Stone's Theorem from Bochner's via Borel Functional Calculus

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1 Fourier Transforms and Unitary Representations

To a (continuous) unitary representation $U(t)$ of the additive group $\mathbb{R}$ on a separable Hilbert space $\mathcal{H}$, a *-representation of the convolution algebra $L^1(\mathbb{R})$ is associated by

$$U(h) = \int_{\mathbb{R}} h(t)U(t)\,dt,$$

where the integration in the right hand side is in the weak sense:

$$(\xi|U(h)\eta) = \int_{\mathbb{R}} h(t)(\xi|U(t)\eta)\,dt \tag{1}$$

for $\xi, \eta \in \mathcal{H}$.

Conversely, given a non-degenerate *-homomorphism $L^1(\mathbb{R}) \ni h \mapsto U(h) \in \mathcal{B}(\mathcal{H})$, a unitary representation $U(t)$ of $\mathbb{R}$ is recovered by

$$U(t)(U(h)\xi) = U(h_t)\xi, \quad h_t(s) = h(s-t).$$

The Fourier transform converts the convolution product into the functional multiplication; a *-homomorphism $L^1(\mathbb{R}) \ni h \mapsto \widehat{h} \in C_0(\mathbb{R})$ is defined by

$$\widehat{h}(x) = \int_{\mathbb{R}} e^{itx} h(t)\,dt.$$

Thus any *-representation of $C_0(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ induces a *-representation of $L^1(\mathbb{R})$, which in turn produces a unitary representation of $\mathbb{R}$ on $\mathcal{H}$. The heart of Fourier analysis is in the fact that the converse holds.

2 Bochner's Theorem

Recall here states on a *-algebra A and the associated GNS-representations.

When A is the group algebra $L^1(G)$ of a locally compact group G, these are equivalently described by positive definite functions on G.

Theorem 2.1 (Bochner)**.** Any positive definite continuous function φ on the additive group $\mathbb{R}$ is expressed in terms of a Borel measure μ on $\mathbb{R}$ by

$$\varphi(t) = \int_{\mathbb{R}} e^{itx}\,\mu(dx).$$

Proof. It suffices to deal with the case $\varphi(e) = 1$ and we shall do the proof in three steps:

(i) For an integrable function f on $\mathbb{R}$,

$$\int_{\mathbb{R}^2} \varphi(s-t) f(s)\overline{f(t)}\,dsdt \geq 0.$$

By L^1-approximation, we may assume that $f \in C_c(\mathbb{R})$. Then

$$\iint_{\mathbb{R}^2} \varphi(s-t) f(s)\overline{f(t)}\,dsdt = \lim_{n\to\infty} \sum_{1\leq j,k\leq n} \varphi(t_j - t_k) f(t_j)\overline{f(t_k)}(t_j - t_{j-1})(t_k - t_{k-1}) \geq 0.$$

(ii) For the choice $f(t) = e^{-\epsilon t^2 - itx}$ ($\epsilon > 0$, $x \in \mathbb{R}$), the above inequality takes the form

$$\rho_\epsilon(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(u) e^{-iux-\epsilon u^2/2}\,du \geq 0$$

with

$$\int_{-\infty}^{\infty} e^{itx}\rho_\epsilon(x)\,dx = \varphi(t)e^{-\epsilon t^2/2}.$$

(iii) Let μ_ϵ be a probability measure on $\mathbb{R}$ defined by $\mu_\epsilon(dx) = \rho_\epsilon(x)dx$ and μ be a limit measure on the extended real line $[-\infty,\infty]$. At this point, the probability measure μ may have point masses at $\pm\infty$. To eliminate this possibility, for $a > 0$, consider the integral $\int_{\mathbb{R}^2} dtdx\,\rho_\epsilon(x)e^{-at^2+itx}$, which gives rise to the relation

$$\int_{\mathbb{R}} e^{-x^2/4a}\rho_\epsilon(x)\,dx = \sqrt{\frac{a}{\pi}}\int_{\mathbb{R}} e^{-at^2-\epsilon t^2/2}\varphi(t)\,dt.$$

