INTEGRALS OF DIFFERENTIAL FORMS AND THE CHANGE OF VARIABLES FORMULA

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Abstract. The vector calculus method is examined to prove the divergence theorem and the change-of-variables formula with one stroke.

1. Introduction

A naive understanding based on linear approximations is not difficult for the change-of-variables formula in multiple integrals (called the Jacobian formula for short) but its strict description is fairly involved with messy estimates. Since the formula is of fundamental importance in working with multiple integrals, there have been lots of proofs invented. According to historical remarks in [7], they are categorized to three groups.

(i) The orthodox estimates method: This approach uses linear approximations to estimate integration volumes and can be traced back to [3]. Note that J. Schwartz himself improved the method by showing that a one-side estimate inequality is sufficient to derive the equality.

(ii) The factorization method: Iterated application of change of variables is adapted to product integrals to prove the formula in small regions. These small-region results are then patched together to a global formula using an argument of partition of unity. This is also old enough to be found in [1], for example.

(iii) The geometric induction method: With the aid of the divergence theorem and coordinate-free description of surface integrals, the formula can be deduced by a dimensional induction, which can be found in [8], for example. As remarked in [7], a naive understanding of the divergence theorem is fairly easy, but its rigorous treatment is definitely not, especially in higher dimensions due to geometric complexities.

Other than these classical ones, we add the following group ([4, 5, 6]) to the list:

(iv) The deformation method: This method is based on the invariance of pullback integrals under deformed differentiable maps, which is checked using a divergence theorem in its primitive form.

We shall here focus on the third method and see how it works in an elementary but rigorous manner. Let us begin with discussing points that must be addressed in the process. Since the divergence theorem is needed in an arbitrary dimension, the right objects for this purpose are differential forms and their integration on hypersurfaces. Notice that its coordinate-free description is equivalent to the validity of the Jacobian formula in that dimension. (Any text on differential forms such as [2] covers these background materials.)

To be explict for precise accounts, let $\varphi : U \to V$ be a diffeomorphism between open subsets in a coordinate space $\mathbb{R}^n$, $D \subset U$ be a compact subset with the boundary ∂D given by a smooth singular chain in $\mathbb{R}^n$ and f be a continuously differentiable function on $E = \varphi(D) \subset V$. Then, with the help of an auxiliary form ω satisfying $f(y)dy = d\omega$, the Jacobian formula for φ is reduced to $(n-1)$-dimensional case through integrals of ω on ∂E and $\varphi^*\omega$ on ∂D, as indicated by the following computation:

$$\int_E f(y)\,dy = \int_{\partial E} \omega = \int_{\partial D} \varphi^*\omega = \int_D d\varphi^*\omega = \int_D \varphi^*(f dy),$$

where we have used the divergence theorem twice and the induction hypothesis once to relate two boundary (lower-dimensional) integrals by the pullback φ^*. Recall that the equality of boundary integrals is nothing but the Jacobian formula for integrals on $n-1$ variables.

Thus the problem is reduced to establishing the divergence theorem on Eulidean domains. As already pointed out, its strict proof is not straightforward at all. Even worse, the standard method relies on the Jacobian formula itself, including the one in $\mathbb{R}^n$, leading to a standstill of tautology.

To get around this pitfall, we derive the divergence theorem on domains of especially well-behaved boundary, based on the ordinary technique of linear approximation by localization. This allows us to establish the Jacobian formula for small regions. After that it is routine work to patch them together by a partition of unity to obtain a global one.

In this way, we obtain the Jacobian formula as well as the integration of differential forms with one stroke, by an induction on the number of integration variables. Since the fundamental formula in higer dimensional calculus is easily obtained once the coordinate-free definition of integration of differential forms is established, the divergence theorem in its genral form is also in our hands at the same time.

