$

Thus $V \ni v \mapsto \pi_{\mathcal{D}}(v) \in \mathcal{L}(\mathcal{D})$ is extended to a *-representation of $\mathcal{A}(V,\sigma)$ on $\mathcal{D}$.

Exercise 23. Compute $\pi_{\mathcal{D}}(e^v)\pi_{\mathcal{D}}(w)\pi_{\mathcal{D}}(e^{-v})$ for $v, w \in V^{\mathbb{C}}$.

4. Covariance Forms

Assume that we are given a state φ of a CCR algebra $\mathcal{A}(V,\sigma)$. Let S be a positive form on $V^{\mathbb{C}}$ defined by $S(x,y) = \varphi(x^*y)$ for $x, y \in V^{\mathbb{C}}$.

Evaluating the commutation relation $x^*y - yx^* = i\sigma(x^*, y)1$ by the functional φ, we have

$$S(x,y) - S(y^*, x^*) = i\sigma(x^*, y) \quad \text{for } x, y \in V^{\mathbb{C}}. \tag{1}$$

Conversely, any positive form S on $V^{\mathbb{C}}$ induces a presymplectic form σ by the formula $i\sigma(x,y) = S(x,y) - S(y,x)$ $(x, y \in V)$.

Definition 4.1. A positive form S on $V^{\mathbb{C}}$ is called a **covariance form** on a presymplectic vector space (V,σ) if it satisfies the equation (1). Let $\mathrm{Cov}(V,\sigma)$ be the set of covariance froms on (V,σ), which is a convex set with an obvious action of $\mathrm{Aut}(V,\sigma)$.

If a presymplectic vector space (V, σ) admits one covariance form S, then there exists plenty of them by adding any non-degenerate inner product on V to S.

Remark 1. If a symplectic vector space (V, σ) has a countable basis, then we can find a canonical basis $\{e_n, f_n\}_{n\geq 1}$ satisfying

$$\sigma(e_k, e_l) = 0 = \sigma(f_k, f_l), \quad \sigma(e_k, f_l) = \delta_{k,l},$$

whence it admits a covariance form.

Related to the existence of covariance forms, the following question seems to be open even for hermitian matrices: Given a hermitian form θ on a complex vector space K, can we find a positive form $(\ |\ )$ satisfying $|\theta(x, y)|^2 \leq (x|x)\,(y|y)$ for $x, y \in K$?

Example 4.2. If $\sigma \equiv 0$, $\mathrm{Cov}(V, \sigma)$ is identified with the set of positive (semidefinite) bilinear forms on V.

Example 4.3. For $V = \mathbb{R}^2$, possible presymplectic forms are parametrized up to choices of bases by the matrix

$$\begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

Then $S \in \mathrm{Cov}(V, \sigma)$ is described by a matrix of the form

$$\begin{pmatrix} z + x & y + i\mu \\ y - i\mu & z - x \end{pmatrix}, \quad x^2 + y^2 + \mu^2 \leq z^2,\ z \geq 0,$$

whence $\mathrm{Cov}(V, \sigma)$ is identified with the region bounded by a half of a two-sheeted hyperboloid ($\mu \neq 0$) or by a cone ($\mu = 0$). Since

$$\mathrm{Aut}(V, \sigma) = \begin{cases} \mathrm{GL}(2, \mathbb{R}) & \text{if } \mu = 0, \\ \mathrm{SL}(2, \mathbb{R}) & \text{otherwise}, \end{cases}$$

orbits in $\mathrm{Cov}(V, \sigma)$ constitute two or three parts according to $\mu \neq 0$ or $\mu = 0$.

Example 4.4. Consider the symplectic vector space $V^{\mathbb{C}} = K \oplus \overline{K}$ in Example 3.3. Let $S : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ be a sesquilinear form. Then S satisfies the equation (1) if and only if

$$\begin{aligned} S(\xi \oplus \overline{\eta}, \xi \oplus \overline{\eta}) &= \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix}^* \begin{pmatrix} 1 + \overline{D} & B \\ B^* & D \end{pmatrix} \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix} \\ &= (\xi|(1 + A)\xi) + (\eta|A\eta) + (\xi|B\overline{\eta}) + (\overline{\eta}|B^*\xi), \end{aligned}$$

where $D : \overline{K} \to \overline{K}$ and $B : \overline{K} \to K$ are bounded maps satisfying $\overline{B} = B^*$. Remark that

$$(\overline{\xi}|D\overline{\eta}) = S(0 \oplus \overline{\xi}, 0 \oplus \overline{\eta}), \qquad (\xi|B\overline{\eta}) = S(\xi \oplus 0, 0 \oplus \overline{\eta}).$$

Thus bounded operators $D : \overline{K} \to \overline{K}$ and $B : \overline{K} \to K$ with $\overline{B} = B^*$ correspond to a covariance form if and only if the operator of matrix form

$$\begin{pmatrix} 1+\overline{D} & B \\ B^* & D \end{pmatrix}$$

is positive. In particular, the obvious choice $D = B = 0$ gives a covariance form.

Note that a right action of $\begin{pmatrix} A & \overline{C} \\ C & \overline{A} \end{pmatrix} \in \mathrm{Aut}(V,\sigma)$ on $\begin{pmatrix} 1+\overline{D} & B \\ B^* & D \end{pmatrix} \in \mathrm{Cov}(V,\sigma)$ is given by

$$\begin{pmatrix} A & \overline{C} \\ C & \overline{A} \end{pmatrix}^* \begin{pmatrix} 1+\overline{D} & B \\ B^* & D \end{pmatrix} \begin{pmatrix} A & \overline{C} \\ C & \overline{A} \end{pmatrix}.$$

For the choice $K = \mathbb{C}$, let $e = (1 \oplus 1)/\sqrt{2}$ and $f = (i \oplus -i)/\sqrt{2}$ as basis vectors in V. Then $\sigma(e,f) = 1$, which corresponds to the parameter $\mu = 1/2$ in the previous example. From the identification

$$S(e,e) = z + x, \quad S(f,f) = z - x, \quad S(e,f) = y + i/2,$$

we have the correspondence of parameters

$$\overline{d} = z - \frac{1}{2}, \qquad b = x + iy.$$

Notice that $d = b = 0$ corresponds to a boundary point $(x,y,z) = (0,0,1/2)$.

Proposition 4.5. Let (V,σ) be a finite-dimensional presymplectic vector space and S be a covariance form on (V,σ). Then we can find a basis $\{d_j, e_k, f_k\}$ of V and sequences $\{\lambda_j\}_{1\leq j\leq m}$ ($\lambda_j \geq 0$), $\{\mu_k\}_{1\leq k\leq n}$ ($0 < \mu_k \leq 1/2$) such that subspaces $\mathbb{C}d_j$, $\mathbb{C}(e_k + if_k)$, $\mathbb{C}(e_k - if_k)$ are mutually $(S+\overline{S})$-orthogonal with $d_j \in \ker\sigma$ and

$$S(d_j,d_j) = \lambda_j, \quad S(e_k \pm if_k, e_k \pm if_k) = 1 \mp 2\mu_k, \quad \sigma(e_k, f_l) = 2\mu_k\delta_{k,l}.$$

Exercise 24. Prove this.

Remark 2. The system $\{e_k, f_k\}$ is $(S+\overline{S})$-orthonomal and S is represented on $\mathbb{C}e_k + \mathbb{C}f_k$ by the matrix

$$\begin{pmatrix} S(e_k,e_k) & S(e_k,f_k) \\ S(f_k,e_k) & S(f_k,f_k) \end{pmatrix} = \begin{pmatrix} 1/2 & i\mu_k \\ -i\mu_k & 1/2 \end{pmatrix}.$$

Lemma 4.6. Given a covariance form S on (V,σ), set $(x,y)_S = S(x,y) + S(y^*,x^*)$ for $x,y \in V^{\mathbb{C}}$. Then $(\ ,\ )_S$ is a positive form on $V^{\mathbb{C}}$ satisfying (i) $(y^*,x^*)_S = (x,y)_S$ and (ii) $|\sigma(x^*,y)|^2 \leq (x,x)_S\,(y,y)_S$ for $x,y \in V^{\mathbb{C}}$.

Conversely, any positive form fulfilling these conditions comes from a covariance form.

