I REMEMBER CLIFFORD ALGEBRA

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Introduction

These are post-lecture notes on quasi-free states of Clifford C*-algebras.
C. Chevalley, Theory of Lie Groups
A. Ikunishi and Y. Nakagami, Introduction to Operator Algebras (Japanese)
J.T. Ottesen, Infinite Dimensional Groups and Algebras in Quantum Physics
Plymen and Robinson, Spinors in Hilbert Space

1. Clifford C*-algebras

Let V be a real Hilbert space and $V^{\mathbb{C}}$ be the accompanied *-Hilbert space with the *-operation in $V^{\mathbb{C}}$ denoted by $\overline{v}$ for $v \in V^{\mathbb{C}}$ and the inner product satisfying $(\overline{v}|\overline{w}) = (w|v)$ for $v, w \in V^{\mathbb{C}}$. Note that there is a one-to-one correspondence between real Hilbert spaces and *-Hilbert spaces. An element v in a *-Hilbert space is said to be **real** if $\overline{v} = v$. For a linear map $L : V^C \to W^{\mathbb{C}}$ with W another real Hilbert space, its complex conjugate is a linear map $\overline{L} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ defined by $\overline{L}x = \overline{L\overline{x}}$. If $\overline{L} = L$, L is said to be real because it corresponds to a real linear map $V \to W$ by restriction and $\mathbb{C}$-linear extension.

The **Clifford C*-algebra** is a unital C*-algebra $C(V)$ linearly generated by $V^{\mathbb{C}}$ with the relations (the canonical anti-commutation relations, the CAR for short)

(i) $x^* = \overline{x}$ for $x \in V^{\mathbb{C}}$ and
(ii) $x^*y + yx^* = (x|y)1$ for $x,\, y \in V$. Here $(x|y)$ denotes the inner product in $V^{\mathbb{C}}$, which is assumed to be linear in the second variable by our convention.

Let $O(V)$ be the group of orthogonal transformations in V. From structural invariance, there is a natural imbedding of the group $O(V)$ into $\mathrm{Aut}(C(V))$.

If θ is an automorphism of $C(V)$, we regard it as an element in the crossed product extension of $C(V)$. Thus $\theta x\theta^{-1} = \theta(x)$ for $x \in C(V)$ and this covariance relation is kept for L^p objects as well. In particular, the induced unitary on $L^2(C(V))$ is denoted by $\mathrm{Ad}\,\theta$, i.e., this is a unitary operator specified by

$$(\mathrm{Ad}\,\theta)\varphi^{1/2} = (\varphi \circ \theta^{-1})^{1/2}.$$

The generating relations being preserved, the conjugation on generators is extended to a conjugate-linear involutive and multicative operation on $C(V)$, which is again denoted by the bar symbol. The bar operation on $C(V)$ commutes with the star operation and, if we introduce the transposed operation by ${}^t x = (\overline{x})^* = \overline{x^*}$,

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it satisfies the common relations shared by transposed operators. As in the case of automorphisms, this conjugation induces an antiunitary operator in $L^2(C(V))$ so that

$$\overline{x\varphi^{1/2}y} = \overline{x}\,\overline{\varphi}^{1/2}\overline{y}$$

with $\overline{\varphi}(x) = \overline{\varphi(\overline{x})}$.

The orthogonal transformation $v \mapsto -v$ ($v \in V$) gives rise to an involutive automorphism ϵ of $C(V)$, which is called the **parity automorphism**. The associated unitary involution $\mathrm{Ad}\,\epsilon$ on $L^2(C(V))$ is also denoted by Π and referred to as the **parity opertator**. The restriction of the parity operator to an invariant subspace is often denoted by the same symbol. For example, given an even state φ of $C(V)$, $\overline{C(V)\varphi^{1/2}}$, $\overline{\varphi^{1/2}C(V)}$ and $\overline{C(V)\varphi^{1/2}C(V)}$ are invariant with their own parity operators denoted by Π as well.

The C*-algebra $C(V)$ is graded by the decomposition $C(V) = C_0(V) + C_1(V)$ according to parity:

$$C_0(V)C_0(V) + C_1(V)C_1(V) \subset C_0(V), \quad C_0(V)C_1(V) + C_1(V)C_0(V) \subset C_1(V).$$

Proposition 1.1. We have the following natural identification.

$$C(V \oplus W) \cong C(V)\widehat{\otimes}C(W).$$

Here $\widehat{\otimes}$ denotes the graded tensor product (anticommutative for products of odd elements) on graded C*-algebras.

Example 1.2.

(i) If $\dim V = 1$, we can find a real element $c \in V$ so that $c^2 = 1$ and

$$C(V) \cong \mathbb{C} \oplus \mathbb{C} \qquad \text{by} \qquad \lambda\frac{1+c}{2} + \mu\frac{1-c}{2} \leftrightarrow \lambda \oplus \mu.$$

(ii) If $\dim V = 2$, we can find an orthonormal basis $\{a, a^*\}$ of $V^{\mathbb{C}}$ and $C(V)$ is linearly spanned by a, a^*, aa^*, a^*a ($aa^* + a^*a = 1$ and $a^2 = 0$), whence $\dim C(V) \leq 4$. By a matrix representation

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^*a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that $\dim C(V) \geq 4$ and the representation gives an isomorphism $C(V) \cong M_2(\mathbb{C})$.

(iii) We have $(v|v)/2 \leq \|v\|_{C^*}^2 \leq (v|v)$ for $v \in V^{\mathbb{C}}$, where $\|v\|_{C^*}$ denotes the norm as an element in $C(V)$. Consequently, the canonical map $V^{\mathbb{C}} \to C(V)$ is injective and any dense subspace of $V^{\mathbb{C}}$ generates $C(V)$ as a C*-algebra.

Exercise 1. Compute the C*-norm of $v \in V^{\mathbb{C}}$ as an element in $C(V)$.

Proposition 1.3. Let $W^{\mathbb{C}} = \langle a, a^* \rangle \oplus V^{\mathbb{C}}$. Then $C(W) \cong M_2(\mathbb{C}) \otimes C(V)$ by the correspondence

$$a \leftrightarrow a \otimes 1, \qquad v \leftrightarrow (aa^* - a^*a) \otimes v$$

with $v \in V^{\mathbb{C}}$.

Proof. This follows from the fact that $aa^* - a^*a$ is a hermitian unitary, which commutes with elements in V and anticommutes with both of a and a^*. $\square$

Corollary 1.4. Let V be finite-dimensional. If $\dim V = 2n$,

$$C(V) \cong M_{2^n}(\mathbb{C}) \cong \bigotimes_1^n M_2(\mathbb{C})$$

and, if $\dim V = 2n+1$,

$$C(V) \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}).$$

Example 1.5. Here are explicit isomorphisms in the above corollary.

(i) If $\dim V = 2n$ and $\{a_j, a_j^*\}_{1\le j\le n}$ is an orthonormal basis of $V^{\mathbb{C}}$, then each $a_j \in C(V)$ is realized by

$$(aa^* - a^*a)^{\otimes(j-1)} \otimes a \otimes 1^{\otimes(n-j)} \in \bigotimes_1^n M_2(\mathbb{C})$$

under the isomorphism $C(V) \cong \bigotimes_1^n M_2(\mathbb{C})$. If we introduce real orthogonal basis $\{h_j\}_{1\le j\le 2n}$ by $h_{2j-1} = a_j^* + a_j$, $h_{2j} = (a_j^* - a_j)/i$, then $(h_j|h_j) = 2$ and $i^k h_1 h_2 \cdots h_{2k-1} h_{2k} = (a_1 a_1^* - a_1^* a_1) \cdots (a_k a_k^* - a_k^* a_k)$ is realized by a matrix tensor product $(aa^* - a^*a)^{\otimes k} \otimes 1 \in \bigotimes_1^n M_2(\mathbb{C})$.

(ii) If $\dim V = 2n+1$ and $h_0, h_1, \dots, h_{2n}$ be a real orthogonal basis satisfying $(h_j|h_j) = 2$ for $0 \le j \le 2n$, then $h_0 h_1 \dots h_{2n}$ is in the center of $C(V)$ and

$$(h_0 h_1 \dots h_{2n})^* = (-1)^n h_0 h_1 \dots h_{2n}, \qquad (h_0 h_1 \dots h_{2n})^2 = (-1)^n$$

shows that $c = i^n h_0 h_1 \dots h_{2n}$ satisfies $c^* = c$ and $c^2 = 1$. Let $V' = \langle h_1, h_2, \dots, h_{2n}\rangle$. Then $C(V') \oplus C(V') \cong C(V)$ by the map

$$C(V') \oplus C(V') \ni x' \oplus y' \mapsto \frac{1+c}{2} x' + \frac{1-c}{2} y' \in C(V).$$

Example 1.6. The above matrix realization also gives explicit forms of the parity automorphism $\epsilon(x)$ as well as the transposed operation ${}^t x$ on $C(V)$.

(i) $\dim V = 2n$: The parity automorphism is inner and given by $\epsilon(x) = uxu^*$, where $u = i^n h_1 h_2 \cdots h_{2n-1} h_{2n}$ is a hermitian unitary and takes the form $\bigotimes_1^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the matrix realization $\bigotimes_1^n M_2(\mathbb{C})$.

Likewise, the transposed operation is described on $\otimes_1^n M_2(\mathbb{C})$ by

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto U({}^t x_1 \otimes \cdots \otimes {}^t x_n) U^*.$$

Here ${}^t x_j$ denotes the ordinary transposed matrix of $x_j \in M_2(\mathbb{C})$ and

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \cdots .$$

(ii) $\dim V = 2n+1$: From $\epsilon(c) = -c$, we see $\epsilon(x' \oplus y') = \epsilon(y') \oplus \epsilon(x')$.

For the transposed operations, in view of ${}^t c = i^n h_{2n} \cdots h_0 = (-1)^n c$, we see

$${}^t(x' \oplus y') = \begin{cases} {}^t x' \oplus {}^t y' & \text{if } n \text{ is even,} \\ {}^t y' \oplus {}^t x' & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 1.7. A real isometry $\phi : V \to W$ is extended to a unital injective homomorphism of $C(V)$ into $C(W)$.

Proof. Non-trivial is the injectivity but $C(W) = C(V)\widehat{\otimes} C(W \ominus V) \supset C(V)$. □

Exercise 2. Let $h \in V$ be a real element satisfying $h^2 = 1$ and W be the orthogonal complement of $\mathbb{R}h$ in V. Then $W \ni w \mapsto ihw \in C(V)$ is extended to an isomorphism $C(W) \cong C_0(V)$ of C*-algebras so that $C(V)$ can be written as a crossed product $C(W) \rtimes \mathbb{Z}_2$.

2. Free States

Given a state φ of $C(V)$, the positive sesquilinear form

$$C(x, y) = \varphi(x^* y), \quad x, y \in V^{\mathbb{C}}$$

satisfies $C(x, y) + \overline{C}(x, y) = (x|y)$ $(x, y \in V^{\mathbb{C}})$ as a result of the CAR. Let us call a positive form C on $V^{\mathbb{C}}$ a covariance form if it satisfies this relation. Given a covariance form C, we can introduce a positive operator $\mathbf{C}$, called the **covariance operator** of C, by the relation

$$C(x, y) = (x|\mathbf{C}y), \qquad x, y \in V^{\mathbb{C}}$$

and $\mathbf{C}$ satisfies $\mathbf{C} + \overline{\mathbf{C}} = I$, which particularily implies $0 \leq \mathbf{C} \leq 1$. In what follows, we shall use the same symbol C to stand for both of a covariance form and the accompanied covariance operator. Clearly the covariance operator of $\overline{\varphi}$ is $\overline{C} = 1 - C$.

Let $\mathrm{Cov}(V)$ be the set of covariance operators, which is a convex set and allows a natural action of the orthogonal transformation group $\mathrm{O}(V)$.

Proposition 2.1. A covariance operator C is extremal in $\mathrm{Cov}(V)$ if and only if $\sigma(C) \subset \{0, 1/2, 1\}$ and $\dim\ker(C - 1/2) \leq 1$.

Proof. Let

$$C = \int_{0 \leq \lambda \leq 1} \lambda E(d\lambda)$$

be the spectral decomposition. Then the relation $C + \overline{C} = 1$ is equivalent to $\overline{E([a, b])} = E([1 - b, 1 - a])$ for $0 \leq a \leq b \leq 1$. Set $A = \frac{1}{2}E(\{1/2\}) + E((1/2, 1])$ and

$$B = \int_{0 \leq \lambda < 1/2} 2\lambda E(d\lambda) + \frac{1}{2}E(\{1/2\}) + \int_{1/2 < \lambda \leq 1} (2\lambda - 1)\, E(d\lambda).$$

Then these are covariance operators satisfying $C = (A + B)/2$. Therefore C is not extremal if $\sigma(C) \not\subset \{0, 1/2, 1\}$.

Assume now that $\sigma(C) \subset \{0, 1/2, 1\}$ and set $E = E([0, 1/2)) = E(\{0\})$ be the kernel projection of C. If $C = (A + B)/2$ with A, B covariance operators, $\xi \in EV^{\mathbb{C}}$ satisfies $A\xi = -B\xi$ and then $(\xi|A\xi) = -(\xi|B\xi)$. Since A and B are positive, this implies $(\xi|A\xi) = (\xi|B\xi) = 0$ and therefore $A\xi = B\xi = A^{1/2}A^{1/2}\xi = B^{1/2}B^{1/2}\xi = 0$. In other words, $AE = BE = 0$ and, in view of $0 = \overline{AE} = (1 - A)\overline{E}$, we have

$$A(E + \overline{E}) = \overline{E} = B(E + \overline{E}).$$

The condition $2C = A + B$ is then reduced to

$$E_{1/2} = AE_{1/2} + BE_{1/2}$$

with $E_{1/2} = E(\{1/2\})$. Thus, if $E_{1/2} = 0$, A and B are forced to be equal to $C = \overline{E}$, proving the extremality of C.

If $\dim\ker(C-1/2)\geq 2$, we can find a unit vector $\xi\in E_{1/2}V^{\mathbb{C}}$ so that $\xi\perp\xi^*$. Let e be the projection to $\mathbb{C}\xi$ and put

$$A=\frac{1}{2}(E_{1/2}-e-\overline{e})+e+\overline{E},\qquad B=\frac{1}{2}(E_{1/2}-e-\overline{e})+\overline{e}+\overline{E}.$$

Then these are covariance operators satisfying $C=(A+B)/2$ and C cannot be extremal.

Finally assume that $\dim\ker(C-1/2)=1$. Then $AE_{1/2}=aE_{1/2}$ with a real scalar and the condition $A+\overline{A}=1$, together with $\overline{E_{1/2}}=E_{1/2}$, implies $a=1/2$. Thus $A=C$, showing the extremality of C. □

Example 2.2. Let $V^{\mathbb{C}}=\mathbb{C}^2$ with the ordinary conjugation. Then covariance operators are of the form

$$C(t)=\begin{pmatrix}1/2 & -it\\ it & 1/2\end{pmatrix}$$

with $-1/2\leq t\leq 1/2$, whereas orbits by the adjoint action of $O(V)=O(2)$ are $\{C(\pm t)\}$ $(0\leq t\leq 1/2)$. The eigenvalues of $C(t)$ are $\frac{1}{2}\pm t$ and extremal $C(t)$ corresponds to $t=\pm 1/2$.

Definition 2.3. A covariance operator is called a **Fock projection** if it is a projection. A state is called a **Fock state** (resp. **pseudo-Fock**) if C is a projection (resp. $\sigma(C)\subset\{0,1/2,1\}$ and $\dim\ker(C-1/2)=1$).

We shall now describe the GNS representation associated to a Fock state in terms of the so-called Fock space. Let E be a Fock projection in $V^{\mathbb{C}}$. Note that $(EV^{\mathbb{C}})^*=\overline{E}V^{\mathbb{C}}$ is the orthogonal complement of $EV^{\mathbb{C}}$ in $V^{\mathbb{C}}$.

Lemma 2.4. We have

$$C(V)=\overline{\mathbb{C}+C(V)(EV)^*+(EV)C(V)}.$$

Proof. This follows from the so-called normal ordering arrangement. □

Corollary 2.5. A Fock state is determined by the covariance operator.

Proof. Let φ be a state of $C(V)$ with the covariance operator given by a projection E and $a\in(EV^{\mathbb{C}})^*=(1-E)V^{\mathbb{C}}$. Then $\varphi(a^*a)=0$ implies $a\varphi^{1/2}=0$ and the previous decomposition shows

$$\varphi(\lambda+\sum_j C(V)a_j+\sum_j a_j^*C(V))=\lambda+\sum_j(\varphi^{1/2}|C(V)a_j\varphi^{1/2})+\sum_j(a_j\varphi^{1/2}|C(V)\varphi^{1/2})=\lambda.$$

□

Let $\mathcal{C}(V)$ be the (algebraic) subalgebra generated by $V^{\mathbb{C}}$. Then by normal ordering arrangement, we have an orthogonal direct sum $\mathcal{C}(V)\varphi^{1/2}=\bigoplus_{n=0}^{\infty}(EV^{\mathbb{C}})^n\varphi^{1/2}$ with the inner product on $(EV^{\mathbb{C}})^n\varphi^{1/2}$ given by

$$(\xi_1\cdots\xi_n\varphi^{1/2}|\eta_1\cdots\eta_n\varphi^{1/2})=\det(\xi_j|\eta_k)$$

for $\xi_j,\eta_j\in EV^{\mathbb{C}}$. In fact,

$$\begin{aligned}\xi^*\eta_1\cdots\eta_n&=\{\xi^*,\eta_1\}\eta_2\dots\eta_n-\eta_1\{\xi^*,\eta_2\}\eta_3\dots\eta_n+\cdots\\&\qquad+(-1)^{n-1}\eta_1\dots\eta_{n-1}\{\xi^*,\eta_n\}+(-1)^n\eta_1\cdots\eta_n\xi^*\end{aligned}$$

is used to get the recursion relation

$$(\xi_1 \cdots \xi_n \varphi^{1/2}|\eta_1 \cdots \eta_n \varphi^{1/2}) = (\xi_2 \cdots \xi_n \varphi^{1/2}|\xi_1^* \eta_1 \cdots \eta_n \varphi^{1/2}) \\ = \sum (-1)^{j-1} (\xi_1|\eta_j)(\xi_2 \cdots \xi_n \varphi^{1/2}|\eta_1 \cdots \widehat{\eta_j} \cdots \eta_n \varphi^{1/2})$$

which is exactly the recursive formula for determinants (hats indicating deletion).