Since the continuous function $e^{-x^2/4a}$ on $[-\infty,\infty]$ vanishes at $\pm\infty$, the limit $\epsilon \to +0$ yields

$$\int_{\mathbb{R}} e^{-x^2/4a}\,\mu(dx) = \sqrt{\frac{a}{\pi}}\int_{\mathbb{R}} e^{-at^2}\varphi(t)\,dt$$

and then, by taking $a \to \infty$, we have $\mu(\mathbb{R}) = \varphi(0) = 1$. Thus μ is supported by $\mathbb{R}$. Now in the identity

$$\int_{\mathbb{R}} e^{itx-x^2/4a}\,\rho_\epsilon(dx) = \sqrt{\frac{a}{\pi}}\int_{\mathbb{R}} e^{-a(t-u)^2-\epsilon u^2/2}\varphi(u)\,du,$$

we take $\epsilon \to +0$ to get

$$\int_{\mathbb{R}} e^{itx-x^2/4a}\,\mu(dx) = \sqrt{\frac{a}{\pi}}\int_{\mathbb{R}} e^{-a(t-u)^2}\varphi(u)\,du,$$

and the claim is proved by taking $a \to +\infty$. □

3 Stone's Theorem

Definition 3.1. Given a topological space X, let $B(X)$ be the Banach *-algebra of bounded Borel functions on X, which contains $C_b(X)$ as a closed *-subalgebra. A sequence $f_n \in B(X)$ is said to converge boundedly to $f \in B(X)$ if we can find $M > 0$ satisfying $\|f_n\|_\infty \leq M$ and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every $x \in X$.

Theorem 3.2 (Borel Functional Calculus)**.** Given a continuous unitary representation $U(t)$ of the additive group $\mathbb{R}$, there exists exactly one *-homomorphism $B(\mathbb{R}) \ni f \mapsto f(U) \in \mathcal{B}(\mathcal{H})$ satisfying the spectral condition: Let μ_ξ be the representing measure of the positive definite function $(\xi|U(t)\xi)$; $(\xi|U(t)\xi) = \int_{\mathbb{R}} e^{itx}\mu_\xi(dx)$. Then

$$(\xi|f(U)\xi) = \int_{\mathbb{R}} f(x)\,\mu_\xi(dx) \quad \text{for } f \in B(\mathbb{R}).$$

Moreover the *-representation $f \mapsto f(U)$ enjoys the following properties:

(i) $f(U) = U(h)$ for $f = \widehat{h}$ with $h \in L^1(\mathbb{R})$.

(ii) If a sequence $\{f_n\} \subset B(\mathbb{R})$ converges boundedly to $f \in B(\mathbb{R})$, then

$$\lim_{n\to\infty} \|f_n(U)\xi - f(U)\xi\| = 0 \quad \text{for every } \xi \in \mathcal{H}.$$

Proof. We first remark the uniqueness: If there exists a *-subalgebra $A \subset B(\mathbb{R})$ which satisfies the spectral condition in the sense that it admits a *-homomorphism of A into $\mathcal{B}(\mathcal{H})$ fulfilling the spectral condition, then $f \mapsto f(U)$ is unique on A because elements in A are linear combinations of hermitian elements and a hermitian operator $f(U)$ with f a real-valued function is uniquely deternmined by $(\xi|f(U)\xi)$ $(\xi \in \mathcal{H})$.