Proof. We shall work with the completion $V_S^{\mathbb{C}}$ of $V^{\mathbb{C}}$ with respect to the (possibly degenerate) inner product $(\ ,\ )_S$. From the obvious inequality $S(x,x) \leq (x,x)_S$, we can find a positive operator $\mathbb{S} \in \mathcal{B}(V_S^{\mathbb{C}})$ such that $S(x,y) = (x, \mathbb{S}y)_S$. The relation $(x,y)_S = S(x,y) + S(y^*, x^*)$ is then equivalent to $\mathbb{S} + \overline{\mathbb{S}} = 1$ ($\overline{\mathbb{S}}x = (\mathbb{S}x^*)^*$). Moreover, we have $i\sigma(x^*, y) = (x, (\mathbb{S} - \overline{\mathbb{S}})y)_S$ and the operator inequality $-1 \leq \mathbb{S} - \overline{\mathbb{S}} \leq 1$ gives

$$|\sigma(x^*, y)| \leq \|x\|_S \, \|(\mathbb{S} - \overline{\mathbb{S}})y\|_S \leq \|x\|_S \|y\|_S.$$

Conversely, assume that a positive form $(\ ,\ )$ on $V^{\mathbb{C}}$ satisfies (i) and (ii) in place of $(\ ,\ )_S$. The inequality (ii) gives rise to a hermitian operator $-1 \leq H \leq 1$ so that $i\sigma(x^*, y) = (x, Hy)_S$, whereas the invariance (i) and the alternating property of σ implies $\overline{H} = -H$. Now set $S(x,y) = (x, \frac{H+1}{2}y)$. Then the identity $S(y^*, x^*) = (x, \frac{\overline{H}+1}{2}y)$ shows that S is a covariance form satisfying $(x,y) = (x,y)_S$. □

Corollary 4.7. Let V_S be the real Hilbert space with respect to the positive form $S + \overline{S}$, i.e., V_S is the completion of $V/\ker(S + \overline{S})$ with respect to the induced inner product. Then V_S is a presymplectic vector space by the presymplectic form σ_S induced from σ and the natural map $V \to V_S$ is presymplectic.

Proposition 4.8. A covariance form S is extremal in $\mathrm{Cov}(V, \sigma)$ if and only if $\mathbb{S}^2 = \mathbb{S}$. Here $\mathbb{S}$ is a positive operator on $V_S^{\mathbb{C}}$ defined by $S(x,y) = (x, \mathbb{S}y)_S$ $(x, y \in V_S^{\mathbb{C}})$.

Exercise 25. Prove this.

Example 4.9. The covariance form

$$S(\xi \oplus \overline{\eta}, \xi \oplus \overline{\eta}) = \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix}$$

appeared in Example 4.4 is extremal.

Remark 3. Even if we start with a symplectic vector space (V, σ), the completed one (V_S, σ_S) with respect to a covariance form S may have a degenerate σ_S.

Exercise 26. Construct an example supporting the above remark.

As a final remark of this section, we record here the following.

Proposition 4.10. Given covariance forms S, T on a presymplectic vector space (V, σ) and a non-degenerate real inner product R on $V^{\mathbb{C}}$ which majorizes S and T, we can find a family of closed separable subspaces $\{V_i\}_{i \in I}$ in the completion V' of V such that $S'(V_i, V_j) = 0 =$

$T'(V_i, V_j)$ if $i \neq j$ and $V' = \overline{\oplus_{i\in I} V_i}$. Here S' and T' are continuous extensions of S and T to $V'^{\mathbb{C}}$.

Exercise 27. Prove this.

In view of this fact, we may restrict ourselves to separable V's to look into mutual relations between free states.

5. Free States

From here on, a presymplectic form σ is supposed to satisfy $\mathrm{Cov}(V, \sigma) \neq \emptyset$. With this assumption, there exists a covariance form S having a trivial kernel; for $v \in V^{\mathbb{C}}$, $S(v, v) = 0$ implies $v = 0$.

Lemma 5.1 (Hadamard-Schur product)**.** Let (a_{jk}) and (b_{jk}) be positive semidefinite matrices of size n. Then the matrix with entries of component-wise multiplication $(a_{jk}b_{jk}))_{1\leq j,k\leq n}$ is positive semidefinite.

Proof. Express positive matrices as convex combinations of positive matrices of rank one. □

Corollary 5.2. For a positive semidefinite matrix (a_{jk}), the matrix $(e^{a_{jk}})$ is positive semidefinite.

Exercise 28. Check these assertions.

Given a covariance form S and a linear functional $\alpha : V \to \mathbb{R}$, the following formula defines a state (called a **free state**) on the C*-algebra $C(V, \sigma)$.

$$\varphi(e^{ix}) = e^{-S(x,x)/2+i\alpha(x)}, \quad x \in V.$$

The positivity of φ is an easy consequence of the above corollary.

When $\alpha = 0$, we simply write φ_S. Note that φ_S and $\varphi_{\alpha,S}$ are related by a gauge automorphism $\theta(e^{ix}) = e^{i\alpha(x)}e^{ix}$ $(x \in V)$: $\varphi_{\alpha,S} = \varphi_S \circ \theta$.

Lemma 5.3. The GNS-vector $\varphi_{\alpha,S}^{1/2}$ is entirely analytic, whence

$$\pi(\mathcal{C}(V, \sigma))\varphi_{\alpha,S}^{1/2} \subset \mathcal{D}, \quad \pi_{\mathcal{D}}(\mathcal{A}(V, \sigma))\varphi_{\alpha,S}^{1/2} \subset \mathcal{D}.$$

Proof. To this end, we introduce a *-algebra $\mathcal{C}(V^{\mathbb{C}}, \sigma)$ generated by the symbols e^v $(v \in V^{\mathbb{C}})$ with relations

$$e^v e^w = e^{i\sigma(v,w)/2} e^{v+w}, \quad (e^v)^* = e^{v^*}.$$

Then a state φ on $\mathcal{C}(V^{\mathbb{C}}, \sigma)$ is defined by

$$\varphi(e^v) = e^{S(v^*,v)/2+\alpha(v)}, \quad v \in V^{\mathbb{C}}$$

and we can see that $V^{\mathbb{C}} \ni v \mapsto e^v\varphi^{1/2}$ is entirely analytic. Since the restriction $e^{ix}\varphi^{1/2}$ is naturally identified with $e^{ix}\varphi_{\alpha,S}^{1/2}$, we have

$\overline{\mathcal{C}(V^{\mathbb{C}},\sigma)\varphi^{1/2}} = \overline{\mathcal{C}(V,\sigma)\varphi_{\alpha,S}^{1/2}}$ and $e^v\varphi^{1/2}$ is an analytic extension of $e^{ix}\varphi_{\alpha,S}^{1/2}$. □

Exercise 29. Let $\varphi_{\alpha,S}$ be a free state on $C(V,\sigma)$ and denote by φ the analytic extension of $\varphi_{\alpha,S}$ to the *-algebra $\mathcal{E}(V,\sigma)$. Then

$$\varphi(x^*y) = S(x,y) + \overline{\alpha(x)}\alpha(y) \quad \text{for } x,y \in V^{\mathbb{C}}.$$

Proposition 5.4. In the situation of Proposition 4.5, if the covariance form S is decomposed according to the decomposition of presymplectic vector space $V = \sum_{j=1}^m \mathbb{R}d_j + \sum_{k=1}^n (\mathbb{R}e_k + \mathbb{R}f_k)$, then the free state φ_S is factorized into the product of one-dimensional gaussian measures

$$\varphi_j(e^{itd_j}) = e^{-\lambda_j t^2/2}$$

and two-dimensional free states

$$\varphi_k(e^{i(xe_k+yf_k)}) = e^{-(x^2+y^2)/2}$$

with $[e_k, f_k] = -2\mu_k$.

Definition 5.5. A free state φ_S is called a **Fock state** if the covariance form S is extremal in $\mathrm{Cov}(V,\sigma)$.

Fock states are basic ones among free states. Let φ_S be a Fock state with the associated GNS representation denoted by π. In view of Proposition 4.8, we may assume that $V = V_S$ and $\sigma = \sigma_S$ from the outset. Then S is of the form $S(x,y) = (x, Py)_S$ with P a projection in $\mathcal{B}(V^{\mathbb{C}})$ satisfying $P + \overline{P} = 1$ and $V^{\mathbb{C}} = K \oplus \overline{K}$ if we set $K = PV^{\mathbb{C}}$, i.e., (V,σ) is the one in Example 3.3.

The following is a key in the analysis of Fock states.

Lemma 5.6. For $a \in \overline{K} \subset \mathcal{A}(V,\sigma)$, $a\varphi_S^{1/2} = 0$.

Proof. For $a \in \overline{K} = \overline{P}V^{\mathbb{C}}$,

$$(a\varphi_S^{1/2}|a\varphi_S^{1/2}) = \varphi_S(a^*a) = S(a,a) = (a, Pa)_S = 0.$$

□

Remark 4. In quantum physics, $\varphi_S^{1/2}$ represents a vacuum and a (resp. a^*) is interpreted as an operator annihilating (resp. creating) a quantum.

For $a, b \in \overline{K}$, we have

$$[a, b^*] = ab^* - b^*a = i\sigma(a,b^*)1 = (a^*, (P-\overline{P})b^*)_S 1 = S(a^*, b^*)1,$$

which is used repeatedly to see

$$\mathcal{A}(V_S,\sigma_S) = \sum_{m,n\geq 0} K^m(\overline{K})^n$$

and then

$$\mathcal{A}(V_S, \sigma_S)\varphi_S^{1/2} = \sum_{n\geq 0} K^n \varphi_S^{1/2}$$

thanks to $\overline{K}\varphi_S^{1/2} = 0$.