Conversely, given a Fock projection E, consider the exterior product $\wedge^n \mathcal{K}$ of $\mathcal{K} = EV^{\mathbb{C}}$ which is a Hilbert space completion of the algebraic exterior product with respect to the inner product specified by

$$(\xi_1 \wedge \cdots \wedge \xi_n|\eta_1 \wedge \cdots \wedge \eta_n) = \det(\xi_j|\eta_k).$$

If we denote the left multiplication of $\xi \in \mathcal{K}$ by $a^*(\xi)$, then it admits the adjoint $a(\xi)$ on the algebraic level by

$$a(\xi)(\eta_1 \wedge \cdots \wedge \eta_n) = \sum_{j=1}^{n} (-1)^{j-1} (\xi|\eta_j) \eta_1 \wedge \cdots \wedge \widehat{\eta_j} \wedge \cdots \wedge \eta_n.$$

It is now immediate to check the CAR:

$$a(\xi)a^*(\eta) + a^*(\eta)a(\xi) = (\xi|\eta)1, \qquad a(\xi)a(\eta) + a(\eta)a(\xi) = 0$$

for $\xi, \eta \in \mathcal{K}$. In particular, $a(\xi)$ is bounded with a bound $\|a(\xi)\| \leq (\xi|\xi)^{1/2}$ and we have a *-representation of $C(V)$ on the Hilbert space $e_2^{\wedge}(\mathcal{K}) = \bigoplus_{n \geq 0} \wedge^n \mathcal{K}$, which is referred to as a Fock representation. The initial vector $1 \in \mathbb{C} = \wedge^0 \mathcal{K}$ (called Fock vacuum) is clearly cyclic and characterized up to scalars by the condition $a(\xi)\Omega = 0$ ($\xi \in \mathcal{K}$).

Exercise 3. Let $P_\wedge : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ be the anti-symmetrizer projection. Then $\wedge^n \mathcal{H}$ is identified with the image of $P_\wedge$ so that the wedge product is extended among $\bigoplus_{n \geq 0} \wedge^n \mathcal{H}$'s by the formula

$$\zeta_1 \wedge \cdots \wedge \zeta_r = \sqrt{\frac{(n_1 + \cdots + n_r)!}{n_1! \cdots n_r!}} P_\wedge(\zeta_1 \otimes \cdots \otimes \zeta_r), \quad \zeta_j \in \wedge^{n_j} \mathcal{H}.$$

Proposition 2.6. Given a Fock projection E, there exists a state φ satisfying

$$\varphi(x^* y) = (x|Ey), \quad x, y \in V^{\mathbb{C}}.$$

Moreover, we have a natural identification $\overline{C(V)\varphi^{1/2}} = \bigoplus_{n=0}^{\infty} \wedge^n EV^{\mathbb{C}}$.

Proof. The Fock state is realized as a vector state with respect to the Fock vacuum. □

Proposition 2.7 (Wick's formula)**.** For a Fock state of $C(V)$, $\varphi(x_1 \dots x_{2n+1}) = 0$ and

$$\varphi(x_1 \dots x_{2n}) = \sum_{\text{pairing with } j < k} \pm \prod \varphi(x_j x_k)$$

for $x_1, \dots, x_{2n+1} \in V^{\mathbb{C}}$. Here $\pm$ indicates the signature of permutation which rearranges $x_1 \dots x_{2n}$ into the product of $x_j x_k$'s.

Proof. By multilinearity, we may assume that $x_j \in EV^{\mathbb{C}}$ or $x_j^* \in EV^{\mathbb{C}}$. It is instructive to check the case $n = 2$:

$$\varphi(x_1 x_2 x_3 x_4) = \varphi(x_1 x_2)\varphi(x_3 x_4) - \varphi(x_1 x_3)\varphi(x_2 x_4) + \varphi(x_1 x_4)\varphi(x_2 x_3).$$

If $x_1 \in EV^{\mathbb{C}}$, both sides of the equation vanishes because x_1 is creating.

If $x_1^* \in EV^{\mathbb{C}}$, apply φ to the identity

$$x_1x_2x_3x_4 = \{x_1, x_2\}x_3x_4 - x_2\{x_1, x_3\}x_4 + x_2x_3\{x_1, x_4\} - x_2x_3x_4x_1$$

to get the desired formula in view of

$$\{x_1, x_j\} = x_1x_j + x_jx_1 = \varphi(x_1x_j + x_jx_1)1 = \varphi(x_1x_j)1.$$

The general case is checked by an induction on n: Again the identity is trivial if $x_1 \in EV^{\mathbb{C}}$. Otherwise,

$$\varphi(x_1x_2 \dots x_{2n}) =$$
$$\varphi(x_1x_2)\varphi(x_3x_4 \dots x_{2n}) - \varphi(x_1x_3)\varphi(x_2x_4 \dots x_{2n}) + \dots + \varphi(x_1x_{2n})\varphi(x_2 \dots x_{2n-1})$$

and the induction stage goes up further. □

Remark 1. In the Wick formula, the summation is taken over pairings of $2n$ objects and the total number of such pairings amounts to be

$$\frac{(2n)!}{2^n n!} = (2n-1)!! = (2n-1) \times (2n-3) \times \dots \times 1.$$

Theorem 2.8. Let φ be a Fock state of covariance operator E. Then the GNS-representation of φ is irreducible and satisfies

$$\langle x\varphi^{1/2}y\varphi^{1/2}\rangle = \varphi(x)\varphi(y)$$

for $x, y \in C(V)$. Consequently the correspondence

$$x\varphi^{1/2}y \mapsto x\varphi^{1/2} \otimes \varphi^{1/2}y$$

is extended to a unitary map between $\overline{C(V)\varphi^{1/2}C(V)}$ and $\mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H} = \overline{C(V)\varphi^{1/2}}$.

Proof. The irreducibility is due to the purity of Fock states, which is a consequence of the fact that the state φ is determined by the condition $\varphi(a^*a) = 0$ for $a \in \overline{E}V^{\mathbb{C}}$.

The splitting property of inner product follows from the density operator realization: Let $\mathcal{H} = \overline{C(V)\varphi^{1/2}}$. Then the GNS-representation of $C(V)$ generates $\mathcal{B}(\mathcal{H})$, whence φ is given by $e\,\mathrm{tr} = \mathrm{tr}\,e$ with $e = |\varphi^{1/2})(\varphi^{1/2}|$ and $\varphi^{1/2}$ by $e\sqrt{\mathrm{tr}}$. With this realization, we have

$$\langle x\varphi^{1/2}y\varphi^{1/2}\rangle = \langle xe\sqrt{\mathrm{tr}}ye\sqrt{\mathrm{tr}}\rangle = \mathrm{tr}(xeye) = \varphi(x)\varphi(y).$$

□

Definition 2.9. A state of $C(V)$ is said to be **free** if it satisfies the Wick's formula. Since a free state φ is determined by its covariance operator C, it is denoted like $\varphi = \varphi_C$.

Proposition 2.10. Let $T : V \to W$ be an isometry of real Hilbert spaces with the associated imbedding $C(V) \to C(W)$ denoted by θ. If φ_C is a free state of $C(W)$ with $C \in \mathrm{Cov}(W)$, then $T^*CT \in \mathrm{Cov}(V)$ (T being identified with the natural extension $V^{\mathbb{C}} \to W^{\mathbb{C}}$) and $\varphi_C \circ \theta = \varphi_{T^*CT}$.

Proof. This is just a structural covariance of free states. □

Exercise 4. For a covariance operator C in $V^{\mathbb{C}}$, show that $\overline{\varphi_C} = \varphi_{\overline{C}}$.

Proposition 2.11. Given a covariance operator C on V, there exits a free state φ such that

$$\varphi(x^*y) = (x|Cy), \quad x, y \in V.$$

Proof. Let $W = V \oplus iV$ with the associated *-operation on $V^{\mathbb{C}} \oplus V^{\mathbb{C}}$ given by $(v_1 \oplus v_2)^* = \overline{v_1} \oplus -\overline{v_2}$. Then

$$P = \begin{pmatrix} C & \sqrt{C(1-C)} \\ \sqrt{C(1-C)} & 1-C \end{pmatrix}$$

is a covariance projection on $W^{\mathbb{C}}$, called the quadrature of C. Let φ be the associated Fock state of $C(W)$. Then the restriction of φ to the subalgebra $C(V)$ induced from the imbedding $V \ni v \mapsto v \oplus 0 \in W$ is a free state of covariance operator C. □

Let $\{a, a^*\}$ be a CAR basis. For $0 \leq t \leq 1$, define a density operator by

$$\rho_t = ta^*a + (1-t)aa^*$$

and let φ_t be the associated state of $C^*(a, a^*)$:

$$\varphi_t(a) = \varphi(a^*) = 0, \quad \varphi(a^*a) = t, \quad \varphi(aa^*) = 1-t.$$

Proposition 2.12. Let $W^{\mathbb{C}} = V^{\mathbb{C}} \oplus \langle a, a^* \rangle$ and assume that a covariance operator C on $W^{\mathbb{C}}$ is of theform $C(v \oplus 0) = Bv \oplus 0$ and $Ca = ta$ with $B \in \mathrm{Cov}(V)$. Then, under the isomorphism $C(W) \cong C(V) \otimes C^*(a, a^*)$ in Proposition 1.3, φ_C is the product state $\varphi_B \otimes \varphi_t$.

Proof. For example, if $v = v_1 \dots v_{2n}$ with $v_j \in V^{\mathbb{C}}$, $v \otimes aa^*$ amounts to

$$v_1(aa^* - a^*a) \dots v_{2n}(aa^* - a^*a)aa^* = v_1 \dots v_{2n}aa^*$$

and $\varphi_C(v_1 \dots v_{2n}aa^*) = \varphi_C(v_1 \dots v_{2n})\varphi_C(aa^*)$. □

Corollary 2.13. The free state $\varphi_{1/2}$ of covariance operator $\frac{1}{2}I$ is tracial, i.e., $\varphi_{1/2}(ab) = \varphi_{1/2}(ba)$ for $a, b \in C(V)$. The evaluation by this state is often expressed by an expectation notation: $\varphi_{1/2}(x) = \langle x \rangle$.

Corollary 2.14. Assume that $\dim V = 2n$ and let

$$C = \sum_j \Big(s_j |a_j)(a_j| + (1-s_j)|\overline{a_j})(\overline{a_j}| \Big), \quad 0 \leq s_j \leq \frac{1}{2}$$

be the spectral decomposition of a covariance operator C with $\{a_j, \overline{a_j}\}$ a Fock basis. Then $\varphi_C(x) = \mathrm{trace}(\rho_C x)$ $(x \in C(V))$ with the density operator ρ_C given by

$$\rho_C = \prod_j ((1-s_j)a_j a_j^* + s_j a_j^* a_j).$$

Here we look into a state φ having an extremal covariance operator C. When C is a Fock projection, φ is free and uniquely determined by C. So assume that $\dim\ker(C - 1/2) = 1$. Let $h \in \ker(C - 1/2)$ be a real vector in V, which is normalized by $(h|h) = 2$, i.e. $h^2 = 1$. Then a state φ of covariance operator C is determined by the value $\lambda = \varphi(h)$ and we shall write φ_λ. Clearly $-1 \leq \lambda \leq 1$ and we can show that any value in this range is realized as a state expectation.

To see this, let E be the projection to $\ker(1 - C)$ and set $W = V \ominus \mathbb{R}h$. Choose Fock vectors $v_\pm$ for the covariance operator E in $W^{\mathbb{C}}$:

$$av_\pm = 0, \qquad a \in \overline{E}V^{\mathbb{C}}.$$

Set $\mathcal{H}_\pm = \overline{C(W)\upsilon_\pm}$ and represent $C(W)$ on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ diagonally. We represent h on $\mathcal{H}$ by

$$h(a_1^* \dots a_n^* \upsilon_\pm) = \pm(-1)^n \upsilon_\pm$$

Then the generators of $C(W)$ together with h meet the anticommutation relations of $C(V)$ when represented on $\mathcal{H}$. Thus $C(V)$ is represented on $\mathcal{H}$ and, if we put $\upsilon = \upsilon_+ \oplus \upsilon_-$, we see that

$$(\upsilon|x^* y \upsilon) = (x|Cy)(\upsilon|\upsilon)$$

for $x, y \in V^{\mathbb{C}}$. Moreover

$$(\upsilon|h\upsilon) = \|\upsilon_+\|^2 - \|\upsilon_-\|^2$$

and, if we require $\|\upsilon\|^2 = \|\upsilon_+\|^2 + \|\upsilon_-\|^2 = 1$, then any value $-1 \leq \lambda \leq 1$ is realized by choosing $\|\upsilon_\pm\|^2 = \frac{1 \pm \lambda}{2}$.

It is also immediate to see

$$C(V)'' = \mathcal{B}(\mathcal{H}_+) \oplus \mathcal{B}(\mathcal{H}_-)$$

for $-1 < \lambda < 1$. Let $\varphi_\pm$ be the states corresponding to $\lambda = \pm 1$, then these are pure and $\varphi_\lambda = \frac{1+\lambda}{2}\varphi_+ + \frac{1-\lambda}{2}\varphi_-$.

We shall now rewrite these in terms of φ_C. Since φ_C is described by the density operator $\frac{1}{2}\big(|\upsilon_+)(\upsilon_+| + |\upsilon_-)(\upsilon_-|\big)$ in $\mathcal{B}(\mathcal{H}_+) \oplus \mathcal{B}(\mathcal{H}_-)$, $\varphi_C^{1/2}$ is associated with

$$\frac{|\upsilon_+)(\upsilon_+| + |\upsilon_-)(\upsilon_-|}{\sqrt{2}},$$

whence

$$\varphi_C^{1/2} = \frac{\varphi_+^{1/2} + \varphi_-^{1/2}}{\sqrt{2}} \quad \text{and} \quad h\varphi_C^{1/2} = \frac{\varphi_+^{1/2} - \varphi_-^{1/2}}{\sqrt{2}} = \varphi_C^{1/2} h.$$

Thus

$$\varphi_\pm^{1/2} = \frac{\varphi_C^{1/2} \pm h\varphi_C^{1/2}}{\sqrt{2}}$$

and

$$\varphi_\pm = \varphi_C \pm h\varphi_C = \varphi_C \pm \varphi_C h.$$

For later use, we point out here that the parity automorphism ϵ exchanges the central components $\mathcal{B}(\mathcal{H}_\pm))$ and is not inner on $\overline{C(V)\varphi_C^{1/2}}$. In fact, the canonical implementation Π of ϵ satisfies $\Pi\varphi_\pm^{1/2} = \varphi_\mp^{1/2}$ and hence interchages the central decomposition $\overline{C(V)\varphi_C^{1/2}C(V)} = \overline{C(V)\varphi_P^{1/2}C(V)} + \overline{C(V)\varphi_-^{1/2}C(V)}$.

3. Quadrature

Let ϵ be the parity automorphism of $C(V)$ with $\Pi = \operatorname{Ad}\epsilon$ the associated unitary involution on $L^2(C(V))$ and define a bounded linear operator $\pi(\xi \oplus \eta)$ on $L^2(C(V))$ by

$$\pi(\xi \oplus \eta)\psi^{1/2} = \xi\psi^{1/2} + (\Pi\psi^{1/2})\eta = \xi\psi^{1/2} + (\psi \circ \epsilon)^{1/2}\eta.$$

Here $\xi, \eta \in V^{\mathbb{C}} \subset C(V)$ and ψ is a state of $C(V)$. Then $\pi(\xi \oplus \eta)^* = \pi(\overline{\xi} \oplus -\overline{\eta})$ and

$$\pi(\overline{\xi} \oplus -\overline{\eta})\pi(\xi' \oplus \eta') + \pi(\xi' \oplus \eta')\pi(\overline{\xi} \oplus -\overline{\eta}) = (\xi|\xi')1 + (\eta|\eta')1$$

show that π is extended to a *-representation of $C(V \oplus iV)$, which is referred to as the **quadrate representation** of $C(V \oplus iV)$. Here iV denote the real part of

$V^{\mathbb{C}}$ with respect to the conjugation $\xi \mapsto -\overline{\xi}$. Note here that, for a state φ of even parity,

$$\pi(C(V \oplus iV))\varphi^{1/2} = C(V)\varphi^{1/2}C(V).$$

In particular, for a free state φ_C of covariance operator C, π leaves the closed central subspace

$$L^2(C) \equiv \overline{C(V)\varphi_C^{1/2}C(V)}$$

invariant. Let π_C be the associated subrepresentation of $C(V \oplus iV)$.

We define the **quadrature**† of a state φ of $C(V)$ to be a state Φ of $C(V \oplus iV)$ given by

$$\Phi(x) = (\varphi^{1/2}|\pi(x)\varphi^{1/2}), \quad x \in C(V \oplus iV).$$

Exercise 5. If V is finite-dimensional, then π is irreducible, irrelevant of the parity of $\dim V$.

Lemma 3.1. The following conditions on a covariance operator C are equivalent.

(i) $\ker C = \{0\}$.
(ii) $\ker(1 - C) = \{0\}$.
(iii) $C = (1+e^H)^{-1}$ with H a self-adjoint operator on $V^{\mathbb{C}}$ satisfying $\overline{H} = -H$.

A covariance operator C is said to be **non-degenerate** if it satisfies one of these conditions.

Proof. This is immediate from functional calculus based on a spectral decomposition of C. □

Theorem 3.2 ([1, Theorem 3])**.** Let H be a self-adjoint operator on $V^{\mathbb{C}}$ satisfying $\overline{H} = -H$. Then $C = (1 + e^H)^{-1}$ is a non-degenerate covariance operator and the free state φ_C is a KMS-state of the automorphism group $\{\theta_t\}$ associated to the orthogonal transformations $\{e^{itH}\}_{t\in\mathbb{R}}$:

$$\varphi_C(x\theta_t(y))\Big|_{t=-i} = \varphi_C(yx)$$

for $x, y \in C(V)$.