2. Change of Variables Formula via Divergence Theorem

The Jacobian formula we shall prove is the following weaker one, which is enough to define integrals of differential forms and immediately extended to Lebesgue integrable functions on V, once the Daniell extension is applied. Let $\varphi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$ as before and let $f \in C_c(V)$ this time. Then we have

$$\int_V f(y)\,dy = \int_U f(\varphi(x))|\det \varphi'(x)|\,dx.$$

Here $\varphi'(x)$ denotes the differential matrix of φ at x.

Owing to the standard machinery of partition of unity, the formula in this form is reduced to localized one: Given any $a \in U$, we can find an open rectangle R (centered) at a inside U so that

$$\int_{\varphi(R)} f(y)\,dy = \int_R f(\varphi(x))|\det\varphi'(x)|\,dx$$

for $f \in C_c(\varphi(R))$, which is further reduced to the subcase that f is continuously differentiable thanks to the uniform approximation based on mollifier (moving average) or the Stone-Weierstrass theorem.

Recall here that the global formula is valid for a composed diffeomorphim $\varphi \circ \psi$ if it holds for both φ and ψ. Notice also that the Jacobian formula holds globally for linear transformations, which can be checked without difficulties by a combination of repeated integrals and factorizations of invertible matrices into elementary ones corresponding to elementary row operations. Consequently it is enough to prove the local version for which $\varphi'(a)$ takes a preassigned form at a given point $a \in U$.

Let us now look into the indunction process. The initial step is just the formula for a signle-variable substitution

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(y)\,dy = \int_\alpha^\beta f(\varphi(x))\varphi'(x)\,dx.$$

As an indunction step, suppose that the Jacobian formula holds up to $n-1$ variables. Then the integration of continuous $(n-1)$-forms over $(n-1)$-dimensional bounded submanifolds of $\mathbb{R}^n$ is well-defined in a coordinate-free fashion and we need to prove the Jacobian formula for n variables in its localized form.

To this end, we introduce some notation and terminology: For a point $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, let $y' = (y_2, \dots, y_n)$ be the removal of the first coordinate from y. Given a subset A of $\mathbb{R}^n$, let A' be the image of A under the projection $y \mapsto y'$ and $A_{y'} \subset \mathbb{R}$ be the cut of A by y'.

We say that $\varphi'(a)$ is transversal to e_1 if images of coordinate hyperplanes under $\varphi'(a)$ are transversal to e_1, i.e., $e_1 \notin \sum_{j\neq i} \mathbb{R}\varphi'(a)e_j$ for each $1 \leq i \leq n$. Here $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$.

Being prepared, we resume the proof of the induction step. Let $F(y) = \displaystyle\int_{-\infty}^{y_1} f(t, y')\,dt$ be an indefinite partial integral of $f \in C_c^1(\varphi(R)) \subset C_c^1(\mathbb{R}^n)$ so that $d(F(y)dy') = f(y)dy$. Note here that F is continuously differentiable on $\mathbb{R}^n$ because so is f. Under the transversality assumption of $\varphi'(a)$ relative to e_1, we shall verify the boundary integral formula

$$\int_{\varphi(R)} f(y)\,dy = \int_{\varphi(\partial R)} F(y)\,dy'$$

for a sufficiently small R.

Once this is established, the computation at the beginning is validated for the choice $\omega = F(y)dy'$, $D = \overline{R}$ and $E = \overline{\varphi(R)}$ to get the localized version of Jacobian formula.