Now we present a typical computation for $a, a_j \in \overline{K}$ $(j = 1, \dots, n)$:

$$\begin{aligned} aa_1^* \cdots a_n^* \varphi_S^{1/2} &= [a, a_1^* \cdots a_n^*]\varphi_S^{1/2} \\ &= \sum_{k=1}^n a_1^* \cdots a_{k-1}^* [a, a_k^*] a_{k+1}^* \cdots a_n^* \varphi_S^{1/2} \\ &= \sum_{k=1}^n S(a^*, a_k^*) a_1^* \cdots a_{k-1}^* a_{k+1}^* \cdots a_n^* \varphi_S^{1/2} \end{aligned}$$

As a result, $aK^n\varphi_S^{1/2} = K^{n-1}\varphi_S^{1/2}$ $(0 \neq a \in \overline{K})$, which is used repeatedly to get for $m, n \geq 1$

$$\overline{K}^m K^n \varphi_S^{1/2} = \begin{cases} K^{n-m}\varphi_S^{1/2} & \text{if } m < n, \\ \mathbb{C}\varphi_S^{1/2} & \text{if } m = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

In particular, we have

$$b_n \cdots b_1 a_1^* \cdots a_n^* \varphi_S^{1/2} = (b_1^* \cdots b_n^* \varphi_S^{1/2} | a_1^* \cdots a_n^* \varphi_S^{1/2})\varphi_S^{1/2}$$

for $a_1, \dots, a_n, b_1, \dots, b_n \in \overline{K}$.

Exercise 30. Let $\{a_j\}_{1\leq j\leq n}$ be an orthonormal system in $\overline{K}$ and $k = (k_1, \cdots, k_n)$ and $l = (l_1, \dots, l_n)$ be multiindices in $\mathbb{Z}_+^n$. Then

$$((a_1^*)^{k_1} \cdots (a_n^*)^{k_n} \varphi_S^{1/2} | (a_1^*)^{l_1} \cdots (a_n^*)^{l_n} \varphi_S^{1/2}) = \delta_{k,l} k!.$$

Theorem 5.7. A free state $\varphi_{\alpha,S}$ is pure if and only if the covariance form S is extremal.

Proof. Since a state φ of a C*-algebra A is pure if and only if $\varphi \circ \theta$ is pure for any *-automorphism θ of A, the problem is reduced to the case of free states of trivial means. We here prove the if part and refer to [16, Theorem 6.10] for the only if part.

Assume that S is extremal and show that the GNS representation π is irreducible.

For a unitary $U \in \pi(C(V, \sigma))'$, we have $U\mathcal{D} = \mathcal{D}$ and $U\pi_{\mathcal{D}}(a)U^* = \pi_{\mathcal{D}}(a)$ for $a \in \mathcal{A}(V, \sigma)$.

Let $a_1, \cdots, a_m, b_1, \cdots, b_n \in \overline{K}$. Then,

$$\begin{aligned}(a_1^* \cdots a_m^* \varphi_S^{1/2} | U(b_1^* \cdots b_n^* \varphi_S^{1/2})) &= (a_1^* \cdots a_m^* \varphi_S^{1/2} | U(\pi(b_1^*) \cdots \pi(b_n^*) \varphi_S^{1/2})) \\ &= (\pi(b_n) \cdots \pi(b_1) a_1^* \cdots a_m^* \varphi_S^{1/2} | U \varphi_S^{1/2}) \\ &= (b_n \cdots b_1 a_1^* \cdots a_m^* \varphi_S^{1/2} | U \varphi_S^{1/2}) \\ &= (\varphi_S^{1/2} | U(a_m \cdots a_1 b_1^* \cdots b_n^* \varphi_S^{1/2}))\end{aligned}$$

vanishes if $m \neq n$ and, for $m = n$,

$$\begin{aligned}(a_1^* \cdots a_n^* \varphi_S^{1/2} | U(b_1^* \cdots b_n^* \varphi_S^{1/2})) &= (b_n \cdots b_1 a_1^* \cdots a_n^* \varphi_S^{1/2} | U \varphi_S^{1/2})) \\ &= (\varphi_S^{1/2} | U \varphi_S^{1/2})(a_1^* \cdots a_n^* \varphi_S^{1/2} | b_1^* \cdots b_n^* \varphi_S^{1/2}).\end{aligned}$$

Consequently

$$(a_1^* \cdots a_m^* \varphi_S^{1/2} | U(b_1^* \cdots b_n^* \varphi_S^{1/2})) = (\varphi_S^{1/2} | U \varphi_S^{1/2})(a_1^* \cdots a_m^* \varphi_S^{1/2} | b_1^* \cdots b_n^* \varphi_S^{1/2})$$

for any m, n. Since $\cup_n K^n \varphi_S^{1/2}$ is dense in the representation space, this implies $U = (\varphi_S^{1/2} | U \varphi_S^{1/2}) 1$. □

6. Transition Probability and Universal Hilbert Spaces

Given a pair $\{\alpha, \beta\}$ of positive sesquilinear forms on a complex vector space D, its commuting representation is an operator relization $(\iota : D \to \mathcal{H}, A, B)$ of them, where $\iota : D \to \mathcal{H}$ is a linear map into a Hilbert space $\mathcal{H}$ with a dense range, A and B are commuting positive operators on $\mathcal{H}$ satisfying $\alpha(x, y) = (\iota(x) | A \iota(y))$ and $\beta(x, y) = (\iota(x) | B \iota(y))$.

Theorem 6.1 (Pusz-Woronowicz)**.** The sesquilinear form $\sqrt{\alpha\beta}(x, y) = (\iota(x) | \sqrt{AB} \iota(y))$ is irrelevant of the choice of commuting representations and characterized by the variational expression

$$\sqrt{\alpha\beta}(x, x) = \sup\{\gamma(x, x); \gamma \text{ is a positive form majorized by } \{\alpha, \beta\}\}.$$

Here majorization means $|\gamma(x, y)|^2 \leq \alpha(x, x)\beta(y, y)$ for $x, y \in D$.

Given a C*-algebra A, we shall put all the left and right GNS-represntations together to construct a single *-bimodule $L^2(A)$ over A. More precisely, we require that $L^2(A)$ is linearly spanned by symbols $\varphi^{1/2} = (\varphi^{1/2})^*$ $(\varphi \in A_+^*)$ so that $(\varphi^{1/2} | a \varphi^{1/2}) = \varphi(a)$ and $(\psi^{1/2} a | a \varphi^{1/2}) = (\psi^{1/2} | a \varphi^{1/2} a^*) \geq 0$ for any $\varphi, \psi \in A_+^*$ and $a \in A$.

To get a hint for the construction, assume $\overline{A\varphi^{1/2}} = \overline{\psi^{1/2} A}$ additionally and think of a (possibly unbounded) operator $\Delta^{1/2}$ defined formally by $\Delta^{1/2}(a\varphi^{1/2}) = \psi^{1/2} a$, which is positive by our requirement.

Thus, if we introduce a pair $\{\varphi_L, \psi_R\}$ of positive sesquilinear forms on A by

$$\varphi_L(x,y) = \varphi(x^*y), \quad \psi_R(x,y) = \psi(yx^*),$$

it is represented on the GNS-space $\overline{A\varphi^{1/2}}$ by the identity operator for φ_L and by the positive operator Δ for ψ_R.

Now their geometric mean $\sqrt{\varphi_L\psi_R}$ is related to the structures in $L^2(A)$ as

$$\sqrt{\varphi_L\psi_R}(x,y) = (x\varphi^{1/2}|\Delta^{1/2}(y\varphi^{1/2})) = (x\varphi^{1/2}|\psi^{1/2}y)$$

for $x, y \in A$.

Theorem 6.2. Let A be a C*-algebra. Then

$$\sqrt{\varphi_L\psi_R}(x^*x, yy^*) \geq 0$$

for $x, y \in A$, and the formal algebraic sum $\sum_{\varphi \in A_+^*} A\varphi^{1/2}A$, which is a *-bimodule by $(a\varphi^{1/2}b)^* = b^*\varphi^{1/2}a^*$, is made into a hilbertian *-bimodule $L^2(A)$ with respect to the inner product defined by

$$\left(\sum_j x'_j \varphi_j^{1/2} x_j \,\middle|\, \sum_k y_k \varphi_k^{1/2} y'_k \right) = \sum_{j,k} \sqrt{(\varphi_j)_L(\varphi_k)_R}(y_k^* x'_j, y'_k x_j^*).$$

Here highly non-trivial is the positivity of the inner product, which turns out to be equivalent to the celebrated Tomita-Takesaki theorem (see [15, §7-§8]).

When φ and ψ are states, $0 \leq (\varphi^{1/2}|\psi^{1/2}) \leq 1$ and it is referred to as a **transition probability**[9] between them.

Example 6.3. When $A = \mathcal{C}(\mathcal{H})$ (the compact operator algebra), $L^2(A)$ is identified with $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{C}_2(\mathcal{H})$ so that each $\varphi^{1/2}$ corresponds to $\rho_\varphi^{1/2} \in \mathcal{C}_2(\mathcal{H})$, where $\rho_\varphi \in \mathcal{C}_1(\mathcal{H})$ denotes the density operator of $\varphi \in A_+^*$, i.e., $\varphi(a) = \mathrm{tr}(\rho a)$ for $a \in \mathcal{C}(\mathcal{H})$.