Proof. Assume that an even state φ satisfies the KMS condition relative to $\{\theta_t\}$:

$$\varphi^{it}\xi\varphi^{-it} = e^{itH}\xi,$$

which is used to get

$$\pi(\xi \oplus \eta)\varphi^{1/2} = (\xi + e^{H/2}\eta)\varphi^{1/2},$$

where η is restricted so that $e^{H/2}\eta$ has a meaning. As a result, we see that $\pi(\xi \oplus \eta)\varphi^{1/2} = 0$ if $\xi + e^{H/2}\eta = 0$. We compare this with the Fock vacuum condition

$$(\overline{P}(\xi \oplus \eta))\varphi_P^{1/2} = 0$$

for the quadrature P of C. From the expression

$$\overline{P} = \begin{pmatrix} \sqrt{1-C} \\ -\sqrt{C} \end{pmatrix} \begin{pmatrix} \sqrt{1-C} & -\sqrt{C} \end{pmatrix},$$

$\xi \oplus \eta$ belongs to the range of $\overline{P}$ if and only if we can find $\zeta \in V$ such that

$$\xi = \sqrt{1-C}\zeta, \qquad \eta = -\sqrt{C}\zeta.$$

†doubling or purification is also used in literature.

This suggests choosing

$$e^H = \frac{1-C}{C} \iff C = \frac{1}{1+e^H}.$$

Moreover, with this choice, C is a covariance operator such that $\ker C = \ker(1-C) = \{0\}$ and we have

$$(\varphi^{1/2}|\pi(x)\varphi^{1/2}) = \varphi_P(x) \quad \text{for } x \in C(V \oplus iV)$$

as a consequence of the characterization of Fock states. By passing to the restriction on $C(V) = C(V \oplus 0) \subset C(V \oplus iV)$, we conclude that $\varphi = \varphi_C$.

Conversely we show that φ_C is a KMS-state of $\{\theta_t\}$. First check the special case:

$$\varphi_C(\xi(e^H\eta)) = (\overline{\xi}|Ce^H\eta) = (\overline{\xi}|\overline{C}\eta) = (\overline{\eta}|C\xi) = \varphi_C(\eta\xi)$$

and

$$\varphi_C((e^H\xi)(e^H\eta)) = (\overline{e^H\xi}|Ce^H\eta) = (\overline{\xi}|e^{-H}Ce^H\eta) = (\overline{\xi}|C\eta) = \varphi_C(\xi\eta).$$

To deal with the general case

$$\varphi_C(\xi_1\dots\xi_m(e^H\eta_1)\dots(e^H\eta_n)) = \varphi_C(\eta_1\dots\eta_n\xi_1\dots\xi_m),$$

we appeal to the Wick formula: only the signature compatibility needs to be checked. Since pairings from $\xi,\dots,\xi_m$ or from $\eta_1,\dots,\eta_n$ produce the same signature, we look at $\dots\xi_j\dots e^H\eta_k\dots$ and $\dots\eta_k\dots\xi_j\dots$. The former amounts to the signature $(-1)^{m-j+k-1}$ while the latter produces $(-1)^{n-k+j-1}$ and these coincide if $(-1)^m = (-1)^n$, i.e., if $m+n$ is even. □

Corollary 3.3. If C is non-degenerate,

$$\varphi_C^{it}\xi\varphi_C^{-it} = (1-C)^{it}C^{-it}\xi$$

for $\xi \in V^{\mathbb{C}}$.

Proposition 3.4. Let C be a covariance operator on $V^{\mathbb{C}}$. Then $\overline{C(V)\varphi_C^{1/2}} = \overline{\varphi_C^{1/2}C(V)}$ if and only if C is non-degenerate.

Proof. If $\ker C(I-C)$ is not trivial, $\overline{C(V)\varphi_C^{1/2}}$ contains an irreducible tensorial component (cf. Proposition 2.12), which prevents interacting left and right multiplications.

If C is non-degenerate, φ_C is a KMS-state, whence we see that left and right GNS-representations give rise to the same Hilbert space. □

Proposition 3.5. Given a covariance operator C on $V^{\mathbb{C}}$, the quadrature of φ_C is equal to φ_P with P the quadrature of C. The representation π_C is therefore irreducible.

Proof. Recall that the Fock vacuum $\varphi_P^{1/2}$ is characterized by the vanishing property under the left multiplication of the range of $\overline{P}$. Since the range of $\overline{P}$ is equal to $\{\sqrt{1-C}\zeta \oplus -\sqrt{C}\zeta; \zeta \in V^{\mathbb{C}}\}$, it suffices to show that

$$\pi_C(\sqrt{1-C}\zeta \oplus -\sqrt{C}\zeta)\varphi_C^{1/2} = 0 \iff (\sqrt{1-C}\zeta)\varphi_C^{1/2} = \varphi_C^{1/2}(\sqrt{C}\zeta) \quad \text{for } \zeta \in V^{\mathbb{C}}.$$

When C is non-degenerate, this follows from the KMS-state analysis discussed in Theorem 3.2.

To deal with the degenerate case, let E be the projection to $\ker C(1-C)$ and write $(1-E)V^{\mathbb{C}} = W^{\mathbb{C}}$ with W a closed real subspace of V. Let φ_W (resp. ψ) be the restriction of φ_C to the C*-subalgebra $C(W) \subset C(V)$ (resp. the C*-subalgebra $C(W^\perp) \subset C(V)$), which is a quasi-free state of the reduced covariance operator $C(1-E)$ (resp. CE). Let u be the unitary operator on the Fock space $\overline{C(W^\perp)\psi^{1/2}}$ defined by

$$u(\eta_1 \cdots \eta_n \psi^{1/2}) = (-1)^n \eta_1 \cdots \eta_n \psi^{1/2} \quad \text{for } \eta_1, \dots, \eta_n \in W^\perp,$$

which implements the parity automorphism of $C(W^\perp)$.

A representation θ of $C(V)$ on $\overline{C(W)\varphi_W^{1/2}} \otimes \overline{C(W^\perp)\psi^{1/2}}$ is then defined by the correspondance

$$\xi + \eta \mapsto \xi \otimes u + 1 \otimes \eta, \quad \xi \in W, \ \eta \in W^\perp$$

on generators, where ξ and η on the right side denote operators by left multiplication. From $u\psi^{1/2} = \psi^{1/2}$ and the Wick formula, we have the equality

$$\begin{aligned}(\varphi_W^{1/2} \otimes \psi^{1/2} | (\xi_1 \cdots \xi_m \otimes u^m \eta_1 \cdots \eta_n)(\varphi_W^{1/2} \otimes \psi^{1/2}))& \\ = \varphi_W(\xi_1 \cdots \xi_m)\psi(\eta_1 \cdots \eta_n) &= \varphi_C(\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n),\end{aligned}$$

which implies that $\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n \varphi_C^{1/2} \mapsto \theta(\xi_1 \cdots \xi_m \eta_1 \cdots \eta_n)(\varphi_W^{1/2} \otimes \psi^{1/2})$ gives rise to an isometry U. Since the operator u is approximated by elements in $C(W^\perp)$ on $\overline{C(W^\perp)\psi^{1/2}}$ thanks to the irreducibility of representation, U is in fact surjective and θ is extended to an isomorphism $C(V)'' \to C(W)'' \otimes \mathcal{B}(\overline{C(W^\perp)\psi^{1/2}})$ of von Neumann algebras so that $\varphi_C = (\varphi_W \otimes \psi)\theta$, which in turn induces an isometric isomorphism

$$\Theta : \overline{C(V)\varphi_C^{1/2}C(V)} \to \overline{C(W)\varphi_W^{1/2}C(W)} \otimes \overline{C(W^\perp)\psi^{1/2}C(W^\perp)}$$

by the relation

$$\Theta(x\varphi_C^{1/2}y) = \theta(x)(\varphi_W^{1/2} \otimes \psi^{1/2})\theta(y), \quad x, y \in C(V).$$

Now, for $\xi + \eta \in V^{\mathbb{C}} = (1-E)V^{\mathbb{C}} + EV^{\mathbb{C}}$, in view of $((1-C)\eta)\psi^{1/2} = 0 = ((C\eta)^*\psi^{1/2})^* = \psi^{1/2}(C\eta)$, we see that

$$\begin{aligned}\Theta(\varphi_C^{1/2}(\sqrt{C}(\xi+\eta)) &= (\varphi_W^{1/2} \otimes \psi^{1/2})\theta(\sqrt{C}(\xi+\eta)) \\ &= (\varphi_W^{1/2} \otimes \psi^{1/2})(\sqrt{C}\xi \otimes u + 1 \otimes \sqrt{C}\eta) \\ &= \varphi_W^{1/2}(\sqrt{C}\xi) \otimes \psi^{1/2} = (\sqrt{1-C}\xi)\varphi_W^{1/2} \otimes \psi^{1/2} \\ &= \theta(\sqrt{1-C}(\xi+\eta))(\varphi_W^{1/2} \otimes \psi^{1/2}) \\ &= \Theta(\sqrt{1-C}(\xi+\eta)\varphi_C^{1/2}).\end{aligned}$$

□

Theorem 3.6 (Dichotomy)**.** *Let C and D be covariance operators on a *-Hilbert space $V^{\mathbb{C}}$ with P and Q their quadratures. Then $L^2(C) \perp L^2(D)$ unless $L^2(C) = L^2(D)$. Moreover, in either case, we have*

$$(\varphi_P^{1/2} | \varphi_Q^{1/2}) = (\varphi_C^{1/2} | \varphi_D^{1/2})^2.$$

Proof. Since the quadrate representation π is irreducible on both of $L^2(C)$ and $L^2(D)$, they are either unitarily equivalent or disjoint as representations of $C(V \oplus iV)$.

Let z_C be the projection to $\overline{\pi(C(V \oplus iV))\varphi_C^{1/2}} = L^2(C)$ and similarly for z_D. Since z_C is in the commutant of the right representation of $C(V)$ on $L^2(C)$, it is approximated by the left multiplication of $C(V)$, i.e., by elements in $\pi(C(V \oplus 0))$. Thus, if a unitary $U : L^2(C) \to L^2(D)$ intertwines π, then

$$U(\xi) = U(z_C\xi) = z_C U(\xi), \quad \xi \in L^2(C)$$

shows that $z_C = z_D$, i.e., $L^2(C) = L^2(D)$. Moreover, as quadratures of φ_C and φ_D, φ_P and φ_Q are vector states associated with $\varphi_C^{1/2}$ and $\varphi_D^{1/2}$ through a single irreducible representation π on $L^2(C) = L^2(D)$, whence

$$(\varphi_P^{1/2}|\varphi_Q^{1/2}) = \mathrm{trace}\Big(|\varphi_C^{1/2})(\varphi_C^{1/2}|\,|\varphi_D^{1/2})(\varphi_D^{1/2}|\Big) = (\varphi_C^{1/2}|\varphi_D^{1/2})^2.$$

Otherwise, π_C and π_D are disjoint, i.e., $z_C \perp z_D$, which implies $(\varphi_P^{1/2}|\varphi_Q^{1/2}) = 0 = (\varphi_C^{1/2}|\varphi_D^{1/2})^2$. ☐

4. Commutant Theorem

We shall now present Araki's commutant theorem [1]. Let $K^{\mathbb{C}}$ be a *-Hilbert space with P a Fock projection and write $\omega = \varphi_P$. Set $\mathcal{H} = \overline{C(K)\omega^{1/2}}$ and let π be a representation of $C(K)$ on $\mathcal{H}$ by left multiplication.

Given a real subspace $V \subset K$, the orthogonal complement $V^\perp$ anticommutes with V in $C(K)$ and, after simple parity modification, we can describe the commutant of $\pi(C(V))$ in terms of $C(V^\perp)$. To see this, introduce a representation π_ϵ of $C(K)$ on $\mathcal{H}$ by

$$\pi_\epsilon(\xi) = i\pi(\xi)\mathrm{Ad}\,\epsilon, \quad \xi \in K,$$

which in fact gives a representation of $C(K)$ because $\pi_\epsilon(\xi)^* = \pi_\epsilon(\overline{\xi})$ and $\{\pi_\epsilon(\xi)\}$ satisfies the correct anticommutation relations. Recall that $\mathrm{Ad}\,\epsilon$ is a unitary involution on $L^2(C(K))$ specified by $(\mathrm{Ad}\,\epsilon)\varphi^{1/2} = (\varphi \circ \epsilon)^{1/2}$ and leaves $\mathcal{H}$ invariant. We then have

$$\pi_\epsilon(\overline{P}K^{\mathbb{C}})\omega^{1/2} = 0, \quad [\pi(V), \pi_\epsilon(V^\perp)] = 0$$

and

$$\pi_\epsilon(\eta_1 \dots \eta_n)\omega^{1/2} = i^{n^2}\eta_1 \dots \eta_n\omega^{1/2}$$

for $\eta_1, \dots, \eta_n \in K^{\mathbb{C}}$.

Remark 2. In the original treatment [1],

$$\pi'(x) = \pi_\epsilon(\alpha(x))$$

is used in stead of our π_ϵ, where $\alpha \in \mathrm{Aut}(C(K))$ is associated to the orthogonal transformation $-i(P - \overline{P})$ of K.

Theorem 4.1 (Araki[1]). For a real subspace V of K, we have

$$\pi(V)' = \pi_\epsilon(V^\perp)''.$$

Corollary 4.2. If V and W are closed real subspaces of K,

$$\pi(V)'' \cap \pi(W)'' = \pi(V \cap W)''.$$

Proof.
$\pi(V)'\vee\pi(W)' = \pi_\epsilon(V^\perp)''\vee\pi_\epsilon(W^\perp)'' = \pi_\epsilon(V^\perp+W^\perp)'' = \pi((V^\perp+W^\perp)^\perp)' = \pi(V\cap W)'$.
□

For a proof of the commutant theorem, it is instructive to consider the case $K = V \oplus iV$ with P the quadrature of a covariance operator S on V. If we further assume $\ker S = \ker(1 - S) = \{0\}$, then the argument in the KMS-state analysis shows that π on $\mathcal{H}$ is identified with the quadrate representation π on $\overline{C(V)\varphi_S^{1/2}C(V)}$ in such a way that $\omega^{1/2}$ corresponds to $\varphi_S^{1/2}$. Here V is identified with $V \oplus 0 \subset K$, $V^\perp$ with $0 \oplus iV$ and

$$\pi_\epsilon(0 \oplus \xi)\psi^{1/2} = i\pi(0 \oplus \xi)(\psi \circ \epsilon)^{1/2} = i\psi^{1/2}\xi$$

if $\psi^{1/2} \in \overline{C(V)\varphi_S^{1/2}C(V)}$. Thus $\pi(C(V\oplus 0))$ is given by left multiplication, whereas $\pi_\epsilon(C(0 \oplus iV))$ by right multiplication on the Hilbert space $\overline{C(V)\varphi_S^{1/2}C(V)}$. It is then a basic fact in modular theory that these generate commutants of each other and we are done.

Let S be the reduction of P to the subspace $V^{\mathbb{C}} \subset K^{\mathbb{C}}$, $L^{\mathbb{C}} = \ker(S)+\ker(1-S)$, and set $W = V \ominus L$ with T the further reduction of S to $W^{\mathbb{C}}$. By an angle operator presentation, K is, up to orthogonal equivalence, of the form

$$K = L \oplus W \oplus iW \oplus M$$

so that $V = L \oplus W \oplus 0 \oplus 0$ and

$$P = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & T & \sqrt{T(1-T)} & 0 \\ 0 & \sqrt{T(1-T)} & 1-T & 0 \\ 0 & 0 & 0 & f \end{pmatrix}.$$

Here e is the projection to the subspace $\ker(1 - S)$, which is a Fock projection for L, and f is a Fock projection for M.

Lemma 4.3. Let X and Y be Fermionic phase spaces with E and F their Fock projections. Let π^X and π^Y be representations of $C(X)$ and $C(Y)$ on Hilbert spaces $\mathcal{H}_X = \overline{C(X)\varphi_E^{1/2}}$, $\mathcal{H}_Y = \overline{C(Y)\varphi_F^{1/2}}$. We denote the parity automorphisms of $C(X)$ and $C(Y)$ by ϵ_X and ϵ_Y respectively.