In fact, let $R = (a_1 - r_1, a_1 + r_1) \times \cdots \times (a_n - r_n, a_n + r_n)$, R_i be an open rectangle in $\mathbb{R}^{n-1}$ obtained from R by removing the i-th component and write $\varphi^*\omega = \sum_i f_i(x)\partial_i dx$, where f_i are continuously differentiable functions on U and $\partial_i dx$ denotes the inner contraction of dx by a basis vector e_i. Then we have

$$\begin{aligned}
\int_{\partial R} \varphi^*\omega &= \sum_i \int_{R_i} \Big(f_i(\dots, a_i + r_i, \dots) - f_i(\dots, a_i - r_i, \dots) \Big) \partial_i dx \\
&= \sum_i \int_{R_i} \int_{a_i - r_i}^{a_i + r_i} \frac{\partial f_i}{\partial x_i}(x) dx_i \partial_i dx = \sum_i \int_R \frac{\partial f_i}{\partial x_i}(x) dx \\
&= \int_R d\left(\sum_i f_i \partial_i dx \right) = \int_R d\varphi^*\omega = \int_R \varphi^*(d\omega) = \int_R \varphi^*(f(x)dx).
\end{aligned}$$

Returning to the proof of the boundary integral formula, a repeated integral is used to have

$$\int_{\varphi(R)} f(y)\, dy = \int_{\varphi(R)} \frac{\partial F}{\partial y_1}(y)\, dy = \int_{\varphi(R)'} dy' \int_{\varphi(R)_{y'}} \frac{\partial F}{\partial y_1}(y)\, dy_1.$$

Notice here that $\varphi(R)'$ is a bounded open subset of $\mathbb{R}^{n-1}$ as a projection of a bounded open subset $\varphi(R)$ of $\mathbb{R}^n$, whereas $\varphi(R)_{y'}$ is a bounded open subset of $\mathbb{R}$ as a cut of $\varphi(R)$ by $y' \in \mathbb{R}^{n-1}$. In general $\varphi(R)_{y'}$ is a disjoint union of countably many open intervals, which makes it non-trivial to be related with boundary integrals.

This kind of complexity, however, disappears if we restrict ourselves to the transversal case. To see this, take an auxiliary open rectangle Q at $\varphi(a)$ inside V and seek for a rectangle R inside $\varphi^{-1}(Q)$.

Let $\pi_i^{\pm} = \{x \in \mathbb{R}^n; x_i = a_i \pm r_i\}$ $(1 \leq i \leq n)$ be hyperplanes supporting each boundary face of R and let $Q_{y'} = (\alpha, \beta)$ for $y' \in Q'$, i.e., $Q = (\alpha, \beta) \times Q'$. Note that the open halfspace $\Pi_i^{\pm}$ in $\mathbb{R}^n$, which contains R and is bordered by $\pi_i^{\pm}$, is described by the inequality $\pm(x_i - a_i) < r_i$ in such a way that $R = \bigcap_i (\Pi_i^+ \cap \Pi_i^-)$.

Now the size of Q (and that of R accordingly) is chosen so small that curves $\varphi^{-1}(y' + te_1)$, corresponding to line segments $y' + te_1$ $(t \in [\alpha, \beta])$ indexed by $y' \in Q'$, have tangent vectors $v(t, y') = \dfrac{\partial}{\partial t}\varphi^{-1}(y' + te_1)$ transversal to any of the hypersubspaces $\pi_i : x_i = 0$ $(1 \leq i \leq n)$, i.e., $\mathbb{R}v(t, y') + \pi_i = \mathbb{R}^n$ $(y' \in Q',\ \alpha \leq t \leq \beta)$.

Then, if the curve $\varphi^{-1}(y' + te_1)$ hits $\pi_i^{\pm}$ at $t = c \in (\alpha, \beta)$, the transversality ensures that partial curves for $[\alpha, c)$ and $(c, \beta]$ are separated by the hyperplane $\pi_i^{\pm}$. Consequently

$$\{t \in [\alpha, \beta]; \varphi^{-1}(y' + te_1) \subset \Pi_i^{\pm}\}$$

is a single interval for each choice of $\pi_i^\pm$ and so is

$$\{t \in [\alpha, \beta]; \varphi^{-1}(y' + te_1) \in R \iff y' + te_1 \in \varphi(R)\} = \varphi(R)_{y'}$$

as their intersection, which is denoted by $(\alpha(y'), \beta(y'))$ $(y' \in \varphi(R)')$.