If ψ is another positive functional, $(\varphi^{1/2}x|y\psi^{1/2}) = \mathrm{tr}(x^*\rho_\varphi^{1/2}y\rho_\psi^{1/2})$ for $x, y \in \mathcal{C}(\mathcal{H})$. Note that $x^*\rho_\varphi^{1/2}y\rho_\psi^{1/2}$ is in the trace class.

The following is a simple consequence of the variational expression of geometric meanss.

Theorem 6.4 (Coarse-Graining Inequality)**.** Let $\phi : \mathcal{A} \to \mathcal{B}$ be a unit-preserving *-homomorphism of unital *-algebras. Then, for bounded positive functionals φ, ψ of $\mathcal{B}$,

$$(\varphi^{1/2}|\psi^{1/2}) \leq ((\varphi \circ \phi)^{1/2}|(\psi \circ \phi)^{1/2}).$$

[9]This is different from the one introduced by A. Uhlmann.

Exercise 31. For hermitian matrices $a_1, \dots, a_m$ and $b_1, \dots, b_m$ in $M_n(\mathbb{C})$, we have the inequality

$$\mathrm{tr}(a_1 b_1 + \cdots + a_m b_m) \leq \mathrm{tr}\Big((a_1^2 + \cdots + a_m^2)^{1/2}(b_1^2 + \cdots + b_m^2)^{1/2}\Big).$$

Hint: Consider the diagonal imbedding $M_n(\mathbb{C}) \to M_{mn}(\mathbb{C})$.

Theorem 6.5 (Approximation[15, Appendix])**.** Let φ and ψ be positive functionals on a C*-algebra A with unit 1_A. Let $\{A_n\}_{n\geq 1}$ be an increasing sequence of C*-subalgebras of A containing 1_A in common and assume that, given any $a \in A$, we can find a sequence $\{a_n \in A_n\}_{n\geq 1}$ satisfying

$$\lim_{n\to\infty} \|a_n \varphi^{1/2} - a\varphi^{1/2}\| = 0 = \lim_{n\to\infty} \|\psi^{1/2} a_n - \psi^{1/2} a\|.$$

Set $\varphi_n = \varphi|_{A_n}, \psi_n = \psi|_{A_n} \in A_n^*$. Then the sequence $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}_{n\geq 1}$ is decreasing and converges to $(\varphi^{1/2}|\psi^{1/2})$.

The *-bimodule $L^2(A)$ bears the following universal character of *-representations.

Theorem 6.6 ([15, Theorem 8.1])**.** Let φ and ψ be positive functionals on a C*-algebra A.

(i) φ and ψ are disjoint if and only if $A\varphi^{1/2}A$ and $A\psi^{1/2}A$ are orthogonal. When $\overline{A\varphi^{1/2}} = \overline{\varphi^{1/2}A}$, this is further equivalent to $(\varphi^{1/2}|\psi^{1/2}) = 0$.

(ii) φ and ψ are quasi-equivalent if and only if $\overline{A\varphi^{1/2}A} = \overline{A\psi^{1/2}A}$.

(iii) φ is pure if and only if $\overline{A\varphi^{1/2}} \cap \overline{\varphi^{1/2}A} = \mathbb{C}\varphi^{1/2}$.

Exercise 32. Check the statements of the theorem for $A = M_m(\mathbb{C}) \oplus M_n(\mathbb{C})$.

7. Finite-Dimensional Analysis

For the CCR algebra associated to a finite-dimensional presymplectic vector space, which is assumed throughout this section unless otherwise stated, fundamental is the following (see [15, Appendix D] for a proof).

Theorem 7.1 (Stone-von Neumann)**.** Let (V, σ) be a finite-dimensional symplectic vector space. Then all the continuous irreducible representations of $C(V, \sigma)$ are unitarily equivalent.

In the following, we shall fix a Lebesgue measure on V once for all. Let π be a continuous *-representation of $C(V, \sigma)$ on a Hilbert space

$\mathcal{H}$. For an integrable function $f \in L^1(V)$, let $\pi(f) \in \mathcal{B}(\mathcal{H})$ be defined by the integral

$$\int_V f(x)\pi(e^{ix})\,dx$$

and introduce a *-algebra structure on $L^1(V)$ so that the above integration gives a *-homomorphism into $\mathcal{B}(\mathcal{H})$:

$$\int_V f(x)\pi(e^{ix})\,dx \int_V g(y)\pi(e^{iy})\,dy \\ = \int_V dy \int_V dx f(x)g(y-x)e^{-i\sigma(x,y)/2}\pi(e^{iy})$$

indicates to define[10]

$$(f \star g)(y) = \int_V f(x)g(y-x)e^{-i\sigma(x,y)/2}\,dx,$$

while

$$\left(\int_V f(x)\pi(e^{ix})\,dx\right)^* = \int_V \overline{f(-x)}\pi(e^{ix})\,dx$$

to set $f^*(x) = \overline{f(-x)}$.

It is immediate to check that these operations in fact make $L^1(V)$ into a Banach *-algebra, which is denoted by $L^1(V,\sigma)$. Let $C^*(V,\sigma)$ be the C*-algebra associated to $L^1(V)$. The image of $f \in L^1(V,\sigma)$ in $C^*(V,\sigma)$ is reasonably denoted by

$$\int_V f(x)e^{ix}\,dx.$$

A gauge automorphism $\theta(e^{ix}) = e^{i\alpha(x)}e^{ix}$ of $C(V,\sigma)$ with $\alpha \in V^*$ then induces a *-automorphism of $C^*(V,\sigma)$ by

$$\theta\left(\int_V f(x)e^{ix}\,dx\right) = \int_V e^{i\alpha(x)}f(x)e^{ix}\,dx$$

for $f \in L^1(V)$.

By our way of definition, any continuous *-representation $\pi(e^{ix})$ of $C(V,\sigma)$ gives rise to a bounded *-representation of $C^*(V,\sigma)$ so that

$$\pi\left(\int_V f(x)e^{ix}\,dx\right) = \int_V f(x)\pi(e^{ix})\,dx.$$

Conversely, any bounded *-representation of $C^*(V,\sigma)$ is of this form. To see this, we realize $\mathcal{C}(V,\sigma)$ as a multiplier algebra of $C^*(V,\sigma)$: For

[10]It is customary to use the convolution notation to avoid confusions with the pointwise multiplication.

$f \in L^1(V)$ and $x \in V$, define $e^{ix}f, fe^{ix} \in L^1(V)$ so that $\pi(e^{ix}f) = \pi(e^{ix})\pi(f)$, $\pi(fe^{ix}) = \pi(f)\pi(e^{ix})$:

$$(e^{ix}f)(y) = e^{-i\sigma(x,y)/2}f(y-x), \quad (fe^{ix})(y) = e^{i\sigma(x,y)/2}f(y-x).$$

The following are also immediate to check.

Lemma 7.2. Let $x, y \in V$ and $f, g \in L^1(V)$. Then

(i) $(e^{ix}f)^* = f^*e^{-ix}$.
(ii) $e^{ix}(e^{iy}f) = e^{-i\sigma(x,y)/2}(e^{i(x+y)}f)$
(iii) $(fe^{ix}) \star g = f \star (e^{ix}g)$.
(iv) $(e^{ix}f)e^{iy} = e^{ix}(fe^{iy})$.

From here on, the star symbols to designate the product on $L^1(V,\sigma)$ are often dropped off to appreciate the associativity maximally. Thus the element in (iii) for example is simply expressed as $fe^{ix}g$.

Proposition 7.3. Given a non-degenerate bounded *-representation π of $L^1(V,\sigma)$ on a Hilbert space $\mathcal{H}$, we have a continuous unitary representation $\pi(e^{ix})$ of $C(V,\sigma)$ such that

$$\pi(e^{ix})\pi(f) = \pi(e^{ix}f), \quad \pi(f)\pi(e^{ix}) = \pi(fe^{ix}).$$

Consequently there is one-to-one correspondence between non-degenerate bounded *-representations of $C^*(V,\sigma)$ and continuous *-representations of $C(V,\sigma)$.

Moreover, left and right multiplications of e^{ix} on $L^1(V)$ are extended to $C^*(V,\sigma)$ so that, for $a \in C^*(V,\sigma)$, $e^{ix}a$ and ae^{ix} belong to $C^*(V,\sigma)$ and they are norm-continuous in $x \in V$.

Corollary 7.4. Bounded positive functionals on $C^*(V,\sigma)$ are identified with continuous positive functionals on $C(V,\sigma)$: Given a continuous positive functional φ on $C(V,\sigma)$,

$$C^*(V,\sigma) \ni \int_V f(x)e^{ix}\,dx \mapsto \int_V f(x)\varphi(e^{ix}) \quad \text{for } f \in f \in L^1(V)$$

defines a bounded positive functional on $C^*(V,\sigma)$ and any bounded positive functional on $C^*(V,\sigma)$ arises this way. The identification is extended to universal Hilbert spaces in such a way that

$$\overline{C(V,\sigma)\varphi^{1/2}C(V,\sigma)} = \overline{C^*(V,\sigma)\varphi^{1/2}C^*(V,\sigma)}$$

and

$$\int_{V\times V} f(x)g(y)(e^{ix}\varphi^{1/2}e^{iy})\,dxdy = \left(\int_V f(x)e^{ix}\,dx\right)\varphi^{1/2}\left(\int_V g(y)e^{iy}\,dy\right)$$

for $\varphi \in C^*(V,\sigma)^*_+$ and $f, g \in L^1(V)$.