Now let $\mathcal{A}$ be the set of *-subalgebras of $B(\mathbb{R})$ satisfying the spectral condition and we shall show that $B(\mathbb{R}) \in \mathcal{A}$. We first notice that the *-subalgebra $\widehat{L^1(\mathbb{R})} \subset C_0(\mathbb{R}) \subset B(\mathbb{R})$ belongs to the class $\mathcal{A}$. In fact, if $f = \widehat{h}$ with $h \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)\mu_\xi(dx) = \iint dt\, e^{itx}h(t)\mu_\xi(dx) = \int_{\mathbb{R}} dt\, h(t)(\xi|U(t)\xi) = (\xi|U(h)\xi)$$

shows that $f(U) = U(h)$ for a hermitian $h \in L^1(\mathbb{R})$ and then for an arbitrary $h \in L^1(\mathbb{R})$ by linearity.

Next, given $A \in \mathcal{A}$ and a sequence $\{f_n\} \subset A$ converging boundedly to $f \in B(\mathbb{R})$, the associated sequence $\{f_n(U)\}$ is convergent in $\mathcal{B}(\mathcal{H})$ with respect to the strong operator topology. In fact, the spectral condition on the *-homomorphism $A \ni a \mapsto a(U) \in \mathcal{B}(\mathcal{H})$ enables us to have the expression

$$\|f_m(U)\xi - f_n(U)\xi\|^2 = \int_{\mathbb{R}} \Big(f_m^*(x)f_m(x) + f_n^*(x)f_n(x) - f_m^*(x)f_n(x) - f_n^*(x)f_m(x)\Big)\mu_\xi(dx)$$

for $\xi \in \mathcal{H}$, which shows that $\{f_n(U)\xi\}$ is a Cauchy sequence in $\mathcal{H}$. In view of the estimate

$$\|f_n(U)\xi\|^2 = (\xi|f_n^*f_n(U)\xi) = \int_{\mathbb{R}} |f_n(x)|^2\,\mu_\xi(dx) \leq \|f_n\|_\infty^2\|\xi\|^2 \leq M^2\|\xi\|^2,$$

we can then define $f(U) \in \mathcal{B}(\mathcal{H})$ by

$$\lim_{n\to\infty} f_n(U)\xi = f(U)\xi, \quad \forall \xi \in \mathcal{H},$$

which satisfies the spectral condition by

$$(\xi|f(U)\xi) = \lim_{n\to\infty} (\xi|f_n(U)\xi) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, \mu_\xi(dx) = \int_{\mathbb{R}} f(x)\, \mu_\xi(dx).$$

Now, if we denote by $\overline{A}$ the *-subalgebra of $B(\mathbb{R})$ whose elements are sequential limits of functions in A with respect to bounded convergence, a linear extension $\overline{A} \to \mathcal{B}(\mathcal{H})$ is well-defined by

$$f(U) = \lim_{n\to\infty} f_n(U)$$

and the spectral condition is satisfied for $\overline{A}$. Since the above limit is in the sense of bounded strong convergence, for $f, g \in \overline{A}$, we have

$$\begin{aligned}(fg)(U) &= \lim_{n\to\infty} (f_n g_n)(U) = \lim_{n\to\infty} f_n(U) g_n(U) \\ &= \Big(\lim_{n\to\infty} f_n(U)\Big) \Big(\lim_{n\to\infty} g_n(U)\Big) = f(U)g(U)\end{aligned}$$

and

$$(\xi|f^*(U)\eta) = \lim_{n\to\infty} (\xi|f_n^*(U)\eta) = \lim_{n\to\infty} (f_n(U)\xi|\eta) = (f(U)\xi|\eta).$$

Thus, the extension is in fact *-homomorphic and we have $\overline{A} \in \mathcal{A}$.