Assume that $K = X \oplus Y$ with $P = E \oplus F$. Then there exists a unitary map $U : \mathcal{H}_X \otimes \mathcal{H}_Y \to \mathcal{H}$ such that

$$U(\varphi_E^{1/2} \otimes \varphi_F^{1/2}) = \varphi_P^{1/2}, \quad U(\mathrm{Ad}\epsilon_X \otimes \mathrm{Ad}\epsilon_Y)U^* = \mathrm{Ad}\epsilon$$

and

$$U^*\pi(x)\pi_\epsilon(y)U = \pi^X(x) \otimes \pi_\epsilon^Y(y), \quad x \in C(X), y \in C(Y).$$

Proof. Since $\pi(x)$ and $\pi_\epsilon(y)$ commute and $\pi(\overline{E}X)\varphi_P^{1/2} = \{0\} = \pi_\epsilon(\overline{F}Y)\varphi_P^{1/2}$, it suffices to check the unitarity for $x = \xi_1 \dots \xi_m$ and $y = \eta_1 \dots \eta_n$ with $\xi_1, \dots, \xi_m \in EX$ and $\eta_1, \dots, \eta_n \in FY$. This follows from

$$\begin{aligned} \|\pi(x)\pi_\epsilon(y)\varphi_P^{1/2}\|^2 &= \varphi_P(\eta_n^* \dots \eta_1^*\xi_m^* \dots \xi_1^*\xi_1 \dots \xi_m\eta_1 \dots \eta_n) \\ &= \det((\xi_i|\xi_j)) \det(\eta_k|\eta_l)) \\ &= \|\pi^X(x)\varphi_E^{1/2}\|^2 \, \|\pi_\epsilon^Y(y)\varphi_F^{1/2}\|^2. \end{aligned}$$

□

We now prove the commutant theorem in two steps:

(i) Let $X = W \oplus iW$ and $Y = M$. Then

$$\pi(W \oplus 0 \oplus 0)'' = U(\pi^X(W \oplus 0)'' \otimes 1)U^*.$$

In view of

$$\pi_\epsilon(0 \oplus 0 \oplus M)'' = U(1 \otimes \pi_\epsilon^Y(M)'')U^* = U(1 \otimes \mathcal{B}(\mathcal{H}_Y))U^*$$

and

$$\pi_\epsilon(0 \oplus \eta \oplus 0) = i\pi(0 \oplus \eta \oplus 0)\mathrm{Ad}\epsilon = U(\pi_\epsilon^X(0 \oplus \eta) \otimes \mathrm{Ad}\epsilon_Y)U^*$$

for $\eta \in W^{\mathbb{C}}$, we observe that

$$\pi_\epsilon(0 \oplus \eta_1 \oplus 0) \dots \pi_\epsilon(0 \oplus \eta_n \oplus 0) = U(\pi_\epsilon^X(\eta_1 \dots \eta_n) \otimes (\mathrm{Ad}\epsilon_Y)^n)U^*,$$

which, together with $\pi_\epsilon(0 \oplus 0 \oplus M)$, generates $U(\pi_\epsilon^X(0 \oplus iW)'' \otimes \mathcal{B}(\mathcal{H}_Y))U^*$, i.e.,

$$\pi_\epsilon(0 \oplus iW \oplus M)'' = U(\pi_\epsilon^X(0 \oplus iW)'' \otimes \mathcal{B}(\mathcal{H}_Y))U^*.$$

Since $\pi^X(W \oplus 0)' = \pi_\epsilon^X(0 \oplus iW)''$ from the preliminary discussion, we get

$$\pi^{X \oplus M}(W \oplus 0 \oplus 0)' = \pi_\epsilon^{X \oplus M}(0 \oplus iW \oplus M)''.$$

(ii) This time, let $X = L$ and $Y = W \oplus iW \oplus M$. From $\pi(L)'' = U(\mathcal{B}(\mathcal{H}_X) \otimes 1)U^*$ and the expression

$$\pi(\eta) = -i\pi_\epsilon(\eta)\mathrm{Ad}\epsilon = -iU(1 \otimes \pi_\epsilon^Y(\eta))(\mathrm{Ad}\epsilon_X \otimes \mathrm{Ad}\epsilon_Y)U^* = U(\mathrm{Ad}\epsilon_X \otimes \pi^Y(\eta))U^*$$

for $\eta \in 0 \oplus W \oplus 0 \oplus 0$, one sees that

$$\begin{aligned}
\pi(L \oplus W \oplus 0 \oplus 0)'' &= U(\mathcal{B}(\mathcal{H}_X) \otimes \pi^Y(W \oplus 0 \oplus 0)'')U^* \\
&= U(\mathcal{B}(\mathcal{H}_X) \otimes \pi_\epsilon^Y(0 \oplus iW \oplus M)')U^* \\
&= U(1 \otimes \pi_\epsilon^Y(0 \oplus iW \oplus M))'U^* \\
&= \pi_\epsilon(0 \oplus 0 \oplus iW \oplus M)'
\end{aligned}$$

and we are done. Here we used the result in the first step at the second line.

5. Quadratic Quantization

Let H be a finite-rank operator on $V^{\mathbb{C}}$. Choose an orthonormal basis $\{\xi_j\}$ and write

$$H = \sum_{j,k} h_{j,k} |\xi_j)(\xi_k|.$$

We define the **quadratic quantization** of H by

$$Q(H) = \frac{1}{2} \sum_{j,k} h_{jk} \xi_j \xi_k^*,$$

which is a quadratic element in the Clifford algebra $C(V)$ and is independent of the choice of orthonormal bases. It then follows that, if $H = \sum_j |\xi_j)(\eta_j|$,

$$Q(H) = \frac{1}{2} \sum_j \xi_j \eta_j^*.$$

Lemma 5.1.

(i)
$$Q(H + {}^tH) = \frac{1}{2}\text{trace}(H) = \frac{1}{2}\sum_j (\eta_j|\xi_j).$$

(ii)
$$[Q(H), \xi] = \frac{1}{2}(H - {}^tH)\xi, \quad \xi \in V^{\mathbb{C}}.$$

Let $o(V) = \{H = \overline{H} = -{}^tH\}$ be the (algebraic) Lie algebra of $O(V)$ and $o(V)^{\mathbb{C}} = \{H = -{}^tH\}$ be its complexification.

Lemma 5.2.

(i) Let $H \in o(V)$. Then $\|Q(H)\| = \frac{1}{4}\|H\|_1$.

(ii) Let $H \in o(V)^{\mathbb{C}}$. Then
$$\frac{1}{8}\|H\|_1 \leq \|Q(H)\| \leq \frac{1}{2}\|H\|_1.$$

Proof. Since $H = \overline{H} = -H^*$, we can find an orthonormal sysytem $\{a_j, a_j^*\}_{1 \leq n}$ and a sequence $\{h_j\}_{1 \leq j \leq n}$ of positive reals such that $H = i\sum_j h_j(|a_j)(a_j| - |a_j^*)(a_j^*|)$, whence
$$Q(H) = \frac{i}{2}\sum_{j=1}^n h_j(a_j a_j^* - a_j^* a_j).$$
Then the spectrum of $2iQ(H)$ is given by
$$\left\{ \sum_{j=1}^n \epsilon_j h_j; \epsilon_j \in \{\pm 1\} \right\}$$
and we have $4\|Q(H)\| = 2\sum_{j=1}^n h_j = \|H\|_1$.

Let $H = A + iB$ with A, $B \in o(V)$. Then $2A = H - H^*$, $2iB = H + H^*$ and
$$\|Q(H)\| \leq \|Q(A)\| + \|Q(B)\| \leq \frac{1}{8}\left(\|H - H^*\|_1 + \|H + H^*\|_1\right) \leq \frac{1}{2}\|H\|_1.$$
For the lower estimate of $\|Q(H)\|$,
$$8\|Q(H)\| \geq 8\max(\|Q(A)\|, \|Q(B)\|) = 2\max(\|A\|_1, \|B\|_1) \geq \|A\|_1 + \|B\|_1$$
is combined with $\|A + iB\|_1 = \text{trace}(U^*(A + iB)) \leq \|A\|_1 + \|B\|_1$ to get $\|H\|_1 \leq 8\|Q(H)\|$. □

Definition 5.3. Let $o_1(V)^{\mathbb{C}}$ be the Banach space of trace class operators H satisfying $H = -{}^tH$ and set $o_1(V) = \{H \in o_1(V)^{\mathbb{C}}; H^* = -H\}$. Then the quantization map is extended to $o_1(V)^{\mathbb{C}} \to C(V)$ by continuity, which is again denoted by Q.

Proposition 5.4. The restriction of Q to $o_1(V)^{\mathbb{C}}$ is injective and, for H, $H' \in o_1(V)^{\mathbb{C}}$, we have
$$[Q(H), \xi] = H\xi, \quad Q(H)^* = Q(H^*), \quad [Q(H), Q(H')] = Q([H, H']).$$

Proposition 5.5. For $H \in o_1(V)^{\mathbb{C}}$, we have
$$\tau(e^{Q(H)}) = \sqrt{\det\left(\frac{e^{H/2} + e^{-H/2}}{2}\right)} = \sqrt{\det\left(\frac{1 + e^H}{2}\right)}.$$
Here $\tau = \varphi_{1/2}$ denotes the normalized trace of $C(V)$.

Proof. To establish the formula, it suffices to check for a hermitian H in view of analytic continuations. With this restriction we can find an orthonormal basis of the form $\{a_j, \overline{a_j}\}$ (a Fock basis) so that $H = \sum_j h_j(|a_j)(a_j| - |\overline{a_j})(\overline{a_j}|)$. Then $Q(H) = \frac{1}{2}\sum_j h_j(a_j a_j^* - a_j^* a_j)$ and

$$e^{Q(H)} = \prod_j \left(e^{h_j/2} a_j a_j^* + e^{-h_j/2} a_j^* a_j\right)$$

(note that $a_j a_j^*$ commute for different j) is evaluated by τ to get

$$\tau(e^{Q(H)}) = \prod_j \cosh(h_j/2) = \sqrt{\det\left(\frac{e^{H/2} + e^{-H/2}}{2}\right)}.$$

The second expression follows from this in view of the fact that $\mathrm{trace}(H) = 0$ and hence $\det(e^{H/2}) = 1$. □

Corollary 5.6. Assume that V is finite dimensional and that $S = (1 + e^H)^{-1}$ with $H \in io(V)$. Then

$$\tau(e^{Q(H)}) = \frac{1}{\sqrt{\det(2S)}} = \frac{1}{\sqrt{\det(2\overline{S})}}$$

and the density operator for φ_S is given by

$$\rho_S = \frac{1}{\mathrm{trace}(e^{Q(H)})} e^{Q(H)}.$$

6. Finite-dimensional Analysis

We work with a finite-dimensional V in this section. Let $H, K \in io(V)$ and set $S = (1 + e^H)^{-1}$, $T = (1 + e^K)^{-1} \in \mathrm{Cov}(V)$. Since density operators are given by $e^{Q(H)}$ and $e^{Q(K)}$ up to scalar multiplication, we see that

$$\varphi_S = \tau(e^{Q(H)})^{-1} e^{Q(H)} \tau, \quad \varphi_T = \tau(e^{Q(K)})^{-1} e^{Q(K)} \tau$$

and therefore

$$\langle \varphi_S^t \varphi_T^{1-t} \rangle = \tau(e^{Q(H)})^{-t} \tau(e^{Q(K)})^{-(1-t)} \tau(e^{tQ(H)} e^{(1-t)Q(K)})$$

for $0 \leq t \leq 1$.

We now assume that H and K are small and write

$$e^{tH} e^{(1-t)K} = e^M$$

with $M = M(t, H, K) \in o(V)^{\mathbb{C}}$ depending on H and K analytically. Since the quadratic quantization $Q : o(V)^{\mathbb{C}} \to C(V)$ is a Lie algebra homomorphism, we have

$$e^{tQ(H)} e^{(1-t)Q(K)} = e^{Q(M)}$$

and hence

$$\tau(e^{tQ(H)} e^{(1-t)Q(K)})^2 = \tau(e^{Q(M)})^2 = \det\left(\frac{1 + e^M}{2}\right) = \det\left(\frac{1 + e^{tH} e^{(1-t)K}}{2}\right).$$

Consequently

$$\begin{aligned}
\langle \varphi_S^t \varphi_T^{1-t} \rangle^2 &= \frac{\det\left(\frac{1+e^{tH}e^{(1-t)K}}{2}\right)}{\det\left(\frac{1+e^H}{2}\right)^t \det\left(\frac{1+e^K}{2}\right)^{1-t}} \\
&= \det\left(\left(\frac{2}{1+e^H}\right)^t \left(\frac{1+e^{tH}e^{(1-t)K}}{2}\right)\left(\frac{2}{1+e^K}\right)^{1-t}\right) \\
&= \det\left(\left(1+e^H\right)^{-t}\left(1+e^{tH}e^{(1-t)K}\right)\left(1+e^K\right)^{t-1}\right) \\
&= \det\left(S^t\left(1+\frac{(1-S)^t}{S^t}\frac{(1-T)^{1-t}}{T^{1-t}}\right)T^{1-t}\right) \\
&= \det\Bigl(S^tT^{1-t}+(1-S)^t(1-T)^{1-t}\Bigr).
\end{aligned}$$

Since both sides in the above identity are analytic in H and K, the relation remains valid for arbitrary H and K in $io(V)$, i.e., as far as they are non-degenerate in the sense that $\ker S = \{0\} = \ker T$. Finally, general S and T are approximated by non-degenerate ones (φ_C is continuous in C in view of Corollary 2.14) and we have proved the following:

Lemma 6.1. Assume that V is finite-dimensional and let S, $T \in \mathrm{Cov}(V)$ be covariance operators. Then, for $0 \leq t \leq 1$,

$$\langle \varphi_S^t \varphi_T^{1-t} \rangle = \sqrt{\det\Bigl(S^tT^{1-t}+(1-S)^t(1-T)^{1-t}\Bigr)}.$$

7. Infinite-dimensional Analysis

Given covariance operators S and T, set

$$M_t(S,T) = S^tT^{1-t}+(1-S)^t(1-T)^{1-t}$$

for $0 \leq t \leq 1$. We want to prove

$$\langle \varphi_S^t \varphi_T^{1-t} \rangle^4 = \det\Bigl(M_t(S,T)M_t(S,T)^*\Bigr),$$

but the case of general exponents is not established yet. We just point out the following fact for a general $0 \leq t \leq 1$ and restrict ourselves to the case $t = 1/2$ in what follows.

Lemma 7.1. The operator $M_t(S,T)$ is a contraction.

Proof. This follows from the expression

$$M_t(S,T) = \begin{pmatrix} S^t & (1-S)^t \end{pmatrix} \begin{pmatrix} T^{1-t} \\ (1-T)^{1-t} \end{pmatrix}.$$

□

Lemma 7.2. Let P and Q be Fock projections. If $P-Q$ is not in the Hilbert-Schmidt class, then $(\varphi_P^{1/2}|\varphi_Q^{1/2}) = 0$.

Proof. Take a countable real orthonormal family $\{\xi_n\}_{n\geq 1}$ such that

$$\sum_{n\geq 1} \|(P-Q)\xi_n\|^2 = \infty$$

Let V_∞ be the closed subspace of V generated by

$$\{\xi_n, P\xi_n, Q\xi_n, PQ\xi_n, QP\xi_n, PQP\xi_n, QPQ\xi_n, \dots\}_{n\geq 1}.$$

Then V_∞ is separable and we may assume that $\{\xi_n\}_{n\geq 1}$ is an orthonormal basis of V_∞ from the outset. Let V_n be the subspace generated by $\{\xi_k\}_{1\leq k\leq n}$ and E_n be the projection to V_n. The sequence $\{E_n\}$ then converges to the projection E_∞ for V_∞. Since V_∞ is invariant under P and Q, E_∞ commutes with P and Q, whence $P_\infty = PE_\infty = E_\infty P$ and $Q_\infty = QE_\infty = E_\infty Q$ are Fock projections of V_∞ satisfying

$$\operatorname{trace}((P_\infty - Q_\infty)^2) = \sum_{n\geq 1} \|(P_\infty - Q_\infty)\xi_n\|^2 = \sum_{n\geq 1} \|(P-Q)\xi_n\|^2 = \infty.$$

Now set $P_n = E_n P E_n$ and $Q_n = E_n Q E_n$, which are covariance operators on V_n but not quadrate generally. Let $M_n = P_n^{1/2} Q_n^{1/2} + \overline{P_n}^{1/2}\overline{Q_n}^{1/2}$. Then contractions $C_n = E_n - M_n M_n^*$ converges in the strong operator topology to

$$\begin{aligned}
C_\infty &= E_\infty - M_\infty M_\infty^* \\
&= E_\infty - (P_\infty Q_\infty + \overline{P_\infty Q_\infty})(Q_\infty P_\infty + \overline{Q_\infty P_\infty}) \\
&= E_\infty - P_\infty Q_\infty P_\infty - \overline{P_\infty Q_\infty P_\infty} \\
&= P_\infty (P_\infty - Q_\infty)^2 + \overline{P_\infty}(\overline{P_\infty} - \overline{Q_\infty})^2 \\
&= (P_\infty - Q_\infty)^2.
\end{aligned}$$

By the formula in the finite-dimensional analysis,

$$(\varphi_P^{1/2}|\varphi_Q^{1/2})^4 \leq (\varphi_{P_n}^{1/2}|\varphi_{Q_n}^{1/2})^4 = \det(E_n - C_n) = \det(E_\infty - C_n),$$

which converges to 0 from the condition that $\operatorname{trace}(C_\infty) = \infty$ (cf. Appendix B). □

Theorem 7.3. Let P and Q be Fock projections on V. Then

$$(\varphi_P^{1/2}|\varphi_Q^{1/2})^4 = \det(1 - (P-Q)^2) = \det(MM^*)$$

with $M = PQ + \overline{PQ}$.

Proof. In view of the previous lemma, we need to establish the formula under the assumption that $P - Q$ is in the Hilbert-Schmidt class. Then, by Appendix A, we can find a canonical system $\{a_j, b_j, a_j^*, b_j^*\}_{j\geq 1}$ in V such that

$$Qa_j = Qb_j = 0, \quad Qa_j^* = a_j^*, \quad Qb_j^* = b_j^*$$

$$Pa_j^* = c_j a_j^* + \sqrt{c_j(1-c_j)}\, b_j, \qquad Pb_j^* = c_j b_j^* - \sqrt{c_j(1-c_j)}\, a_j,$$

$$Pa_j = (1-c_j)a_j - \sqrt{c_j(1-c_j)}\, b_j^*, \qquad Pb_j = (1-c_j)b_j + \sqrt{c_j(1-c_j)}\, a_j^*$$

with $0 < c_j < 1$ and $P\wedge Q + (1-P)\wedge(1-Q) + P\wedge(1-Q) + (1-P)\wedge Q$ is the projection to the orthogonal complement of $\{a_j, b_j, a_j^*, b_j^*\}$.

Note that the condition $\|P - Q\|_{HS} < \infty$ is equivalent to requiring that

$$\sum_j (1 - c_j) < \infty$$

and $P\wedge(1-Q) + (1-P)\wedge Q$ is of finite rank.