Proposition 7.5. When $\sigma \equiv 0$, the Gelfand spectrum of $C^*(V,\sigma) = C^*(V)$ is identified with the dual vector space V^* in such a way that each $v^* \in V^*$ gives a continuous one-dimensional *-representation of $C(V,\sigma)$ defined by $e^{ix} \mapsto e^{i\langle x,v^*\rangle}$. Bounded positive functionals φ on $C^*(V,\sigma)$ are in one-to-one correspondence with finite positive Radon measures μ on V^* by the relation

$$\varphi(e^{ix}) = \int_{V^*} e^{i\langle x,v^*\rangle}\mu(dv^*).$$

Corollary 7.6. Let ν be the Radon measure associated to another $\psi \in C^*(V)^*_+$. Then

$$(\varphi^{1/2}|e^{ix}\psi^{1/2}) = \int_{V^*} e^{i\langle x,v^*\rangle}\sqrt{\mu(dv^*)\nu(dv^*)} \quad \text{for } x \in V.$$

Example 7.7. Let $V = \mathbb{R}$ and set $\varphi = \varphi_{\alpha,S}$ with $S(x,x) = sx^2$ $(s \geq 0)$ and $\alpha \in \mathbb{R}^* = \mathbb{R}$. The correspondent Radon measure $\mu_{\alpha,S}$ is then given by

$$\mu_{\alpha,S}(dy) = \begin{cases} \frac{1}{\sqrt{2\pi s}} e^{-(y-\alpha)^2/2s}\,dy & \text{if } s > 0,\\ \delta(y-\alpha)dy & \text{if } s = 0,\end{cases}$$

which is used to see that

$$\begin{aligned}(\varphi_{\alpha,S}^{1/2}|e^{ix}\varphi_{\beta,T}^{1/2}) &= \frac{1}{(4\pi^2 st)^{1/4}}\int_{\mathbb{R}} e^{ixy}\exp\left(-\frac{(y-\alpha)^2}{4s} - \frac{(y-\beta)^2}{4t}\right)dy\\ &= \sqrt{\frac{2\sqrt{st}}{s+t}}\exp\left(-\frac{st}{s+t}x^2 + i\frac{\alpha t+\beta s}{s+t}x - \frac{(\alpha-\beta)^2}{4(s+t)}\right)\end{aligned}$$

for $st > 0$. If $(s,\alpha) \neq (t,\beta)$ and $st = 0$, the measures $\mu_{\alpha,S}$ and $\mu_{\beta,T}$ have disjoint supports, whence $(\varphi_{\alpha,S}^{1/2}|e^{ix}\varphi_{\beta,T}^{1/2}) = 0$ for $x \in V$.

Lemma 7.8. Let S and T be covariance forms on a not necessarily finite-dimensional presymplectic vector space (V,σ). Assume that $x \in V$ satisfies $T(x,x) > 0$. Then, for any linear functionals α, β on V,

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2})^2 \leq \frac{2}{\sqrt{S(x,x)/T(x,x)} + \sqrt{T(x,x)/S(x,x)}}.$$

Proof. Let $\phi : C(\mathbb{R},0) \to C(V,\sigma)$ be a *-homomorphism associated with a presymplectic map $\mathbb{R} \ni \lambda \mapsto \lambda x \in V$ and set $r = \sqrt{S(x,x)/T(x,x)} = \sqrt{s/t}$. Then

$$\begin{aligned}(\varphi_{S,\alpha}^{1/2}|\varphi_{T,\beta}^{1/2}) &\leq ((\varphi_{S,\alpha}\circ\phi)^{1/2}|(\varphi_{T,\beta}\circ\phi)^{1/2})\\ &= \sqrt{\frac{2\sqrt{st}}{s+t}}\exp\left(-\frac{(\alpha(x)-\beta(x))^2}{s+t}\right) \leq \sqrt{\frac{2}{r+r^{-1}}}.\end{aligned}$$

□

Corollary 7.9. Let $S, T \in \mathrm{Cov}(V, \sigma)$ and α, β be linear functionals on V. Then
$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = 0$$
unless positive forms $S + \overline{S}$ and $T + \overline{T}$ on $V^{\mathbb{C}}$ are topologically equivalent.

Remark 5. The topological inequivalence of $S + \overline{S}$ and $T + \overline{T}$ in fact implies the disjointness of $\varphi_{\alpha,S}$ and $\varphi_{\beta,T}$.

We return to the finite-dimensional case. To describe free states on $C^*(V, \sigma)$ and the associated W*-algebraic stuffs, it is convenient to work with a dense *-algebra of $L^1(V)$. We shall adopt here the Schwartz space $\mathcal{S}(V)$ of V as such a *-algebra [11], which admits a special trace functional defined by
$$\tau : \mathcal{S}(V) \ni g \mapsto g(0) \in \mathbb{C}.$$
Formally this is equivalent to requiring $\tau(e^{ix}) = \delta(x)$ for $x \in V$ ($\delta(x)$ being the delta function with respect to the preassigned Lebesgue measure).

Let $\mathcal{H} = \overline{\mathcal{S}(V, \sigma)\tau^{1/2}}$, which is identified with $L^2(V)$ by the relation
$$(g\tau^{1/2}|g\tau^{1/2}) = (g^* \star g)(0) = \int_V |g(v)|^2\, dv$$
and a *-representation $\{\pi(e^{ix})\}$ of $C(V, \sigma)$ is defined on $\mathcal{H}$ by the formula $\pi(e^{ix})(g\tau^{1/2}) = (e^{ix}g)\tau^{1/2}$ ($g \in \mathcal{S}(V, \sigma)$), which is continuous by the continuity of
$$\tau(g^* e^{ix} h) = \int_V \overline{g(y)} h(y - x) e^{-i\sigma(x,y)/2}\, dy$$
in $x \in V$.

The identity
$$(g \star h)\tau^{1/2} = \int_V g(x)(e^{ix}h)\tau^{1/2}\, dx.$$
for $g, h \in \mathcal{S}(V)$ shows that the GNS-representation $\mathcal{S}(V, \sigma)$, which is also denoted by π, is boundend as an integration of $g(x)\pi(e^{ix})$ and non-degenerate because of
$$\lim_{n\to\infty} \pi(\delta_n) = 1_{\mathcal{H}} \text{ in the strong operator topology}$$
for an approximate delta function $\{\delta_n\}_{n\geq 1}$ in $\mathcal{S}(V)$.

[11] The *-subalgebra is denoted by $\mathcal{S}(V, \sigma)$ to indicate the dependence on σ.

Let $\alpha : V \to \mathbb{R}$ be a linear functional of V and consider a covariance form S such that $S + \overline{S}$ is non-degenerate for the moment. Since the free state $\varphi_{\alpha,S}$ is continous in the sense that $\varphi_{\alpha,S}(e^{ix})$ is continuous in $x \in V$ (see §3), it gives rise to the bounded positive functional of $C^*(V,\sigma)$ specified by

$$\int_V f(x)e^{ix}\,dx \mapsto \int_V f(x)e^{-S(x,x)/2+i\alpha(x)}\,dx, \quad f \in \mathcal{S}(V),$$

which turns out to split via $\pi(C^*(V,\sigma))$ from the relation

$$\int_V f(x)e^{-S(x,x)/2+i\alpha(x)}\,dx = \tau(\rho_{\alpha,S} \star f).$$

Here the density operator $\rho_{\alpha,S}$ relative to the trace τ is given by a function $\rho_{\alpha,S}(x) = e^{-S(x,x)/2-i\alpha(x)}$ in $\mathcal{S}(V)$ which satisfies $\pi(\rho_{\alpha,S}) \geq 0$ because $(f\tau^{1/2}|\pi(\rho_{\alpha,S})f\tau^{1/2})$ is equal to

$$\tau(\rho_{\alpha,S} \star f \star f^*) = \iint f(x)\overline{f(y)}\varphi_{\alpha,S}(e^{ix}e^{-iy})\,dxdy \geq 0.$$

(The last integration is positive as a limit of positive definite sums.)

Since $\rho_{\alpha,S} = \theta(\rho_S)$ with θ a gauge automorphism given by $\theta(e^{ix}) = e^{-i\alpha(x)}e^{ix}$ $(x \in V)$, the problem of finding the square root of $\rho_{\alpha,S}$ is reduced to the case $\alpha = 0$: $\rho_{\alpha,S}^{1/2} = \theta(\rho_S^{1/2})$.

To get a formula for the square root of ρ_S in $C^*(V,\sigma)$, we try to find it as a 'function' $h : V \to \mathbb{C}$ satisfying $h(-x) = \overline{h(x)}$ and

$$e^{-S(x,x)/2} = \int_V h(y)h(x-y)e^{i\sigma(x,y)/2}\,dy = (h \star h)(x)$$

for $x \in V$.