Let $R_i^\pm = \{x \in \mathbb{R}^n; a_j - r_j < x_j < a_j + r_j (j \neq i), x_i = a_i \pm r_i\}$ be the open faces of R. Then, by construction, both $(\alpha(y'), y')$ and $(\beta(y'), y')$ in $\mathbb{R}^n$ are on the boundary $\partial\varphi(R) = \varphi(\partial R)$ of $\varphi(R)$ and vary continuously with $y' \in \varphi(R)'$. Moreover, if y' belongs to the open face $\varphi(R_i^\pm)$ of $\varphi(R)$, these intervals depend on y' in a continuously differentiable fashion, thanks to the transversality.

Thus $(\alpha(y'), y')$ and $(\beta(y'), y')$ give rise to a smooth parametrization of the boundary face of $\varphi(R)$. More precisely, when $(\beta(y'), y')$ moves around $\varphi(R_i^+)$ for example, the hypersurface $\varphi(R_i^+)$ is parametrized by $\varphi(R_i^+)' \ni y' \mapsto (\beta(y'), y')$.

These observations are now combined with the repeated integral expression to have

$$\int_{\varphi(R)} f(y)\, dy = \int_{\varphi(R)'} \Big(F(\beta(y'), y') - F(\alpha(y'), y')\Big)\, dy' = \int_{\varphi(\partial R)} F(y)\, dy'$$

and the whole thing is done.

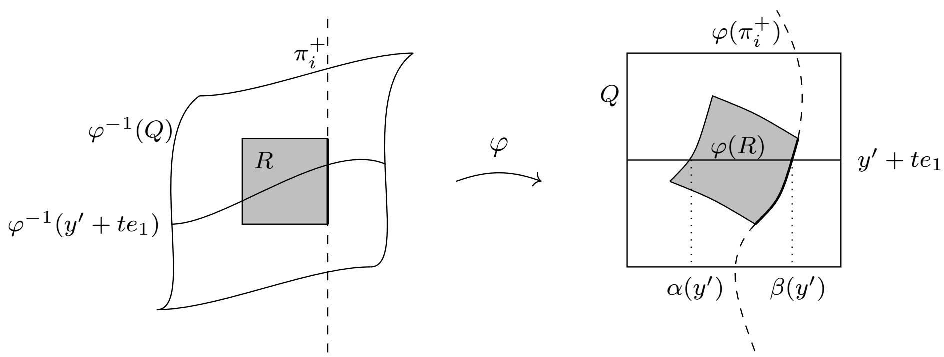

FIGURE 1. Ruled Domain

Remark 1. The frame of $\varphi(R)$ (the second boundary of $\varphi(R)$) consists of submanifolds having dimensions less than $n - 1$, whence its image is a compact negligible subset of $\varphi(R)' \subset \mathbb{R}^{n-1}$ (an easy part of Saad's theorem). Consequently adding and removing its subsets from a subregion of $\varphi(R)'$ does not affect integrals defined over them.

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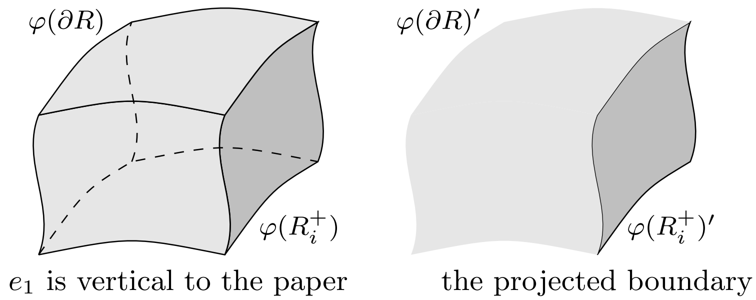

FIGURE 2. Tofu Boundary and Shadow

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