Now introduce a sequence of Fock projections $\{P_n\}$ by $P_n\xi = \xi$ for $\xi \in (P \wedge Q + P \wedge (1-Q))V$, $P_n\xi = 0$ for $\xi \in ((1-P) \wedge Q + (1-P) \wedge (1-Q))V$,

$$P_n a_j^* = c_j a_j^* + \sqrt{c_j(1-c_j)}b_j, \qquad P_n b_j^* = c_j b_j^* - \sqrt{c_j(1-c_j)}a_j,$$

$$P_n a_j = (1-c_j)a_j - \sqrt{c_j(1-c_j)}b_j^*, \qquad P_n b_j = (1-c_j)b_j + \sqrt{c_j(1-c_j)}a_j^*$$

for $1 \leq j \leq n$, and

$$P_n a_j^* = a_j^*, \qquad P_n b_j^* = b_j^*, \qquad P_n a_j = P_n b_j = 0$$

for $j > n$. Then $(P-Q)^2 - (P_n - Q)^2 = 1 - c_j$ on the subspace $\langle a_j, b_j, a_j^*, b_j^* \rangle$ $(j > n)$ and vanishes otherwise. Thus $(P_n - Q)^2 \leq (P-Q)^2$ and

$$\lim_{n\to\infty} \mathrm{trace}((P-Q)^2 - (P_n - Q)^2) = 0.$$

By repeated applications of Proposition 2.12 or a combination of this with Lemma 6.1 (note that the quadrature of P is $P \oplus 1 - P$), we see that

$$(\varphi_{P_n}^{1/2}|\varphi_Q^{1/2})^4 = \det(1 - (P_n - Q)^2)$$

and then, by the continuity of determinants in the trace norm, we obtain

$$\begin{aligned}(\varphi_P^{1/2}|\varphi_Q^{1/2})^4 &= \lim_{n\to\infty}(\varphi_{P_n}^{1/2}|\varphi_Q^{1/2})^4 = \lim_{n\to\infty}\det(1-(P_n-Q)^2) \\ &= \det(1-(P-Q)^2).\end{aligned}$$

This coincides with $\det(MM^*)$ in view of

$$MM^* = PQP + \overline{PQP} = I - (P-Q)^2.$$

□

Remark 3. With the notation in the above proof, the distinct parts are $Q = \sum_j \Big(|a_j^*)(a_j^*| + |b_j^*)(b_j^*|\Big)$ and

$$\begin{aligned}P = \sum_j |\sqrt{c_j}a_j^* + \sqrt{1-c_j}b_j)(\sqrt{c_j}a_j^* + \sqrt{1-c_j}b_j| \\ + |\sqrt{c_j}b_j^* - \sqrt{1-c_j}a_j)(\sqrt{c_j}b_j^* - \sqrt{1-c_j}a_j|\end{aligned}$$

with the associated density operators given by $\rho_Q = \prod_j a_j a_j^* b_j b_j^*$ and

$$\rho_P = \prod_j \left(c_j a_j a_j^* b_j b_j^* + (1-c_j) b_j^* b_j a_j^* a_j + \sqrt{c_j(1-c_j)} b_j^* a_j^* + \sqrt{c_j(1-c_j)} a_j b_j \right),$$

which are expressed in terms of the infinite tensor product of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for ρ_Q and the infinite tensor product of projection matrices

$$\begin{aligned}c_j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (1-c_j) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ + \sqrt{c_j(1-c_j)} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \sqrt{c_j(1-c_j)} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

for ρ_P with the correspondence

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$(\varphi_P^{1/2}|\varphi_Q^{1/2}) = \prod_j c_j,$$

whereas $\det(1-(P-Q)^2) = \prod_j c_j^4$ because of

$$1-(P-Q)^2 = \bigoplus_j \begin{pmatrix} c_j & 0 \\ 0 & c_j \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 7.4. Let S and T be covariance operators on V. Then

$$(\varphi_S^{1/2}|\varphi_T^{1/2})^4 = \det(MM^*)$$

with $M = S^{1/2}T^{1/2} + (1-S)^{1/2}(1-T)^{1/2}$.

Proof. Let P and Q be the quadratures of S and T respectively. Then

$$MM^* = \begin{pmatrix} \sqrt{S} & \sqrt{1-S} \end{pmatrix} \begin{pmatrix} \sqrt{T} \\ \sqrt{1-T} \end{pmatrix} \begin{pmatrix} \sqrt{T} & \sqrt{1-T} \end{pmatrix} \begin{pmatrix} \sqrt{S} \\ \sqrt{1-S} \end{pmatrix}$$

is similar to PQP, which together with the identity

$$I-(P-Q)^2 = PQP + \overline{PQP}$$

shows that

$$\det(MM^*) = \sqrt{\det(I-(P-Q)^2)}.$$

Now, Corollary 3.6 is used to get

$$(\varphi_S^{1/2}|\varphi_T^{1/2}) = (\varphi_P^{1/2}|\varphi_Q^{1/2})^{1/2} = \det(I-(P-Q)^2)^{1/8} = \det(MM^*)^{1/4}.$$

□

Corollary 7.5. Let $S = S' \oplus S''$ and $T = T' \oplus T''$ be splitted covariance operators on $V' \oplus V''$. Then

$$(\varphi_S^{1/2}|\varphi_T^{1/2}) = (\varphi_{S'}^{1/2}|\varphi_{T'}^{1/2})\,(\varphi_{S''}^{1/2}|\varphi_{T''}^{1/2}).$$

Remark 4. In the finite-dimensional case, $\det(M)$ already gives the transition amplitude between states. It is therefore interesting to see what is happening in the infinite-dimensional case. Since we are concerned with non-vanishing amplitudes, assume that $P-Q$ is in the Hilbert-Schmidt class. Then

$$P = \bigoplus_j \begin{pmatrix} c_j I & -\sqrt{c_j(1-c_j)}J \\ \sqrt{c_j(1-c_j)}J & (1-c_j)I \end{pmatrix}, \qquad Q = \bigoplus_j \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$M = PQ + \overline{PQ} = \bigoplus_j \begin{pmatrix} \sqrt{c_j}I & \sqrt{1-c_j}J \\ \sqrt{1-c_j}J & \sqrt{c_j}I \end{pmatrix} \begin{pmatrix} \sqrt{c_j}I & 0 \\ 0 & \sqrt{c_j}I \end{pmatrix}$$

shows that the phase part U (this is in fact an orthogonal operator on V) of M is a direct sum of rotational matrices but $I-U$ has an expression

$$I-U = \bigoplus_j \begin{pmatrix} (1-\sqrt{c_j})I & -\sqrt{1-c_j}J \\ -\sqrt{1-c_j}J & (1-\sqrt{c_j})I \end{pmatrix}.$$

Since the diagonal part is supposed to be in the trace class, $I - U$ belongs to the trace class if and only if

$$\bigoplus_j \begin{pmatrix} 0 & -\sqrt{1-c_j}J \\ -\sqrt{1-c_j}J & 0 \end{pmatrix}$$

is in the trace class, i.e., if and only if $P - Q$ is in the trace class.

Lemma 7.6 ([1, Lemma 5.1])**.**

$$\|S - T\|_{HS} \leq 2\|\sqrt{S} - \sqrt{T}\|_{HS}.$$

$$\|\sqrt{S(1-S)} - \sqrt{T(1-T)}\|_{HS} \leq 2\|\sqrt{S} - \sqrt{T}\|_{HS}.$$

$$\frac{1}{2}\|P - Q\|_{HS} \leq \sqrt{2}\|S^{1/2} - T^{1/2}\|_{HS} \leq \|P - Q\|_{HS}.$$

Proof. The first inequality follows from

$$2(S-T) = (\sqrt{S} - \sqrt{T})(\sqrt{S} + \sqrt{T}) + (\sqrt{S} + \sqrt{T})(\sqrt{S} - \sqrt{T})$$

and the second one from

$$\sqrt{S(1-S)} - \sqrt{T(1-T)} = (\sqrt{S} - \sqrt{T})\sqrt{1-S} + \sqrt{T}(\sqrt{1-S} - \sqrt{1-T})$$

if one notices $\sqrt{1-S} - \sqrt{1-T} = \overline{\sqrt{S} - \sqrt{T}}$.

The lower inequality in the last line follows from

$$P - Q = \begin{pmatrix} \sqrt{S} - \sqrt{T} \\ \sqrt{1-S} - \sqrt{1-T} \end{pmatrix} \begin{pmatrix} \sqrt{S} & \sqrt{1-S} \end{pmatrix} + \begin{pmatrix} \sqrt{T} \\ \sqrt{1-T} \end{pmatrix} \begin{pmatrix} \sqrt{S} - \sqrt{T} & \sqrt{1-S} - \sqrt{1-T} \end{pmatrix}.$$

Finally the upper inequality in the last line is a special case of the inequality ([1, Lemma 5.2])

$$\||X| - |Y|\|_{HS} \leq \|X - Y\|_{HS}$$

for bounded hermitian operators X, Y if one notices

$$\left| P - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{pmatrix} \sqrt{S} & 0 \\ 0 & \sqrt{S} \end{pmatrix}, \quad \left| Q - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{pmatrix} \sqrt{T} & 0 \\ 0 & \sqrt{T} \end{pmatrix}.$$

□

Proposition 7.7. For covariance operators S and T on V,

$$\frac{2\|\sqrt{S} - \sqrt{T}\|_{HS}}{\sqrt{1 + 2\|\sqrt{S} - \sqrt{T}\|_{HS}^2}} \leq \|\varphi_S^{1/2} - \varphi_T^{1/2}\| \leq 4\sqrt{2}\|\sqrt{S} - \sqrt{T}\|_{HS}.$$

Proof. We first observe that

$$\|\varphi_S^{1/2} - \varphi_T^{1/2}\|^2 = 2(1 - (\varphi_S^{1/2}|\varphi_T^{1/2})) = 2(1 - \det(1 - (P-Q)^2)^{1/8}).$$

From the expression

$$1 - \det(1 - (P-Q)^2)^{1/8} = \frac{1 - \det(1 - (P-Q)^2)}{1 + d + \cdots + d^7}, \quad d = \det(1 - (P-Q)^2)^{1/8},$$

$$\frac{1}{8}(1 - \det(1 - (P-Q)^2)) \leq 1 - \det(1 - (P-Q)^2)^{1/8} \leq 1 - \det(1 - (P-Q)^2).$$

For the upper estimate, we use

$$1 - \det(1 - (P-Q)^2) \le \|P-Q\|_{HS}^2 e^{\|P-Q\|_{HS}^2},$$

which is further estimated linearly by

$$2\|P-Q\|_{HS}^2$$

from the inequality $\min(1, te^t) \le 2t$ $(t \ge 0)$.

For the lower estimate, we use

$$\frac{1}{\det(1-C)} = \det(1 + C + C^2 + \dots) \ge 1 + \mathrm{trace}(C)$$

for $0 \le C \le 1$. Then

$$1 - \det(1-C) \ge 1 - \frac{1}{1+\mathrm{trace}(C)} = \frac{\mathrm{trace}(C)}{1+\mathrm{trace}(C)},$$

which is combined with the inequality $\mathrm{trace}((P-Q)^2) \ge 2\|\sqrt{S} - \sqrt{T}\|_{HS}^2$ in the previous lemma. □

Corollary 7.8. Let S, T be covariance operators on V with P and Q their quadratures. Then $(\varphi_S^{1/2}|\varphi_T^{1/2}) > 0$ if and only if (i) $P \wedge (1-Q) = 0$ and (ii) $\sqrt{S} - \sqrt{T}$ is in the Hilbert-Schmidt class.

Proof. If $\|\sqrt{S}-\sqrt{T}\|_{HS} = \infty$, the lower estimate shows $(\varphi_S^{1/2}|\varphi_T^{1/2}) = 0$. Otherwise, $P-Q$ is in the Hilbert-Schmidt class by Lemma 7.6 and the transition probability is evaluated by the formula

$$(\varphi_S^{1/2}|\varphi_T^{1/2})^8 = \det(I - (P-Q)^2),$$

which vanishes if and only if $P-Q$ has a non-trivial kernel, i.e., $P \wedge (1-Q) \ne 0$. □

Theorem 7.9. Let S and T be covariance operators on a separable *-Hilbert space V. Then $\overline{C(V)\varphi_S^{1/2}C(V)} = \overline{C(V)\varphi_T^{1/2}C(V)}$ if $\sqrt{S} - \sqrt{T}$ is in the Hilbert-Schmidt class and, otherwise, $\overline{C(V)\varphi_S^{1/2}C(V)} \perp \overline{C(V)\varphi_T^{1/2}C(V)}$.

Proof. By the separability of V, we can find a trace class operator $iH \in o_1(V)$ such that $e^{iH}Se^{-iH}$ is a non-degenerate covariance operator. From the equality

$$\varphi_{e^{iH}Se^{-iH}}(e^{iQ(H)}(\cdot)e^{-iQ(H)}) = \varphi_S(\cdot),$$

we see

$$\overline{C(V)\varphi_{e^{iH}Se^{-iH}}^{1/2}C(V)} = \overline{C(V)e^{iQ(H)}\varphi_S^{1/2}e^{-iQ(H)}C(V)} = \overline{C(V)\varphi_S^{1/2}C(V)}$$

and similarly for $\overline{C(V)\varphi_T^{1/2}C(V)}$. Since $\sqrt{e^{iH}Se^{-iH}} - \sqrt{S} = e^{iH}\sqrt{S}e^{-iH} - \sqrt{S}$ is in the trace clase, the problem is reduced to standard S and T. Then $\overline{C(V)\varphi_S^{1/2}C(V)} = \overline{C(V)\varphi_S^{1/2}}$ and $\overline{C(V)\varphi_T^{1/2}C(V)} = \overline{\varphi_T^{1/2}C(V)}$, which shows that $\overline{C(V)\varphi_S^{1/2}C(V)}$ is orthogonal to $\overline{C(V)\varphi_T^{1/2}C(V)}$ if and only if $(\varphi_S^{1/2}|\varphi_T^{1/2}) = 0$, i.e., $\|\sqrt{S}-\sqrt{T}\|_{HS} = \infty$.

Otherwise, by Theorem 3.6, we have $L^2(S) = L^2(T)$. □

Corollary 7.10 (Powers-Størmer-Araki)**.** Free states φ_S and φ_T are quasi-equivalent if and only if $\sqrt{S} - \sqrt{T}$ is in the Hilbert-Schmidt class.

Corollary 7.11. Let S be a covariance operator on V and θ be an automorphism of $C(V)$ induced from an orthogonal transformation g of V. Then the unitary operator $\mathrm{Ad}\,\theta$ leaves $\overline{C(V)\varphi_S^{1/2}C(V)}$ invariant if and only if $gS^{1/2}g^{-1} - S^{1/2}$ is in the Hilbert-Schmidt class.

Lemma 7.12. Suppose that P and Q are Fock projections with $P - Q$ in the Hilbert-Schmidt class. Let $\{c_k\}_{1 \leq k \leq n}$ be an orthonormal basis of $[(1-P) \wedge Q]$ and $\{a_j^*, b_j^*, a_j, b_j\}$ be a reducing Fock basis of $\{c_k, c_k^*\}^\perp$ described in Appendix A. Then

$$\lim_{n \to \infty} \prod_{j=1}^{n} (\cos\theta_j - \sin\theta_j a_j^* b_j^*) c_1 \dots c_n \varphi_Q^{1/2}$$

exists in norm topology and gives a P-Fock vector; it is vanished by $\overline{P}V^{\mathbb{C}}$.

Proof. By the way of choice, the family $\{b_j \cos\theta_j - a_j^* \sin\theta_j, a_j \cos\theta_j + b_j^* \sin\theta_j, c_k\}$ gives an orthonormal basis of $\overline{P}([P \wedge Q + (1-P) \wedge (1-Q)]^\perp)$, the Dirac see construction suggests as a P-Fock vector

$$\prod_j N_j^{-1} (b_j \cos\theta_j - a_j^* \sin\theta_j)(a_j \cos\theta_j + b_j^* \sin\theta_j) c_1 \dots c_n \varphi_Q^{1/2}.$$

Here $N_j > 0$ denotes renormalization constants to be fixed in a moment. Since $(b_j \cos\theta_j - a_j^* \sin\theta_j)(a_j \cos\theta_j + b_j^* \sin\theta_j)$ commutes with each other and with any c_k, the relevant term is

$$\begin{aligned}(b_j \cos\theta_j - a_j^* \sin\theta_j)(a_j \cos\theta_j + b_j^* \sin\theta_j)\varphi_Q^{1/2} &= (b_j b_j^* \cos\theta_j \sin\theta_j - a_j^* b_j^* \sin^2\theta_j)\varphi_Q^{1/2} \\ &= \sin\theta_j (\cos\theta_j - a_j^* b_j^* \sin\theta_j)\varphi_Q^{1/2},\end{aligned}$$

which suggests the choice $N_j = 1/\sin\theta_j$ and the problem is reduced to the convergence of

$$\prod_{j=1}^{\infty} (\cos\theta_j - a_j^* b_j^* \sin\theta_j) c_1 \dots c_n \varphi_Q^{1/2}$$

under the assumption that $\sum_j \theta_j^2 < \infty$ with $0 < \theta_j < \pi/2$. Since

$$0 \geq \sum_j \log\cos\theta_j > -\infty,$$

the problem is further reduced to the convergence of

$$\prod_j (1 - a_j^* b_j^* \tan\theta_j)\varphi_Q^{1/2}.$$

In view of $(a_j^* b_j^*)^m = 0$ for $m \geq 2$,

$$\prod_j (1 - a_j^* b_j^* \tan\theta_j)\varphi_Q^{1/2} = \sum_F (-\tan\theta)^F (a^* b^*)^F \varphi_Q^{1/2},$$

where F runs through all finite subsets of $\{j\}$ and

$$(-\tan\theta)^F = \prod_{j \in F} (-\tan\theta_j), \quad (a^* b^*)^F = \prod_{j \in F} a_j^* b_j^*.$$

Since the family $\{(a^* b^*)^F \varphi_Q^{1/2}\}$ is orthonormal, the sum is norm-convergent by

$$\sum_F (\tan^2\theta)^F = \prod_j (1 + \tan^2\theta_j) < \infty.$$

(The limit Fock vector turns out to be normalized.) $\square$

Corollary 7.13. Let P and Q be Fock projections such that $P-Q$ is in the Hilbert-Schmidt class and $[P\wedge(1-Q)]$ is even-dimensional. Let $H=H^*$ be the bounded Hilbert-Schmidt class operator on V considered in Appendix A ($iH=(U-U^*)\arcsin\sqrt{C}+ih$). Let $H=\sum_j h_j(E_j-\overline{E_j})$ be the spectral decomposition with $h_j>0$ and set $H_n=\sum_{j=1}^n h_j(E_j-\overline{E_j})$. Then

$$\lim_{n\to\infty} e^{itQ(H_n)}$$

exists for $t\in\mathbb{R}$ in strong operator topology and gives a one-parameter group of unitary operators (denoted by $e^{itQ(H)}$) such that

$$e^{itQ(H)}\varphi_Q^{1/2}=\lim_{n\to\infty}\prod_{j=1}^n(\cos(t\theta_j)-\sin(t\theta_j)a_j^*b_j^*)\varphi_Q^{1/2}.$$