7.1. **Square Roots of Density Operators.** From here on σ is assumed to be non-degenerate for the time being. Let $\mathbb{S}$ be the operator representing the covariance form S with respect to the inner product $(\ ,\ )_S$. From the relation $\mathbb{S} + \overline{\mathbb{S}} = 1$, we can read off the following spectral property of $\mathbb{S}$: If ξ is an eigenvector of eigenvalue λ, then so is $\overline{\xi}$ with eigenvalue replaced by $1-\lambda$, i.e., $\mathbb{S}\overline{\xi} = (1-\lambda)\overline{\xi}$. Moreover, by the non-degeneracy condition on σ, $\lambda = 1/2$ is not an eigenvalue.

To do the spectral decomposition within real subspaces of V, we restrict eigenvalues to the range $0 \leq \lambda < 1/2$ and normalize eigenvectors ξ so that $(S+\overline{S})$-orthonormal vectors in V are obtained by

$$e = \frac{\xi + \overline{\xi}}{\sqrt{2}}, \quad f = \frac{\xi - \overline{\xi}}{\sqrt{2}i}, \quad \xi = \frac{e + if}{\sqrt{2}}.$$

Relative to the basis $\{e, f\}$, $\mathbb{S}$ is represented on the two dimensional subspace $\mathbb{C}e + \mathbb{C}f = \mathbb{C}\xi + \mathbb{C}\overline{\xi}$ by the matrix

$$\mathbb{S} = \begin{pmatrix} 1/2 & i\mu \\ -i\mu & 1/2 \end{pmatrix} \quad \text{with} \quad \overline{\mathbb{S}} = \begin{pmatrix} 1/2 & -i\mu \\ i\mu & 1/2 \end{pmatrix}$$

and

$$\mathbb{S} - \overline{\mathbb{S}} = i \begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix},$$

where $2\mu \equiv 1 - 2\lambda$.

Consequently the canonical (Liouville) measure is of the form $2\mu dsdt$ with respect to the (partial) coordinates $(s, t) \in \mathbb{R}^2$ representing the vector $se + tf$ in a two-dimensional subspace of V, whereas the preassigned reference measure is of the form $2mdsdt$ with $m > 0$.

In terms of this basis, the relevant forms are expressed by

$$S(se + tf, se + tf) = \frac{1}{2}(s^2 + t^2), \quad i\sigma(se + tf, s'e + t'f) = 2i\mu(st' - s't)$$

and the equation to determine h takes the form

$$e^{-(s^2+t^2)/4} = 2m \int_{\mathbb{R}^2} h(s', t')h(s - s', t - t')e^{i\mu(st'-s't)}\, ds'dt'$$

with the hermiticity condition given by $h(-s, -t) = \overline{h(s, t)}$. We shall deal with a slightly more general situation: for $g, h \in L^1(\mathbb{R}e + \mathbb{R}f)$, consider

$$(g \star h)(se + tf) = 2m \int_{\mathbb{R}^2} g(s', t')h(s - s', t - t')e^{i\mu(st'-s't)}\, ds'dt'.$$

If we write

$$g(s, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{g}(\xi, \eta)e^{is\xi+it\eta}\, d\xi d\eta$$

with the Fourier transform $\widehat{g}$ defined by

$$\widehat{g}(\xi, \eta) = \int_{\mathbb{R}^2} g(s, t)e^{-is\xi-it\eta}\, dsdt$$

and similarly for h, then

$$(g \star h)(s, t) = \frac{2m}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{g}(\xi, \eta)\widehat{h}(\xi - \mu t, \eta + \mu s)e^{is\xi+it\eta}\, d\xi d\eta.$$

For the choice

$$\widehat{g}(\xi, \eta) = Ae^{-a(\xi^2+\eta^2)/2\mu}, \quad \widehat{h}(\xi, \eta) = Be^{-b(\xi^2+\eta^2)/2\mu}$$

with A, B, a, b positive reals, explicit computations are worked out by Gaussian integrals: The results are

$$g(s, t) = \frac{\mu A}{2\pi a}e^{-\mu(s^2+t^2)/2a}, \quad h(s, t) = \frac{\mu B}{2\pi b}e^{-\mu(s^2+t^2)/2b}$$

and

$$(g \star h)(s,t) = \frac{ABm\mu}{\pi(a+b)} e^{-\mu(s^2+t^2)/2(a*b)},$$

where

$$a * b = \frac{a+b}{ab+1}.$$

Recall here that the function $h(s,t) = \exp(-\mu(s^2+t^2)/2c)$ with $c > 0$ gives a positive element

$$2m \int_{\mathbb{R}^2} h(s,t) e^{i(se+tf)}\, dsdt$$

in $C^*(\mathbb{R}e + \mathbb{R}f, \sigma)$ if and only if the associated positive functional $\tau(h\cdot)$ is a free state[12], which means that

$$\frac{\mu}{c}(s^2+t^2) = \begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} z & i\mu \\ -i\mu & z \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

with $z = \mu/c > 0$ satisfying $\mu^2 \leq z^2$, namely $c \leq 1$.

In particular, for the choice $a = b = c > 0$ and $A = B = C > 0$,

$$(h \star h)(se+tf) = \frac{C^2 m\mu}{2\pi c} e^{-\mu(s^2+t^2)/2(c*c)},$$

where $c * c = 2c/(c^2+1) \leq 1$ for any $c > 0$ as being expected (the square of a hermitian element being positive). Thus, equating this with $e^{-(s^2+t^2)/4}$, we reach the solution

$$h(s,t) = \sqrt{\frac{\mu}{2\pi cm}} e^{-\mu(s^2+t^2)/2c}$$

satisfying $0 < c \leq 1$ (recall $0 < 2\mu \leq 1$) by the choice

$$c = \frac{2\mu}{1+\sqrt{1-4\mu^2}}, \quad C = \frac{\sqrt{2\pi c}}{\sqrt{m\mu}}.$$

From the local expression

$$\mathbb{S} + \overline{\mathbb{S}} + 2\sqrt{\mathbb{S}\overline{\mathbb{S}}} = \begin{pmatrix} 1+\sqrt{1-4\mu^2} & 0 \\ 0 & 1+\sqrt{1-4\mu^2} \end{pmatrix},$$

we see that

$$\frac{\mu}{2c}(s^2+t^2) = \frac{1}{4} \begin{pmatrix} s & t \end{pmatrix} \left(\mathbb{S} + \overline{\mathbb{S}} + 2\sqrt{\mathbb{S}\overline{\mathbb{S}}} \right) \begin{pmatrix} s \\ t \end{pmatrix}.$$

---

[12] cf. the uniqueness of *-representations of $C^*(V,\sigma)$.

The last relation immediately gives a coordinates-free expression for $\rho_S^{1/2}$ $(= \int_V \rho_S^{1/2}(x)e^{ix}\,dx)$:

$$\rho_S^{1/2}(x) = \frac{1}{\sqrt{N_S}} \exp\left(-\frac{1}{2}A(x)\right), \quad A = \frac{1}{2}\left(S + \overline{S} + 2\sqrt{S\overline{S}}\right)$$

with the normalization constant $N_S = \int_V e^{-A(x)}\,dx$ determined by the requirement

$$1 = \rho_S(0) = \int_V \rho_S^{1/2}(x)\rho_S^{1/2}(x)\,dx$$

It is then immediate to check that the expression is valid for degenerate σ as long as $S + \overline{S}$ is non-degenerate. Note that $\rho_S^{1/2} \in \mathcal{S}(V,\sigma)$ by the non-degeneracy of $S + \overline{S}$. As a gauge shift of $\varphi_S^{1/2}$, we obtain $\rho_{\alpha,S}^{1/2}(x) = e^{-i\alpha(x)}\rho_S^{1/2}(x)$ in view of

$$\rho_{\alpha,S}^{1/2} = \theta(\rho_S^{1/2}) = \int_V \rho_S^{1/2}(x)\theta(e^{ix})\,dx = \int_V \rho_S^{1/2}(x)e^{-i\alpha(x)}e^{ix}\,dx.$$

Now the transition probability is calculated,in view of Theorem 6.5, by

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = \tau(\rho_{\alpha,S}^{1/2}\rho_{\beta,T}^{1/2}) = \frac{1}{\sqrt{N_S N_T}} \int_V e^{-\frac{1}{2}A(x)-i\alpha(x)}e^{-\frac{1}{2}B(x)-i\beta(x)}\,dx,$$

where $B = \frac{1}{2}(T^{1/2} + \overline{T}^{1/2})^2$.