By a transfinite induction, we can find a maximal $B \supset \widehat{L^1(\mathbb{R})}$ in $\mathcal{A}$. We then have $\overline{B} = B$ by maximality. Since $C_0(\mathbb{R}) \subset \widehat{L^1(\mathbb{R})}$ and $\mathbb{R}$ is a metric space, B must contain all bounded Borel functions; $B = B(\mathbb{R})$. □

We shall now introduce a projection-valued measure E on $\mathbb{R}$. For a Borel set S in $\mathbb{R}$, $E(S) = 1_S(U)$ is a projection operator and $S = \sqcup_{n\geq 1} S_n$ implies a bounded point-wise convergence $1_S = \sum_n 1_{S_n}$, whence

$$E(S) = \sum_{n\geq 1} E(S_n)$$

in the strong operator topology; $E(S)$ gives a projection-valued measure on $\mathbb{R}$. In terms of the projection-valued measure E, we have the following expression for $f(U) \in \mathcal{B}(\mathcal{H})$ ($f \in B(\mathbb{R})$),

$$f(U) = \int_{\mathbb{R}} f(x) E(dx).$$

Particularly, by choosing $f = \widehat{h}$ with $h \in L^1(\mathbb{R})$, the relation $\int dt\, h(t) U(t) = \iint dt\, h(t) e^{itx} E(dx)$ holds and, by making $h(t)$ converge to $\delta(t-s)$, we get the celebrated Stone's theorem:

$$U(s) = \int_{\mathbb{R}} e^{isx} E(dx).$$

4 Generalizations

The method of the proof described above is ready to be generalized to locally compact abelian groups (mainly due to M.A. Naimark, W. Ambrose and R. Godement independently).

Let G be a locally compact separable abelian group and $B(\widehat{G})$ be the *-algebra (by point-wise operations) of bounded Borel functions on the Pontryagin dual $\widehat{G}$. In the following, the duality pairing is denoted by $\langle g, \chi \rangle$ ($g \in G$, $\chi \in \widehat{G}$) and Haar measures by dg and $d\chi$ respectively.

Theorem 4.1. Given a positive definite continuous function φ on G, we can find a Borel measure μ on $\widehat{G}$ so that

$$\varphi(g) = \int_{\widehat{G}} \langle g, \chi \rangle \mu(d\chi).$$

Theorem 4.2. Given a continuous unitary representation U of G, there exists exactly one *-homomorphism $B(\widehat{G}) \ni f \mapsto f(U) \in \mathcal{B}(\mathcal{H})$ satisfying the spectral condition: Let μ_ξ be the representing measure of the positive definite function $(\xi|U(g)\xi)$; $(\xi|U(g)\xi) = \int_{\widehat{G}} \langle g, \chi \rangle \mu_\xi(d\chi)$. Then

$$(\xi|f(U)\xi) = \int_{\widehat{G}} f(\chi)\, \mu_\xi(d\chi) \quad \text{for } f \in B(\widehat{G}).$$

Moreover the *-representation $f \mapsto f(U)$ enjoys the following properties:

(i) $f(U) = U(h)$ for $f = \widehat{h}$ with $h \in L^1(G)$. Here $U(h) = \int_G h(g)U(g)\, dg$ and $\widehat{h}(\chi) = \int_G \langle g, \chi \rangle\, dg$.

(ii) If a sequence $\{f_n\} \subset B(\widehat{G})$ converges boundedly to $f \in B(\widehat{G})$, then

$$\lim_{n\to\infty} \|f_n(U)\xi - f(U)\xi\| = 0 \quad \text{for every } \xi \in \mathcal{H}.$$

Theorem 4.3. Given a continuous unitary representation U of G on a separable Hilbert space $\mathcal{H}$, we can find a projection-valued measure E on $\widehat{G}$ so that

$$U(g) = \int_{\widehat{G}} \langle g, \chi \rangle E(d\chi).$$

5 Comments

See Hewitt-Ross' Abstract Harmonic Analysis, §33 Notes for historical comments on the subject. Stone's theorem is derived there from Bochner's via the absorbing property of regular representations. A more operator-algebraic approach can be found in Abstract Harmonic Analysis by Loomis, where a Borel extension of the Gelfand transform is utilized.

Although both of these methods are quite universal in its applicability, we have focussed here on the original situation and tried a direct approach to the problem so that the core of proof can be understood plainly.