Proof. From the expression

$$iH=\sum_j\theta_j\Big(|b_j)(a_j^*|-|a_j)(b_j^*|+|b_j^*)(a_j|-|a_j^*)(b_j|\Big),$$

$$iQ(H_n)=\frac{1}{2}\sum_{j=1}^n\theta_j(b_ja_j-a_jb_j+b_j^*a_j^*-a_j^*b_j^*)=\sum_{j=}^n\theta_j(b_ja_j+b_j^*a_j^*).$$

Thus the convergence of $e^{iQ(H_n)}\varphi_Q^{1/2}$ is reduced to checking $e^{\theta_j(b_ja_j+b_j^*a_j^*)}\varphi_Q^{1/2}=(\cos\theta_j-\sin\theta_ja_j^*b_j^*)\varphi_Q^{1/2}$. From

$$(ba+b^*a^*)^n\varphi_Q^{1/2}=\begin{cases}(-1)^m\varphi_Q^{1/2} & \text{if } n=2m,\\ (-1)^m b^*a^*\varphi_Q^{1/2} & \text{if } n=2m+1,\end{cases}$$

$$\lim_{n\to\infty}e^{iQ(H_n)}\varphi_Q^{1/2}=\sum_{n\geq 0}\frac{\theta^n}{n!}(ba+b^*a^*)^n\varphi_Q^{1/2}=\cos\theta\varphi_Q^{1/2}+\sin\theta b^*a^*\varphi_Q^{1/2}.$$

If a finite family $\{x_l\}$ is taken from the algebraic sum of $\{a_j^*,b_j^*,a_j,b_j\}\subset V$ and $y\in C(\{a_j^*,b_j^*,a_j,b_j\}^\perp)\subset C(V)$, then

$$\lim_{n\to\infty}e^{iQ(H_n)}x_1\dots x_my\varphi_Q^{1/2}=\lim_{n\to\infty}(e^{iH}x_1)\dots(e^{iH}x_m)ye^{iQ(H_n)}\varphi_Q^{1/2}$$

exists and hence the limit defines an isometry $e^{iQ(H)}$ on $\overline{C(V)\varphi_Q^{1/2}}$ and $e^{iQ(H)}$ intertwines the automorphism $\theta_{e^{iH}}$ on $C(V)$.

Since $\overline{C(V)e^{iQ(H)}\varphi_Q^{1/2}}\cong\overline{C(V)\varphi_P^{1/2}}$ as left $C(V)$-modules and $\mathrm{Ad}(\theta_{e^{iH}})$ gives a unitary map $\overline{C(V)\varphi_Q^{1/2}}\to\overline{C(V)\varphi_P^{1/2}}$, we can find a unitary map $U:\overline{C(V)\varphi_Q^{1/2}}\to\overline{C(V)e^{iQ(H)}\varphi_Q^{1/2}}$ such that $Ue^{iQ(H)}:\overline{C(V)\varphi_Q^{1/2}}\to\overline{C(V)\varphi_Q^{1/2}}$ intwetwines the left action of $C(V)$. By the irreducibility, this implies the unitarity of $Ue^{iQ(H)}$, whence $e^{iQ(H)}$ itself is a unitary. $\square$

Theorem 7.14. Let P and Q be Fock projections on $V^{\mathbb{C}}$. Then the following conditions are equivalent.

(i) $\overline{C(V)\varphi_P^{1/2}}$ and $\overline{C(V)\varphi_Q^{1/2}}$ are unitarily equivalent as left $C(V)$-modules.
(ii) $L^2(P)=L^2(Q)$.

(iii) $P-Q$ is in the Hilbert-Schmidt class.

(iv) We can find a P-Fock vector inside $\overline{C(V)\varphi_Q^{1/2}}$.

Proof. Since Fock representations are irreducible, unitary equivalence between them follows from their quasi-eqivalence, i.e, the equality $\overline{C(V)\varphi_P^{1/2}C(V)} = \overline{C(V)\varphi_Q^{1/2}C(V)}$, which holds if and only if $P-Q$ is in the Hilbert-Schmidt class by Theorem 7.9.

(iv) $\Longrightarrow$ (ii): Let $\mathcal{H} = \overline{C(V)\varphi_P^{1/2}}$ and identify $L^2(P)$ with $\mathcal{H}\otimes\overline{\mathcal{H}}$. Let $\xi \in \mathcal{H}$ be a unit vector annihilated by left multiplication of $\overline{Q}V^{\mathbb{C}} \subset C(V)$. Then $\xi\otimes\overline{\xi}$ is identified with $\varphi_Q^{1/2}$ and hence $\varphi_Q^{1/2} \in L^2(P)$. Thus $L^2(Q) = L^2(P)$ by dichotomy and (ii) follows.

(iii) implies (iv) by the previous lemma. □

Corollary 7.15 (Powers-Størmer)**.** Let P be a Fock projection on $V^{\mathbb{C}}$ and $g \in O(V)$ be an orthogonal transformation. Then the automorphism θ_g has a unitary implementation on $\overline{C(V)\varphi_P^{1/2}}$ if and only if $gPg^{-1} - P$ is in the Hilbert-Schmidt class.

Example 7.16. A unitary implementation of the parity automorphism ϵ (which corresponds to the choice $g=-1$) is given by

$$\Pi : x\varphi_P^{1/2} \mapsto \epsilon(x)\varphi_P^{1/2}, \quad x \in C(V).$$

Thus an implementer of ϵ can be found in $\mathrm{End}(\overline{C(V)\varphi_P^{1/2}C(V)}_{C(V)})$, which is given by

$$x\varphi_P^{1/2}y \mapsto \epsilon(x)\varphi_P^{1/2}y$$

with $x, y \in C(V)$.

Let $\Omega \in \overline{C(V)\varphi_Q^{1/2}}$ be a P-Fock vector. Then by Lemma 7.12,

$$\Pi\Omega = (-1)^{\dim\ker(P\wedge(1-Q))}\Omega.$$

According to Araki, the index between equivalent Fock projections is defined by

$$\mathrm{index}(P,Q) = (-1)^{\dim\ker(P\wedge(1-Q))}.$$

8. Factoriality of Free States

We shall investigate factoriality of free states according to [1, §10]. A major step on sufficiency is worked out here, however, a full characterization is postponed until § 10.

Lemma 8.1. Let C be a covariance operator. Then we can find a covariance operator D for any given $\epsilon > 0$ such that D has pure point spectra and $\|\sqrt{C} - \sqrt{D}\|_2 \leq \epsilon$.

Proof. By subtracting the subspace $\ker(C-1/2)$ from V, we may assume that $\ker(C-1/2) = \{0\}$. Let E be the spectral projection of C corresponding to the spectral range $[0,1/2)$ and set $A = EC^{1/2}$. Then $0 \leq A \leq 1/\sqrt{2}E$ and we can find $0 \leq B \leq 1/\sqrt{2}E$ such that $A-B$ has arbitrarily small Hilbert-Schmidt norm. Let $D = B^2 + \overline{E} - \overline{B^2}$, which has pure point positive spectra and satisfies $D + \overline{D} = E + \overline{E} = 1$. Moreover the Hilbert-Schmidt norm of

$$\sqrt{C} - \sqrt{D} = A - B + \sqrt{\overline{E} - \overline{A^2}} - \sqrt{\overline{E} - \overline{B^2}}$$

is estimated by $c\|A-B\|_2$ with $c>0$ a universal constant. $\square$

Proposition 8.2. Let C be a covariance operator on V and suppose that $\ker(C-1/2)$ is not odd-dimensional. Then $\mathrm{End}({}_{C(V)}L^2(C)_{C(V)}) = \mathbb{C}$.

Proof. Since the quadrate representation of $C(V\oplus iV)$ is irreducible on $L^2(C)$, the factoriality follows if we can show that $\pi(C(V\oplus iV))|_{L^2(C)}$ is approximated by the biaction of $C(V)$.

In view of the stability of $L^2(C)$ under the Hilbert-Schmidt perturbation on $C^{1/2}$, we may assume that C has only point spectra by the previous lemma. Since $\ker(C-1/2)$ is assumed to be even-dimensional, we can find a Fock basis $\{a_j, a_j^*\}_{j\in L}$ consisting of eigenvectors of C. For a finite subset $F\subset L$, let $h_F = \prod_{l\in F} h_l$ be a product of mutually commuting hermitian unitaries $h_j = a_j^*a_j - a_ja_j^* = (a_j+a_j^*)(a_j-a_j^*)$ in $C(V)$ and define a unitary operator U_F on $L^2(C(V))$ by $U_F = \mathrm{Ad}\, h_F$. By the way of definition, h_F commutes with φ_C and

$$h_F a_l h_F^* = -a_l, \quad h_F a_l^* h_F^* = -a_l^*,$$

if $l\in F$, which are utilized to check the strong-operator convergence

$$\lim_{F\to L} U_F = \Pi$$

on $L^2(C)$, whence Π is approximated by the biaction of $C(V)$ and so is $\pi(\xi\oplus\eta) = l(\xi)+r(\eta)\Pi$ for $\xi,\eta\in V$. $\square$

Let P be a Fock projection on $V^{\mathbb{C}}$ and $W = V\oplus\mathbb{R}h$ be a fermionic phase space with one hermitian element $\overline{h}=h$ added to V. We choose $(h|h)=2$ so that $h^2=1$ in $C(W)$. Let φ_S be a pseudo-Fock state of $C(W)$ with $S=P\oplus 1/2$ an extremal covariance operator. Let π be the quadrate representation of $C(W\oplus iW)$ on $L^2(S) = \overline{C(W)\varphi_S^{1/2}C(W)}$. Since $h\varphi_S^{1/2} = \varphi_S^{1/2}h$ by Lemma 3.5, we see

$$C(W)\varphi_S^{1/2}C(W) = C(V)\varphi_S^{1/2}C(V) + C(V)\varphi_S^{1/2}hC(V),$$

which is an orthogonal sum (the partner for h missing when evaluated) with each component isometrically isomorphic to $C(V)\varphi_P^{1/2}C(V)$ in view of the invariance of φ_S under the parity automorphism ϵ (see the key lemma for more information). Under this identification, the left and right multiplications of h are given by

$$\begin{pmatrix} 0 & l \\ l & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix},$$

respectively. Here unitaries l, r on $\overline{C(V)\varphi_P^{1/2}C(V)}$ are defined by

$$l(x\varphi_P^{1/2}y) = \epsilon(x)\varphi_P^{1/2}y, \qquad r(x\varphi_P^{1/2}y) = x\varphi_P^{1/2}\epsilon(y).$$

Since any operator on $L^2(S)$ which intertwines the irreducible biaction of $C(V)$ is realized by a two-by-two matrix of scalar components, it intertwines the biaction of $C(W)$ if and only if its matrix representation is of the form

$$\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}.$$

Thus the center $\mathrm{End}({}_{C(W)}L^2(S)_{C(W)})$ is isomorphic to $\mathbb{C}\oplus\mathbb{C}$ and generated by the flip operator

$$x_1\varphi_S^{1/2}y_1 + x_2\varphi_S^{1/2}hy_2 \mapsto x_1\varphi_S^{1/2}hy_1 + x_2\varphi_S^{1/2}y_2.$$

For the GNS-representation, $\overline{C(W)\varphi_S^{1/2}}$ is decomposed into two equivalent irreducible components of the left action of $C(V)$ by

$$\overline{C(W)\varphi_S^{1/2}} = \overline{C(V)\varphi_S^{1/2}} + \overline{C(V)\varphi_S^{1/2}h},$$

which is isometrically isomorphic to

$$\overline{C(V)\varphi_S^{1/2}} \oplus \overline{C(V)\varphi_P^{1/2}}$$

by

$$x\varphi_S^{1/2} + y\varphi_S^{1/2} \mapsto x\varphi_P^{1/2} + y\varphi_P^{1/2}.$$

Note that, in the last realization, the left multiplication of h is expressed by

$$\begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix},$$

where Π denotes the parity operator on $\overline{C(V)\varphi_P^{1/2}}$. Thus the projection to irreducible components of $\overline{C(W)\varphi_S^{1/2}}$ is given by

$$e_\pm(x\varphi_S^{1/2} + y\varphi_S^{1/2}h) = \frac{x \pm \epsilon(y)}{2}\varphi_S^{1/2} + \frac{y \pm \epsilon(x)}{2}\varphi_S^{1/2}h$$

with $he_\pm = \pm e_\pm$ and

$$2e_\pm\varphi_S^{1/2} = \varphi_S^{1/2} \pm \varphi_S^{1/2}h$$

are P-Fock vectors for $C(V)$. The corresponding pure states $\varphi_\pm$ given by

$$\varphi_\pm(x) = \frac{1}{2}(\varphi_S^{1/2} \pm \varphi_S^{1/2}h|x(\varphi_S^{1/2} \pm \varphi_S^{1/2}h)) = \varphi_S(x) \pm \frac{1}{2}\varphi_S(hx + xh), \quad x \in C(W)$$

are, however, not of the form of free state in view of $\varphi_\pm(h) = \pm 1$ or $\varphi_\pm \circ \epsilon = \varphi_\mp$.

9. Orthogonal Transformations and Automorphisms

Let M be a von Neumann algebra generated by a representation of a C*-algebra A. Then we have a natural imbedding of $L^2(M)$ into $L^2(A)$. Let θ be an automorphism of A. Then θ is extended to a normal automorphism of M if and only if θ leaves $L^2(M) \subset L^2(A)$ invariant.

Here we shall investigate the automorphic action induced from orthogonal transformations. Let θ be an automorphism of $C(V)$ induced from $g \in O(V)$ and denote by $U = \mathrm{Ad}\theta$ the induced unitary operator on $L^2(C(V))$. Let C be a covariance operator and investigate the condition that U leaves $L^2(C)$ invariant. Since θ commutes with the parity automorphism ϵ, we see that $U\pi(x \oplus y) = \pi(gx \oplus gy)U$ on $L^2(C)$. Thus the invariance of $L^2(C)$ under U implies the stronger property that U on $L^2(C)$ implements the automorphism of $C(V \oplus iV)$ induced from $g \oplus g \in O(V \oplus iV)$. Since π_C is the GNS-representation of its quadrature φ_P, the problem is reduced to unitary implementations on Fock representations.

Let P and Q be Fock projections for a phase space V.

Lemma 9.1.

(i) The group $O(V)$ acts transitively on the set of Fock projections.

(ii) Assume that $\ker(P \wedge (1-Q))$ is even or infinite dimensional. Then we can find a hermitian operator h on $V^{\mathbb{C}}$ such that $\overline{h} = -h$ (whence $e^{ih} \in O(V)$), $-\pi \leq h \leq \pi$ and $e^{ih}Pe^{-ih} = Q$.

We say P and Q are equivalent if $P - Q$ is in the Hilbert-Schmid class. Choose and keep an equivalence class of Fock projections once for all in this section.

Theorem 9.2. For $g \in O(V)$, the following conditions are equivalent.

(i) The automorphism θ_g leaves $L^2(P) = \overline{C(V)\varphi_P^{1/2}C(V)}$ invariant.
(ii) We can find a unitary operator U on $\overline{C(V)\varphi_P^{1/2}}$ satisfying $Ux\xi = \theta_g(x)U\xi$ for $x \in C(V)$ and $\xi \in \overline{C(V)\varphi_P^{1/2}}$.
(iii) We can find a gPg^{-1}-Fock vector inside $\overline{C(V)\varphi_P^{1/2}}$.
(iv) $Pg(1-P)$ is in the Hilbert-Schmidt class.
(v) $gP - Pg$ is in the Hilbert-Schmidt class.

Proof. (i) $\iff$ (v): Let $Q = gPg^{-1}$. Since $\varphi_Q^{1/2} \in L^2(P)$ by (i), the dichotomy shows that $L^2(Q) = L^2(P)$, which is equivalent to (v) due to the Power-Störmer criterion.

(iv) $\iff$ (v) follows from $Pg(1-P) = (Pg - gP)(1-P)$ and $gP - Pg = \overline{Pg(1-P)} - Pg(1-P)$.

(ii) $\Longrightarrow$ (iii): For $\xi \in \overline{P}V \subset C(V)$, $\theta_g(\xi) \in g\overline{P}g^{-1}V$ and

$$\theta_g(\xi)U\varphi_P^{1/2} = U\xi\varphi_P^{1/2} = 0.$$

(iii) $\Longrightarrow$ (v): Let $\mathcal{H}_P = \overline{C(V)\varphi_P^{1/2}}$ and identify $L^2(P)$ with $\mathcal{H}_P \otimes \overline{\mathcal{H}_P}$. Let $\xi \in \mathcal{H}_P$ be a unit vector annihilated by left multiplication of $g\overline{P}g^{-1} \subset V \subset C(V)$. Then $\xi \otimes \overline{\xi}$ is identified with $\varphi_Q^{1/2}$ and hence $\varphi_Q^{1/2} \in L^2(P)$. Thus $L^2(Q) = L^2(P)$ and (v) follows.