To dispose of the last integral, choose an auxiliary inner product $(\ |\ )$ on V which meets the following conditions ($(\ ,\ )_S + (\ ,\ )_T$ multiplies by a positive real can be used for example): (i) A and B are represented by commuting positive inverible operators $\mathbf{A}$ and $\mathbf{B}$ respectively (ii) The reference measure is the Lebesgue measure with respect to the inner product $(\ |\ )$. Then

$$N_S = \int_V e^{-(x|\mathbf{A}x)}\,dx = \frac{\pi^{n/2}}{\sqrt{\det(\mathbf{A})}}$$

($n = \dim V$) and we have

$$\begin{aligned}(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) &= \frac{(\det \mathbf{AB})^{1/4}}{\pi^{n/2}} \int_V e^{-(x|\mathbf{A}x)/2-(x|\mathbf{B}x)/2-i\alpha(x)}\,dx \\ &= \frac{\det^{1/4}(\mathbf{AB})}{\sqrt{\det(\mathbf{A}/2+\mathbf{B}/2)}}e^{-(v|(\mathbf{A}+\mathbf{B})^{-1}v)/2} \\ &= \sqrt{\det\left(\frac{2\sqrt{\mathbf{AB}}}{\mathbf{A}+\mathbf{B}}\right)}e^{-(v|(\mathbf{A}+\mathbf{B})^{-1}v)/2},\end{aligned}$$

where $v \in V$ is defined by $(v|x) = \alpha(x)$ $(x \in V)$.

With the help of Pusz-Woronowicz' functional calculus, the last formula takes a coordinates-free (independent of the choice of auxiliary inner products) expression:

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = \sqrt{\det\left(\frac{2\sqrt{AB}}{A+B}\right)} e^{-\frac{1}{2}(A+B)^{-1}(\beta-\alpha)}.$$

Recall that, given a positive sesquilinear form Q of a vector space K, the inverse form Q^{-1} (which is a quadratic form on the algebraic dual K^*) is defined as follows: Let $\alpha : K \to \mathbb{C}$. If we can find $a \in K_Q$ (the completion of K relative to Q) satisfying $\alpha(x) = Q(a, x)$ for $x \in K$, then $Q^{-1}(\alpha)$ is set to be $Q(a, a)$ and otherwise $Q^{-1}(\alpha) = +\infty$. Note that Q^{-1} satisfies the paralellelogram law.

By looking into supporting subspaces for degenerate $S + \overline{S}$ or $T + \overline{T}$, it turns out out that the last formula remains valid without restrictions on S and T. Thus we have reached the following.

Theorem 7.10. Let (V, σ) be a finite-dimensional presymplectic vector space. For covariance forms S and T of (V, σ) and linear functionals α, β of V, we have

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = \sqrt{\det\left(\frac{2\sqrt{AB}}{A+B}\right)} e^{-\frac{1}{2}(A+B)^{-1}(\alpha-\beta)},$$

where positive quadratic forms A, B are defined by

$$2A = (\sqrt{S} + \sqrt{\overline{S}})^2, \quad 2B = (\sqrt{T} + \sqrt{\overline{T}})^2.$$

Notice that $2\sqrt{AB} \leq A+B$ (geometric mean is majorized by arithmetic mean).

Example 7.11. Let σ be defined on $V = \mathbb{R}^2$ by the matrix $\begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix}$ with $\mu \in \mathbb{R}$ and consider a covariance form S in the boundary of $\mathrm{Cov}(V, \sigma)$, i.e., S is represented by a matrix of the form

$$\begin{pmatrix} z+x & y+i\mu \\ y-i\mu & z-x \end{pmatrix}, \quad z = \sqrt{x^2+y^2+\mu^2}.$$

From the extremality analysis before, we see that

$$\sqrt{S\overline{S}} = \begin{cases} 0 & \text{if } \mu \neq 0, \\ S & \text{if } \mu = 0 \end{cases}$$

and the associated quadratic form A is given by the matrix

$$\begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix}$$

for $\mu \neq 0$, whereas it is multiplied by a factor 2 for $\mu = 0$.

Note that the matrix $\begin{pmatrix} z+x & y+i\mu \\ y-i\mu & z-x \end{pmatrix}$ $(z^2 = x^2+y^2+\mu^2)$, does not commute with its complex conjugate unless $(x, y, \mu) = (0, 0, 0)$.

Let S' be another boundary covariance form with the associated quadratic form denoted by A'. Then both of A and A' are non-degenerate for $\mu \neq 0$ and we have

$$\begin{aligned}(\varphi_S^{1/2}|\varphi_{S'}^{1/2}) &= 2\frac{(z^2-x^2-y^2)^{1/4}(z'^2-x'^2-y'^2)^{1/4}}{\sqrt{(z+z')^2-(x+x')^2-(y+y')^2}} \\ &= \frac{2|\mu|}{\sqrt{(z+z')^2-(x+x')^2-(y+y')^2}}.\end{aligned}$$

It is ready to establish transition probability formula in the commutative case $\sigma \equiv 0$.

Theorem 7.12. Let V be an arbitrary (not necessarily finite-dimensional) real vector space. Let S, T be covariance forms for $\sigma \equiv 0$ and $\alpha, \beta : V \to \mathbb{R}$ be linear functionals. Then

$$(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = \sqrt{\det\left(\frac{2\sqrt{ST}}{S+T}\right)}e^{-\frac{1}{4}(S+T)^{-1}(\alpha-\beta)}.$$

Proof. If $\mathrm{tr}\frac{(S^{1/2}-T^{1/2})^2}{S+T} = \infty$, we can find a finite-dimensional subspace $W \subset V$ such that $\mathrm{tr}\frac{(\sqrt{S|_W}-\sqrt{T|_W})^2}{S|_W+T|_W}$ is arbitrarily large. Then

$$\begin{aligned}(\varphi_{S,\alpha}^{1/2}|\varphi_{T,\beta}^{1/2}) &\leq (\varphi_{S|_W,\alpha|_W}^{1/2}|\varphi_{T|_W,\beta|_W}^{1/2}) \\ &= \sqrt{\det\left(\frac{2\sqrt{S|_WT|_W}}{S|_W+T|_W}\right)}\; e^{-\frac{1}{4}(S|_W+T|_W)^{-1}(\alpha|_W-\beta|_W)} \\ &\leq \sqrt{\det\left(\frac{2\sqrt{S|_WT|_W}}{S|_W+T|_W}\right)}\end{aligned}$$

is arbitrarily small. Thus $(\varphi_{S,\alpha}^{1/2}|\varphi_{T,\beta}^{1/2}) > 0$ implies $\mathrm{tr}\frac{(S^{1/2}-T^{1/2})^2}{S+T} < \infty$ and the Gaussian integrals on each eigenspace of the trace class operator $\frac{(S^{1/2}-T^{1/2})^2}{S+T}$ are worked out to get the determinant formula. $\square$

Remark 6. When $\sigma \equiv 0$, $\overline{S} = S$ and $2A = (S^{1/2} + \overline{S}^{1/2})^2 = 4S$.

Notice here that simple restrictions to finite-dimensional algebras are not enough to get the probabilistic disjointness for non-trivial σ because it involves the non-linear transformation such as $2A = (S^{1/2} + \overline{S}^{1/2})^2$.

8. Infinite-Dimensional Analysis

We shall establish the determinant formula for the transition problity between free states when V is infinite-dimensional.

We first make the meaning clear for the relevant determinants. Let R and Q be quadratic forms on a real vector space V and assume that $Q(v) \leq R(v)$ for $v \in V$. Let V_R be the real Hilbert space induced from R, i.e., V_R is a completion of $V/\ker R$. Let $(\ |\ )$ be an inner product on V_R which gives the topology of V_R and write $Q(v) = (\dot{v}|\mathbf{Q}\dot{v})$, $R(v) = (\dot{v}|\mathbf{R}\dot{v})$ with $\mathbf{Q}$, $\mathbf{R}$ positive bounded linear operators on V_R. By our assumptions, $\mathbf{Q} \leq \mathbf{R}$ and $\mathbf{R}$ has a bounded inverse. We set

$$\det\left(\frac{Q}{R}\right) = \begin{cases} \det(\mathbf{R}^{-1}\mathbf{Q}) & \text{if } \mathbf{R} - \mathbf{Q} \text{ is in the trace class,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\det(1+T)$ with T in the trace class is the Fredholm determinant and we can easily check that this definition does not depend on the choice of an inner product $(\ |\ )$.

In view of the structural identity $(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) = (\varphi_{\alpha-\beta,S}^{1/2}|\varphi_T^{1/2})$, we may set $\beta = 0$.

Thanks to the coarse-graining inequality and the finite-dimensional formula, we can show that $(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) = 0$ unless $S \overset{q}{\sim} T$ and $(A+B)^{-1}(\alpha) < \infty$. Here $S \overset{q}{\sim} T$ means that A and B are equivalent as quadratic forms and $(A+B)\backslash(A^{1/2} - B^{1/2})^2$ is in the trace class.

Thus the problem is reduced to the case that $S \overset{q}{\sim} T$ and $(A+B)^{-1}(\alpha) < \infty$. The approximation theorem for transition probability then allows us to move into the hilbertian space $V_A = V_B$, i.e., we may assume that $V = V_A = V_B$ and $\alpha : V \to \mathbb{R}$ is continuous from outset.