(v) $\Longrightarrow$ (ii): (v) implies $L^2(P) = L^2(Q)$, whence φ_P and φ_Q are quasi-equivalent. Since these are pure states, $\overline{C(V)\varphi_P^{1/2}}$ and $\overline{C(V)\varphi_Q^{1/2}}$ are unitarily equivalent and we can find a unitary map $U : \mathcal{H}_P \to \mathcal{H}_Q$ intertwining left actions of $C(V)$. Since θ_g gives rise to a unitary map $\overline{C(V)\varphi_Q^{1/2}} \to \overline{C(V)\varphi_P^{1/2}}$ by $x\varphi_Q^{1/2} \mapsto \theta_g(x)\varphi_P^{1/2}$, $U_g = \theta_g U$ is a unitary operator on $\mathcal{H}$ and satisfies

$$U_g x = \theta_g x U = \theta_g(x)U_g$$

for $x \in C(V)$. □

Corollary 9.3. Let θ be the automorphism of $C(V)$ associated to an orthogonal transformation $g \in O(V)$ and let C be a covariance operator. Then θ leaves $L^2(C)$ if and only if $\sqrt{C}g\sqrt{1-C} - \sqrt{1-C}g\sqrt{C}$ is in the Hilbert-Schmidt class.

Proof. Let P be the quadrature of C. Then, from the condition (iv) in the theorem, θ leaves $L^2(C)$ if and only if

$$\begin{pmatrix} \sqrt{C} & \sqrt{1-C} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \sqrt{1-C} \\ -\sqrt{C} \end{pmatrix} = \sqrt{C}g\sqrt{1-C} - \sqrt{1-C}g\sqrt{C}$$

is in the Hilbert-Schmidt class. □

If C has a bounded inverse, then the condition is equivalent to requiring that $gC - Cg$ is in the Hilbert-Schmidt class. In fact, assume that $\sqrt{C}g\sqrt{1-C} - \sqrt{1-C}g\sqrt{C}$ is in the Hilbert-Schmidt class. Then

$$[g, \sqrt{(1-C)/C}] = C^{-1/2}(\sqrt{C}g\sqrt{1-C} - \sqrt{1-C}g\sqrt{C})C^{-1/2}$$

is in the Hilbert-Schmidt class and then so is

$$g\frac{1-C}{C} - \frac{1-C}{C}g = [g, \sqrt{(1-C)/C}]\sqrt{\frac{1-C}{C}} + \sqrt{\frac{1-C}{C}}[g, \sqrt{(1-C)/C}].$$

Thus $[C, g] = C[g, (1-C)/C]C$ is in the same class.

Conversely, assume that $[g, C]$ is a Hilbert-Schmidt operator. If we set $h = 1-2C$ with $\|h\| < 1$, then

$$\begin{aligned} 2(\sqrt{C}g\sqrt{1-C} - \sqrt{1-C}g\sqrt{C}) &= \sqrt{1-h}g\sqrt{1+h} - \sqrt{1+h}g\sqrt{1-h} \\ &= [\sqrt{1-h}, g]\sqrt{1+h} + \sqrt{1+h}[\sqrt{1-h}, g] \\ &\quad + [g, \sqrt{1-h^2}] \end{aligned}$$

is in the Hilbert-Schmidt class by Taylor expansion.

Definition 9.4. Let $O_C(V)$ be the subgroup of $O(V)$ consisting of $g \in O(V)$ satisfying the equivalent conditions in the corollary. Note that $O_C(V) = O_D(V)$ if $\sqrt{C} - \sqrt{D}$ is in the Hilbert-Schmidt class.

Let $g \in O_P(V)$, U be a unitary on $\mathcal{H} = \overline{C(V)\varphi_P^{1/2}}$ satisfying $\theta_g = \mathrm{Ad}\, U$ and Π be the implementer of ϵ on $\mathcal{H}$. By irreducibility, such an operator is unique up to scalar multiplication and $\epsilon\theta_g\epsilon = \theta_g$ shows that $\Pi U \Pi$ is again an implementer of θ_g. As a result, we can find $\epsilon_P(g) \in \mathbb{C}$ such that $\Pi U \Pi = \epsilon_P(g) U$. From $\Pi^2 = 1$, $\epsilon_P(g)^2 = 1$.,e, $\epsilon_P(g) = \pm 1$, which is referred to as a **signature** of $g \in O_P(V)$. Clearly, $\epsilon_P : O_P(V) \to \{\pm 1\}$ is a group homomorphism. Note that $\epsilon_P(g) = \mathrm{index}(g^{-1}Pg, P)$.

Proposition 9.5. For $g \in O_P(V)$, $\ker(PgP) \cap PV^{\mathbb{C}}$ is finite-dimensional and $\epsilon_P(g) = \pm 1$ according to the parity of its dimension d:

$$\epsilon_P(g) = (-1)^d = \begin{cases} 1 & \text{if } d \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Let $Q = g^{-1}Pg$. Then $\xi \in [P \wedge (1-Q)]$ if and only if $P\xi = \xi$ and $Q\xi = 0$, i.e., $PgP\xi = 0$, hence $[P \wedge (1-Q)] = \ker(PgP) \cap PV$. Since $P - Q$ is in the Hilbert-Schmidt class and $(P-Q)\xi = \xi$ for $\xi \in [P \wedge (1-Q)]$, $d = \dim[P \wedge (1-Q)] < \infty$.

Let U be a unitary implementation of θ_g on $\overline{C(V)\varphi_P^{1/2}}$. Then

$$\Pi U \varphi_P^{1/2} = (-1)^{\dim[P \wedge (1-Q)]} U \varphi_P^{1/2}$$

by Lemma 7.12, which we compare with $\Pi U \varphi_P^{1/2} = \epsilon_P(g) U \Pi \varphi_P^{1/2} = \epsilon_P(g) U \varphi_P^{1/2}$. □

10. Blattner's Theorem

Let C be a covariance operator on V and $M_C = \mathrm{End}(L^2(C)_{C(V)})$ be the associated central cut of $C(V)^{**}$ so that $L^2(C) = L^2(M_C)$. Note that M_C is equal to the W*-algebra generated by the GNS-representation of φ_C. An automorphism θ of $C(V)$ leaves M_C invariant if and only if $\mathrm{Ad}\,\theta$ leaves $L^2(C)$ invariant. A characterization of such transformations is given in the previous section. If we can further find a unitary $u \in M_C$ such that $uxu^* = \theta(x)$ for $x \in M_C$, θ is said to be inner relative to C or simply C-inner. An orthogonal transformation $g \in O(V)$ is said to be C-inner if so is the induced automorphism θ_g of $C(V)$.

Let $O_C(V)$ be the subgroup of $O(V)$ consisting of $g \in (V)$ such that θ_g leaves $L^2(C)$ invariant and let $O'_C(V)$ be the normal subgroup of $O_C(V)$ consisting of C-inner transformations. Remark that, for a Fock projection P, $O_P(V) = O'_P(V)$.

For a C-inner automorphism θ with $u \in M_C$ a unitary implementer, we remark $\theta(u) = uuu^* = u$ and $u\varphi_C^{1/2}u^* = (\varphi_C \circ \theta^{-1})^{1/2}$.

We shall investigate the condition of C-innerness on g. When $C = 1/2$ and V is infinite-dimensional, this was solved by Blattner in the following form: $g \in O(V)$ is inner relative to $C = 1/2$ if and only if one of the following alternatives holds:

(i) $g - 1$ is in the Hilbert-Schmidt class and $\ker(g + 1)$ is not odd-dimensional.

(ii) $g + 1$ is in the Hilbert-Schmidt class and $\ker(g - 1)$ is odd-dimensional.

We shall write $O_2(V)$ instead of $O'_{1/2}(V)$ and call it the Blattner group, which turns out to be consisting of universally inner transformations.

Let $\Pi = \operatorname{Ad}\epsilon$ be the parity operator, which is a unitary involution on $L^2(C(V))$, and Π_C be the restriction of Π to the subspace $L^2(C) \subset L^2(C(V))$.

Let $g \in O(V)$ be C-inner and implemented by a unitary $u \in M_C$. Since $\operatorname{Ad}\Pi$ makes M_C invariant, $\Pi u \Pi$ is also a unitary implementer of θ_g and hence $u^*\Pi u\Pi$ is in the center of M_C. Therefore, if M_C is a factor, we can find $z \in \mathbb{T}$ such that $\Pi\, u \Pi = zu$ and $\Pi^2 = 1$ implies $z = \pm 1$. In accordance with the case of Fock projections, if we set $\epsilon_C(g) = z$, ϵ_C gives a group homomorphism $O'_C(V) \to \{\pm 1\}$ for a factorial C.

With these observations in mind, we consider the quadrate representation π of $C(V \oplus iV)$ and its restriction π_C to the invariant subspace $L^2(C)$. Thus π_C is naturally identified with the GNS-representation of φ_P with P the quadrature of C (Proposition 3.5), which converts Π_C into the restriction of Π_P to $\overline{C(V \oplus iV)\varphi_P^{1/2}} \subset L^2(P)$ as can be seen from $\Pi_C\varphi_C^{1/2} = \varphi_C^{1/2}$ and

$$\Pi_C\pi_C(x \oplus y)\psi^{1/2} = -x\Pi_C\psi^{1/2} - \psi^{1/2}y = \pi_C(-x \oplus -y)\Pi_C\psi^{1/2}$$

for $x, y \in V^{\mathbb{C}}$.

The following argument (converting C-innerness into P-innerness) is due to A.L. Carey ([3], cf. [6] also). Since $\Pi_C\pi_C(0 \oplus y)$ with $y \in V^{\mathbb{C}}$ is given by a right multiplication of $-y$ on $L^2(C)$, $u \in M_C$ means

$$u\Pi_C\pi_C(0 \oplus y) = \Pi_C\pi_C(0 \oplus y)u \quad \text{for } y \in V^{\mathbb{C}},$$

which is equivalent to

$$u\pi_C(0 \oplus y)u^* = \pi_C(0 \oplus \pm y) \quad \text{for } y \in V^{\mathbb{C}},$$

owing to the above observation. In view of $u\pi_C(x \oplus 0)u^* = \pi_C(gx \oplus 0)$ for $x \in V^{\mathbb{C}}$, the condition is further equivalent to

$$u\pi_C(x \oplus y)u^* = \pi_C(gx \oplus \pm y) \quad \text{for } x \oplus y \in V^{\mathbb{C}} \oplus V^{\mathbb{C}}.$$

Since π_C is equivalent to the GNS-representation of φ_P and since

$$g_\pm = \begin{pmatrix} g & 0 \\ 0 & \pm 1 \end{pmatrix} \in O(V \oplus iV),$$

one of the following alternatives is sufficient to ensure the C-innerness of $g \in O(V)$:

(i) θ_{g_+} is unitarily implementable on $\overline{C(V \oplus iV)\varphi_P^{1/2}}$ and $\epsilon_P(g_+) = 1$ or

(ii) θ_{g_-} is unitarily implementable on $\overline{C(V \oplus iV)\varphi_P^{1/2}}$ and $\epsilon_P(g_-) = -1$.

Moreover, under the factoriality assumption on M_C (this is the case if $\ker(C-1/2)$ is even-dimensional by Proposition 8.2), this is also necessary.

By Theorem 9.2 and Proposition 9.5, these conditions are equivalent to requiring that

$$\begin{pmatrix} g & 0 \\ 0 & \pm 1 \end{pmatrix} P \begin{pmatrix} g^{-1} & 0 \\ 0 & \pm 1 \end{pmatrix} - P = \begin{pmatrix} gCg^{-1} - C & (\pm g - 1)\sqrt{C(1-C)} \\ \sqrt{C(1-C)}(\pm g^{-1} - 1) & 0 \end{pmatrix}$$

is in the Hilbert-Schmidt class and the parity of the kernel dimension of

$$\begin{pmatrix} \sqrt{C} & \sqrt{1-C} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \sqrt{C} \\ \sqrt{1-C} \end{pmatrix} = \sqrt{C} g \sqrt{C} \pm (1-C)$$

is equal to ± 1.

Theorem 10.1. Let C be covariance operator on V. Then $g \in O(V)$ is C-inner, if it satisies one of the following alternatives. Operators $gC - Cg$ and $(\pm g - 1)\sqrt{C(1-C)}$ are in the Hilbert-Schmidt class and

$$(-1)^{\dim \ker(\sqrt{C} g \sqrt{C} \pm (1-C))} = \pm 1.$$

Conversely, if g is C-inner, it satisfies one of the above alternatives under the extra assumption of factoriality on M_C.

Corollary 10.2 (Blattner)**.** Let V be even or infinite dimensional. Then, for the choice $C = 1/2$, M_C is a factor and $O_2(V) = O'_C(V)$ consists of $g \in O(V)$ which satisfies the Blattner's condition. Moreover we have

$$\epsilon_{1/2}(g) = \begin{cases} 1 & \text{if } \ker(g+1) \text{ is finite and even-dimensional,} \\ -1 & \text{if } \ker(g-1) \text{ is finite and odd-dimensional.} \end{cases}$$

Proof. The factoriality follows from Proposition 8.2 and we just apply the theorem to get the result. □

Remark 5.

(i) When V is finite-dimensional, in view of the form of spectral decomposition of orthogonal transformations, the parity of V is equal to that of $\dim \ker(g+1) + \dim \ker(g-1)$.

(ii) Let $SO_2(V)$ be the set of orthogonal transformations satisfying the even Blattner condition. Then it is a normal subgroup of $O_2(V)$ such that $O_2(V)/SO_2(V) \cong \mathbb{Z}_2$.

Example 10.3. Assume that M_C is a factor. Then the parity automorphism ϵ is C-inner if and only if $C(1-C)$ is in the trace class and $\ker(C-1/2)$ is even-dimensional. (The factoriality assumption on M_C is in fact a consequence of other conditions. See Theorem 10.6.)

Lemma 10.4. Let $g \in O(V)$ satisfy the even Blattner condition; $g-1$ is in the Hilbert-Schmidt class and $\dim \ker(g+1)$ is even. Then g is C-inner with an even implementer for any covariance operator C.

Proof. Since $g-1$ is in the Hilbert-Schmidt class, $\ker(g+1)$ is finite-dimensional. Then 1 is the unique accumulation point of $\sigma(g)$ and the spectral decomposition of g takes the form

$$g = e_0 + \sum_{n \geq 1} (e^{i\theta_n} e_n + e^{-i\theta_n} \overline{e_n})$$

with $\dim e_n = 1$, $0 < \theta_n \leq \pi$ for $n \geq 1$ (consequently $e_0 = \ker(g-1)$) and

$$\|g-1\|_{HS}^2 = \sum_{n\geq 1}(|e^{i\theta_n}-1|^2 + |e^{-i\theta_n}-1|^2) = 8\sum_{n\geq 1}\sin^2(\theta_n/2) < \infty.$$

If we set

$$h = \sum_{n\geq 1}\theta_n(e_n - \overline{e_n}),$$

then it belongs to $o_2(V)$ and $g = e^{ih}$. Since $e^{ith} \in O_2(V)$ with

$$\|e^{ith}-1\|_{HS}^2 = 8\sum_{n\geq 1}\sin^2(t\theta_n/2)$$

and $\ker(e^{ith}+1) = \{0\}$ for $0 \leq t < 1$, the problem is reduced to the case of arbitrarily small $\|g-1\|_{HS}$ with trivial $\ker(g+1)$.

Let P be the quadrature of C and set $Q = g_+ P g_+^{-1}$. Then $P - Q$ is in the Hilbert-Schmidt class and $[P \wedge (1-Q)]$ is even-dimensional, whence we can find an operator $iH \in o_2(V)$ such that $Q = e^{iH}Pe^{-iH}$ with $e^{iQ(H)}$ an even unitary operator on $L^2(C) = \overline{C(V \oplus iV)\varphi_P^{1/2}}$ satisfying $e^{iQ(H)}(x \oplus y)e^{-iQ(H)} = e^{iH}(x \oplus y)$ for $x, y \in V^{\mathbb{C}}$.

Since $e^{-iH}g_+$ commutes with P, the unitary operator $\mathrm{Ad}\theta_{e^{-iH}g_+}$ induced from leaves $L^2(C) = \overline{C(V \oplus V)\varphi_P^{1/2}} \subset L^2(P)$ invariant. If we define an even unitary operator U on $L^2(C)$ by $U = e^{iQ(H)}\left(\mathrm{Ad}\theta_{e^{-iH}g_+}|_{L^2(C)}\right)$, then it intertwines θ_{g_+}:

$$U\left(\pi(w_1)\dots\pi(w_n)\varphi_C^{1/2}\right) = \pi(g_+w_1)\dots\pi(g_+w_n)\varphi_C^{1/2}$$

for $w_1, \dots, w_n \in V^{\mathbb{C}} \oplus V^{\mathbb{C}}$.

In other words, $U \in M_C$ and

$$U(x_1 \dots x_n\varphi_C^{1/2}) = (gx_1)\dots(gx_n)\varphi_C^{1/2}$$

for $x_1, \dots, x_n \in V^{\mathbb{C}}$. $\square$

The following reveals the universal nature of the Blattner condition in the inner implementability problem.

Theorem 10.5 (Carey)**.** For an arbitrary covariance operator C, we have $O_2(V) \subset O'_C(V)$ and, for $g \in O_2(V)$,

$$\epsilon_C(g) = \begin{cases} 1 & \text{if } g \text{ satisfies the even Blattner condition,} \\ -1 & \text{if } g \text{ satisfies the odd Blattner condition.} \end{cases}$$

If V is infinite-dimensional and $0 \notin \sigma(C)$, then we have $O_2(V) = O'_C(V)$.

Proof. By the previous lemma, $SO_2(V) \subset O'_C(V)$.

Choose $\{a, a^*\} \subset V$ and set $u = a + a^*$. Then the automorphism $\theta = \mathrm{Ad}u$ leaves V invariant in such a way that $\theta(w) = -w$ for $w \in \{a, a^*\}^\perp$ and $\theta(a) = a^*$. If we set $h = \theta|_V$, then $h+1$ is of finite-rank and $\ker(h-1) = \mathbb{C}u$ is one-dimensional. Thus h meets the odd Blattner condition and at the same time it is implemented by the odd unitary $u \in C(V)$. Now let $g \in O(V)$ satisfy the odd Blattner condition. Then gh satisfies the even Blattner condition and hence C-inner. Since $h \in O_2(V) \cap O'_C(V)$, g is also C-inner.