Since $S + \overline{S}$, $T + \overline{T}$, A and B are all equivalent as quadratic forms, V is decomposed into a direct sum of separable subspaces V_j so that

$$S(V_j, V_k) = T(V_j, V_k) = 0 \quad \text{for } j \neq k.$$

The problem is therefore further reduced to the case that V is separable. For any choice of an increasing sequence of finite-dimensional subspaces V_n of V such that $V = \overline{\cup_n V_n}$, let S_n, T_n and α_n be restrictions of S, T

and α respectively. Again the approximation theorem is applied to see

$$(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) = \lim_{n\to\infty}(\varphi_{\alpha_n,S_n}^{1/2}|\varphi_{T_n}^{1/2})$$

and the problem boils down to showing

$$\lim_{n\to\infty}\det\left(\frac{2\sqrt{A_nB_n}}{A_n+B_n}\right) = \det\left(\frac{2\sqrt{AB}}{A+B}\right),$$

$$\lim_{n\to\infty}(A_n+B_n)^{-1}(\alpha_n) = (A+B)^{-1}(\alpha).$$

These approximation formulae on determinants are, however, not so obvious. The major difficulty is that $A_n = (S_n^{1/2} + \overline{S}_n{}^{1/2})^2/2$ is not the restriction of A to the subspace V_n. Of course A is approximated by A_n in the weak sense:

$$\lim_{n\to\infty} A_n(v) = A(v) \quad \text{for } v \in \bigcup_n V_n.$$

Question: With this weak pproximation property for A_n and B_n, is that true or not for the following convergence?

$$\lim_{n\to\infty}\det\left(\frac{2\sqrt{A_nB_n}}{A_n+B_n}\right) = \det\left(\frac{2\sqrt{AB}}{A+B}\right).$$

To circumvent the difficulty, we appeal to the old trick called purification: Given a positive functional φ on a C*-algebra A, its purification is a positive functional Φ on $A \otimes A^\circ$ defined by

$$\Phi(a \otimes b^\circ) = (\varphi^{1/2}|a\varphi^{1/2}b).$$

Here A° denotes the opposite algebra: A° is a replica of A with the multiplication reversed in the order as $a^\circ b^\circ = (ba)^\circ$ for $a, b \in A$.

We apply this to $A = C(V,\sigma)$ and $\varphi = \varphi_{\alpha,S}$. The result is that $C(V,\sigma) \otimes C(V,\sigma)^\circ = C(V \oplus V, \sigma \oplus -\sigma)$ and $\Phi = \varphi_{\alpha\oplus\alpha,P}$, where the purified covariance form P is defined by

$$P = \begin{pmatrix} S & \sqrt{S\overline{S}} \\ \sqrt{S\overline{S}} & \overline{S} \end{pmatrix}$$

Let Q be the purification of T. It turns out that $(V\oplus V)_P = (V \oplus V)_Q$ and $\varphi_{\alpha\oplus\alpha,P}|_C$, $\varphi_Q|_C$ are equivalent as Radon measures on the central C*-subalgebra $C = C(\ker(\sigma \oplus -\sigma), 0)$ of $C(V \oplus V, \sigma \oplus -\sigma)$. When C is trivial, life is easy: $\sqrt{P\overline{P}} = 0 = \sqrt{Q\overline{Q}}$ (P and Q being extremal) and the major difficulty disappears.

Thus, to get the maximal benefit from purification, we perform the direct integral decomposition of $\varphi_{\alpha\oplus\alpha,P}$ and φ_Q over the Gelfand spectrum Ω of C. For an explicit description of decomposed components, we replace Ω with a Borel subset $\Omega_0 \cong V_0^*$ which supports relevant measures (cf. discussions in §1).

Now the proof of the determinant formula goes like this (see [16, §11] for details):

$$
\begin{aligned}
(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) &= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)\,(\varphi_{\alpha,S,\omega}^{1/2}|\varphi_{T,\omega}^{1/2}) \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)(\varphi_{\dot\alpha+\Delta\omega,\dot S}^{1/2}|\varphi_{\dot T}^{1/2}) \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)(\varphi_{\dot\alpha+\Delta\omega\oplus\dot\alpha+\Delta\omega,\dot P}^{1/2}|\varphi_{\dot Q}^{1/2})^{1/2} \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)\det\left(\frac{\left(\frac{\dot Q+\overline{\dot Q}}{\dot P+\overline{\dot P}}\right)^{1/2}+\left(\frac{\dot P+\overline{\dot P}}{\dot Q+\overline{\dot Q}}\right)^{1/2}}{2}\right)^{-1/4} \\
&\qquad\qquad \times e^{-\frac{1}{2}\dot G^{-1}(\dot\alpha+\Delta\omega\oplus\dot\alpha+\Delta\omega)} \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)\sqrt{\det\left(\frac{2\sqrt{\dot A\dot B}}{\dot A+\dot B}\right)}e^{-\frac{1}{2}(\dot A+\dot B)^{-1}(\dot\alpha+\Delta\omega)} \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)\,(\varphi_{\dot\alpha+\Delta\omega,\dot A/2}^{1/2}|\varphi_{\dot B/2}^{1/2}) \\
&= \int_{V_0^*} \sqrt{\nu_{\alpha,S}\nu_T}(d\omega)\,(\varphi_{\alpha,A/2,\omega}^{1/2}|\varphi_{B/2,\omega}^{1/2}) \\
&= (\varphi_{\alpha,A/2}^{1/2}|\varphi_{B/2}^{1/2}) = \sqrt{\det\left(\frac{2\sqrt{AB}}{A+B}\right)}e^{-\frac{1}{2}(A+B)^{-1}(\alpha)}.
\end{aligned}
$$

Here dots indicate that it concerns the symplectic quotients such as $\dot V = V/\ker\sigma$.

Corollary 8.1. The condition $(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2}) > 0$ is an equivalence relation on $V^* \times \mathrm{Cov}(V,\sigma)$.

9. Kakutani's Dichotomy

The method of purification along with the central decomposition is also useful in the quasi-equivalence analysis. Recall that, two states φ and ψ on a C*-algebra A are quasi-equivalent or disjoint according to $\overline{A\varphi^{1/2}A} = \overline{A\psi^{1/2}A}$ or $\overline{A\varphi^{1/2}A} \perp \overline{A\psi^{1/2}A}$.

By utilizing the determinant formula, we see that the condition $(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) > 0$ quarantees $(\varphi^{1/2}_{\dot\alpha+\Delta\omega\oplus\dot\alpha+\Delta\omega,\dot P}|\varphi_{\dot Q}^{1/2}) > 0$ for every $\omega \in V_0^*$. Since $\varphi_{\dot\alpha+\Delta\omega\oplus\dot\alpha+\Delta\omega,\dot P}$ and $\varphi_{\dot Q}$ are pure states, non-vanishing of the transition probability between them implies their unitary equivalence, i.e.,

$$\overline{C(\dot V,\dot\sigma)\varphi^{1/2}_{\dot\alpha+\Delta\omega\oplus\dot\alpha+\Delta\omega,\dot P}} = \overline{C(\dot V,\dot\sigma)\varphi_{\dot Q}^{1/2}},$$

which are direct-integrated over ω to get

$$\overline{C(V,\sigma)\varphi_{\alpha,S}^{1/2}C(V,\sigma)} = \overline{C(V,\sigma)\varphi_T^{1/2}C(V,\sigma)}.$$

Note here the measures used for integrations are equivalent thanks to $(\varphi_{\alpha,S}^{1/2}|\varphi_T^{1/2}) > 0$ again.

Conversely, assume that the transition probality vanishes. For notational simplicity, write $A = C(V,\sigma)$, $\varphi = \varphi_{\alpha,S}$. By a Hilbert-Schmidt perturbation, we can find T' such that $(\varphi_T^{1/2}|\varphi_{T'}^{1/2}) > 0$ and $\overline{A\varphi_{T'}^{1/2}A} = \overline{\varphi_{T'}^{1/2}A}$. Write $\psi = \varphi_{T'}$. Since non-vanishing is an equivalence relation, $(\varphi^{1/2}|\varphi_T^{1/2}) = 0$ is equivalent to $(\varphi^{1/2}|\psi^{1/2}) = 0$. It implies $\varphi^{1/4}\psi^{1/4} = 0$ and then $\varphi^{1/2}\psi^{1/2} = 0$ as an element in A^*, which is used to get

$$(\varphi^{1/2}|\psi^{1/2}a) = (\varphi^{1/2}\psi^{1/2})(a) = 0$$

for $a \in A$. Since $\overline{\psi^{1/2}A} = \overline{A\psi^{1/2}A}$, this further implies the orthogonality of $\varphi^{1/2}$ with $\overline{A\psi^{1/2}A}$, whence

$$\overline{A\varphi^{1/2}A} \perp \overline{A\psi^{1/2}A} = \overline{A\varphi_T^{1/2}A}.$$

Theorem 9.1. Two free states $\varphi_{\alpha,S}$ and $\varphi_{\beta,T}$ on a CCR C*-algebra $C(V,\sigma)$ are quasi-equivalent or disjoint according to non-vanishing or vanishing of their transition probability $(\varphi_{\alpha,S}^{1/2}|\varphi_{\beta,T}^{1/2})$.

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