Now assume that $\dim V = \infty$ and C has a bounded inverse. Then M_C is a factor by the factoriality characterization below. Apply Theorem 10.1 to a C-inner $g \in O(V)$ to see that $g-1$ or $g+1$ is in the Hilbert-Schmidt class. So we need to check the parity condition to have $g \in O_2(V)$: Suppose contrarily that $g-1$ (resp. $g+1$) is in the Hilbert-Schmidt class and that $\ker(g+1)$ is odd-dimensional (resp. even-dimensional). Then $-g \in O_2(V)$ and $-g$ is C-inner, whence $-1 = g^{-1}(-g)$ is C-inner as well, i.e., the parity automorphism is C-inner. By Example 10.3, this implies that $C(1-C)$ is in the trace class, which contradicts with $\dim V = \infty$ because C has a bounded inverse. □

In this way, the characterization of C-innerness is almost completed aside from the facoriality assumption. To discuss the case of non-factor M_C, we here establish the characterization of factoriality of M_C first.

Assume that M_C is not a factor. Then $\dim(C-1/2)$ is finite and odd by Proposition 8.2. Choose a real vector $h \in \ker(C-1/2)$ such that $(h|h) = 2$. Then $h = h^* = h^{-1}$ in $C(V)$ and $h\varphi_C = \varphi_C h$. Let $W = V \ominus \mathbb{R}h$ and $D = C|_{W^\mathbb{C}}$ be the restricted covariance operator on $W^\mathbb{C}$. By a natural isometry $L^2(D) \ni x\varphi_D^{1/2}y \mapsto x\varphi_C^{1/2}y \in L^2(C)$, we can identify $L^2(D)$ with a $C(W)$-subbimodule of $L^2(C)$ so that $L^2(C) = L^2(D) + L^2(D)h$. Note that $L^2(D)h = hL^2(D)$. Since $\ker(D-1/2)$ is even-dimensional, M_D is a factor by Proposition 8.2, i.e., $\mathrm{End}({}_{C(W)}L^2(D)_{C(W)}) = \mathbb{C}1$. Since the adjoint of h on $C(W)$ is the parity automorphism ϵ, we have $C(V) \cong C(W) \rtimes_\epsilon \mathbb{Z}_2$ and

$$\mathrm{End}({}_{C(W)}L^2(C)_{C(W)}) = \mathbb{C} \oplus \mathbb{C}$$

if ϵ is not D-inner. Since multiplications by h exchange these two components, we see $\mathrm{End}({}_{C(V)}L^2(C)_{C(V)}) = \mathbb{C}1$, i.e., M_C is a factor.

Thus the non-factor case of M_C is narrowed to the case that ϵ is D-inner. Apply Theorem 10.1 to $g = -1$: ϵ is D-inner if and only if $\sqrt{D(1-D)}$ is in the Hilbert-Schmidt class. (By the even dimensionality of $\ker(D-1/2)$, $\epsilon_Q(-1) = 1$ with Q the quadrature of D, whereas the possibility $\epsilon_Q(-1) = -1$ is excluded by a mismatch of parity.)

Let $\{d_j\}_j$ be all eigenvalues of D counting multiplicity with $\sum_j d_j(1-d_j) < \infty$. Since $\ker(D-1/2)$ is even-dimensional, we can find a Fock projection for $\ker(D-1/2)$, which is added to the spectral projection for $D > 1/2$ to get a Fock projection P on W so that $\sqrt{D} - P$ is in the Hilbert-Schmidt class. Consequently $L^2(D) = L^2(P)$ and we can find a unitary $u \in M_D = M_P$ which implements ϵ. Since u^2 implements the identity automorphism and M_D is a factor (of type I), u^2 is a scalar operator. By adjusting phase factors, we may assume that $u^2 = 1$. Then $L^2(D)uh \ni \xi uh \mapsto \xi \in L^2(D)$ is an isomorphism between $C(W)$-bimodules and

$$\mathrm{End}({}_{C(W)}L^2(C)_{C(W)}) \cong M_2(\mathbb{C})$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \xi + \eta uh \mapsto a\xi + b\eta + (c\xi + d\eta)uh$$

for $\xi, \eta \in L^2(D)$. It is immediate to see that operators of this form commute with left and right multiplications of h if and only if $a = d$, $b = c$:

$$\mathrm{End}({}_{C(V)}L^2(C)_{C(V)}) \cong \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} ; a, b \in \mathbb{C} \right\} \cong \mathbb{C} \oplus \mathbb{C}.$$

Theorem 10.6 (Araki)**.** Let C be a covariance operator for V. The following are equivalent.

(i) The W*-algebra $\mathrm{End}(L^2(C)_{C(V)})$ is not a factor.
(ii) The bimodule ${}_{C(V)}L^2(C)_{C(V)}$ is reducible.
(iii) The operator $\sqrt{C(1-C)}$ is in the Hilbert-Schmidt class and $\ker(C-1/2)$ is odd-dimensional.
(iv) We can find a non-Fock extremal covariance operator E such that $\sqrt{C}-\sqrt{E}$ is in the Hilbert-Schmidt class.

If this is the case, we have

$$\mathrm{End}(L^2(C)_{C(V)}) \cong \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H}), \quad \mathrm{End}({}_{C(V)}L^2(C)_{C(V)}) \cong \mathbb{C} \oplus \mathbb{C}.$$

Corollary 10.7. Let C be a covariance operator on V with M_C the associated W*-algebra.

(i) The parity automorphism is inner on M_C if and only if we can find a Fock projection F such that $\sqrt{C}-F$ is in the Hilbert-Schmidt class.
(ii) M_C is not a factor if and only if we can find a non-Fock extremal covariance operator E such that $\sqrt{C}-\sqrt{E}$ is in the Hilbert-Schmidt class.

Theorem 10.8. Let C be a covariance operator with M_C not a factor. Then $g \in O(V)$ is C-inner if and only if $gC - Cg$ is in the Hilbert-Schmidt class and $\det(g) = 1$, i.e., g belongs to the connected component $SO(V)$ in $O(V)$ of the identity transformation.

Proof. By the previous theorem, we can find a non-Fock extremal covariance operator E such that $\sqrt{C}-\sqrt{E}$ is in the Hilbert-Schmidt class. Let g be C-inner. Then θ_g leaves $L^2(C) = L^2(E)$ leaves invariance, whence $\sqrt{E}g\sqrt{1-E} - \sqrt{1-E}g\sqrt{E}$ is in the Hilbert-Schmidt class by Corollary 9.3. Let $E = P + \frac{1}{2}|h)(h|$ be the spectral decomposition with $h = \overline{h} \in V$ satisfying $(h|h) = 2$ and set $W = (P+\overline{P})V$. Then $\sqrt{E} = P + \frac{1}{\sqrt{2}}|h)(h|$ and $\sqrt{1-E} = \overline{P} + \frac{1}{\sqrt{2}}|h)(h|$ are used to see that $Pg - gP$ is in the Hilbert-Schmidt class. On the other hand, $\sqrt{C} = P + \mathrm{HS}$ implies $C = P + \mathrm{HS}$ and hence $gC - Cg$ is in the Hilbert-Schmidt class.

If h is an eigenvector of g, $gh = \pm h$ and g commutes with $P + \overline{P} = 1 - |h)(h|$. Then the reduced transformation g_W is P-inner and we can find an implementing unitary u satisfying $u^2 = 1$.

In terms of the decomposition $\overline{C(V)\varphi_E^{1/2}} = \overline{C(W)\varphi_E^{1/2}} \oplus \overline{C(W)\varphi_E^{1/2}}uh$, $M_E = \mathrm{End}(\overline{C(V)\varphi_E^{1/2}}_{C(V)})$ is realized by the matrix representation

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

with $x, y \in M_P$ so that the left multiplication by h is represented by $x = 0$, $y = 1$. The unitary operator $\mathrm{Ad}\theta_g$ is then represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} x & \pm y \\ \pm y & x \end{pmatrix}$$

shows that θ_g leaves central projections

$$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

of M_E invariant if and only if $gh = h$. Thue θ_g is C-inner or not C-inner according to $gh = \pm h$. Note here that under the invariance of $\mathbb{C}h$ by g, the condition $gh = h$ is equivalent to the condition $g \in SO(V)$.

Now assume that h and gh are linearly independent. Then we can find a rotation operator on $\langle h, gh\rangle$ which maps gh into h. So, if we extend it by the identity on $\langle h, gh\rangle^\perp$, then we have a rotation transformation $r \in SO(V)$ such that rg preserves $\mathbb{C}h$ and $r - 1$ is of rank 2. Cleraly r is E-inner and the E-innerness of g is reduced to that of rg, i.e., whether rg belongs to $SO(V)$ or not, which is equivalent to $g \in SO(V)$ or not because $r \in SO(V)$. □

Investigate the state of density operator

$$\exp(haa^* + ka^*a + za + \overline{z}a^*).$$

11. Free States on Even Subalgebras

Recall that the CAR algebra $C(V)$ is $\mathbb{Z}_2$-graded and we have the parity decomposition $C(V) = C_0(V) + C_1(V)$ with $C_0(V) = C(V)^{\mathbb{Z}_2}$ the even subalgebra. We shall here investigate the restriction of free states to $C_0(V)$ and their representations.

Let $E : C(V) \ni x \mapsto (x + \epsilon(x))/2 \in C_0(V)$ be the conditional expectation. An isometric embedding $L^2(C_0(V)) \to L^2(C(V))$ is then defined by $y\omega^{1/2}y' \mapsto y(\omega \circ E)^{1/2}y'$, where $y, y' \in C_0(V)$ and ω is a state of $C_0(V)$.

Given an even state φ of $C(V)$ with the restriction to $C_0(V)$ denoted by φ_0, we have an orthogonal decomposition

$$\overline{C(V)\varphi^{1/2}C(V)} = (\overline{C(V)\varphi^{1/2}C(V)})_0 + (\overline{C(V)\varphi^{1/2}C(V)})_1$$

with

$$(\overline{C(V)\varphi^{1/2}C(V)})_0 = \overline{C_0(V)\varphi^{1/2}C_0(V) + C_1(V)\varphi^{1/2}C_1(V)},$$
$$(\overline{C(V)\varphi^{1/2}C(V)})_1 = \overline{C_0(V)\varphi^{1/2}C_1(V) + C_1(V)\varphi^{1/2}C_0(V)}$$

and $\overline{C_0(V)\varphi^{1/2}C_0(V)}$ isometrically isomorphic to $\overline{C_0(V)\varphi_0^{1/2}C_0(V)}$. This simple observation is already enough to see that, if two even states φ and ψ are disjoint, then so are φ_0 and ψ_0. Thus, for the analysis of representations free states on $C_0(V)$, we may suppose that they are not disjoint as states on $C_(V)$, which is equivalent to assuming that they are quasi-equivalent thanks to the dichotomy.

Lemma 11.1. Let S be a covariance operator for V and assume that S is not a projection. Then

$$\overline{C_0(V)\varphi^{1/2}C_0(V)} = \overline{C_1(V)\varphi^{1/2}C_1(V)}.$$

Proof. Let $W^{\mathbb{C}} = \ker(S(I - S))^\perp$ as in the proof of Proposition (main). By using the expression $C_1(V) = \overline{C_1(W^\perp)C_0(W) + C_0(W^\perp)C_1(W)}$ and its adjoint relation, we need to deal with four subspaces

$$\begin{aligned}
&C_1(W^\perp)C_0(W)\varphi^{1/2}C_0(W)C_1(W^\perp),\\
&C_0(W^\perp)C_1(W)\varphi^{1/2}C_0(W)C_1(W^\perp),\\
&C_1(W^\perp)C_0(W)\varphi^{1/2}C_1(W)C_0(W^\perp),\\
&C_0(W^\perp)C_1(W)\varphi^{1/2}C_1(W)C_0(W^\perp).
\end{aligned}$$

Since $\overline{C(W)\varphi^{1/2}} = \overline{\varphi^{1/2}C(W)}$ as witnessed in the proof of Proposition (main or quadrature), $\overline{C_0(W)\varphi^{1/2}} = \overline{\varphi^{1/2}C_0(W)}$ and $\overline{C_1(W)\varphi^{1/2}} = \overline{\varphi^{1/2}C_1(W)}$ as even and odd parts of this, which is then used to get (note that $W \neq \{0\}$)

$$\overline{C_0(W)\varphi^{1/2}C_0(W)} = \overline{C_0(W)\varphi^{1/2}} = \overline{C_1(W)C_1(W)\varphi^{1/2}} = \overline{C_1(W)\varphi^{1/2}C_1(W)}$$

and

$$\overline{C_1(W)\varphi^{1/2}C_0(W)} = \overline{C_0(W)\varphi^{1/2}C_1(W)}, \overline{C_1(W)\varphi^{1/2}C_1(W)} = \overline{C_0(W)\varphi^{1/2}C_0(W)}.$$

Thus the four subspaces are included in $\overline{C_0(V)\varphi^{1/2}C_0(V)}$ and the claim is proved. □

Corollary 11.2. Let S and T be non-Fock covariance operators for V. Then the restricted states $(\varphi_S)_0$ and $(\varphi_T)_0$ of $C_0(V)$ are quasi-equivalent if and only if so are φ_S and φ_T. Furthermore, $(\varphi_S)_0$ and $(\varphi_T)_0$ are disjoint unless they are quasi-equivalent.

Proof. Since $\psi^{1/2}$ is an even vector and belongs to $\overline{C(V)\varphi^{1/2}C(V)}$ by the quasi-equivalence assumption, it in fact belongs to $\overline{C_0(V)\varphi^{1/2}C_0(V)}$ and then $\overline{C_0(V)\psi^{1/2}C_0(V)} \subset \overline{C_0(V)\varphi^{1/2}C_0(V)}$. By symmetry, the reverse inclusion and therefore the equality $\overline{C_0(V)\psi^{1/2}C_0(V)} = \overline{C_0(V)\varphi^{1/2}C_0(V)}$ hold, which means $\overline{C_0(V)\psi_0^{1/2}C_0(V)} = \overline{C_0(V)\varphi_0^{1/2}C_0(V)}$. □

Before going into the exceptional case of Fock projections, we study the factoriality of restricted states first. Recall that, if a compact group G acts continuously on a W*-algebra N by a group homomorphism $\theta : G \to \mathrm{Aut}(N)$, then the crossed product $N \rtimes G$ is represented on $L^2(N)$ through the covariance representations of $\{\mathrm{Ad}(\theta_g)\}_{g\in G}$ together with the left or right action of N (the reason: the trivial representaion is contained in the regular representation of G as a subrepresentation). Since the fixed point subalgebra N^G is identified with $\mathrm{End}(L^2(N)_{N\rtimes G})$, we have an imprimitivity bimodule ${}_{N^G}L^2(N)_{N\rtimes G}$, which gives a kind of duality between N^G and $N \rtimes G$. For example, if N is a factor, G is finite and the action θ is outer, then $N \rtimes G$ is a factor again, whence N^G is a factor as well. If we apply this observation to the W*-algebra M_C generated from a free state φ_C of $C(V)$ and the parity automorphism, we obtain the following (Example 10.3? and Araki's characterization of factoriality of M_C):

Proposition 11.3. Let C be a covariance operator and assume that φ_C is not quasi-equivalent to either a Fock state or a pseudo-Fock state. Then M_C is a factor and the parity automorphism is outer on M_C, whence $(\varphi_C)_0$ is a factor state of $C_0(V)$.

Proposition 11.4. Let P be a Fock projection for V. Then the left representations of $C_0(V)$ on $\overline{C_0(V)\varphi_P^{1/2}}$ and on $\overline{C_1(V)\varphi_P^{1/2}}$ are irreducible and inequivalent.

Let φ_S be a pseudo-Fock state of $C(V)$. Then the left representations of $C_0(V)$ on $\overline{C_0(V)\varphi_S^{1/2}}$ and on $\overline{C_1(V)\varphi_S^{1/2}}$ are irreducible and equivalent.

Proof. As a commutant of the parity operator Π in $M_P = \mathcal{B}(\overline{C(V)\varphi_P^{1/2}})$, $C_0(V)$ generates the W*-algebra isomorphic to $\mathcal{B}(\overline{C_0(V)\varphi_P^{1/2}}) \oplus \mathcal{B}(\overline{C_1(V)\varphi_P^{1/2}})$.

Let h be a real central element of V such that $h^2 = 1$. Then $\overline{C_0(V)\varphi_S^{1/2}}$ is isometrically isomorphic to $\overline{C_1(V)\varphi_S^{1/2}}$ by the right multiplication of h and, under this identification, we see that M_S consists of elements of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with a, $b \in \mathcal{B}(\overline{C_0(V)\varphi_S^{1/2}})$. □

Now assume that C is a covariance operator such that $C^{1/2} - P$ is in the Hilbert-Schmidt class but C is not a projection. Then $\overline{C(V)\varphi_P^{1/2}C(V)} = \overline{C(V)\varphi_C^{1/2}C(V)}$ and then by extracting the even part

$$\overline{C_0(V)\varphi_P^{1/2}C_0(V)} \oplus \overline{C_1(V)\varphi_P^{1/2}C_1(V)} = \overline{C_0(V)\varphi_C^{1/2}C_0(V)}$$

with $\overline{C_0(V)\varphi_P^{1/2}C_0(V)}$ and $\overline{C_1(V)\varphi_P^{1/2}C_1(V)}$ inequivalent components. Thus the even part of $L^2(C)$ contains two inequaivalent factorial components, one of which is equal to $\overline{C_0(V)\varphi_P^{1/2}C_0V)}$; the dichotomy breaks down here.

Choose an orthonormal basis $\{h_j\}_{j\geq 1}$ in V and consider the orthogonal transformation $\mathrm{Ad}(h_1)$; $\mathrm{Ad}(h_1)h_j = -h_j$ for $j \neq 1$ and $\mathrm{Ad}(h_1)h_1 = h_1$. Let Q be another Fock projection defined by

$$Q = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix} P \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} P \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}.$$

Then $\overline{C_0(V)(\varphi_P)_0^{1/2}C_0(V)}$ and $\overline{C_0(V)(\varphi_Q)_0^{1/2}C_0(V)}$ are orthogonal and

$$\overline{C_0(V)(\varphi_C)_0^{1/2}C_0(V)} = \overline{C_0(V)(\varphi_P)_0^{1/2}C_0(V)} + \overline{C_0(V)(\varphi_Q)_0^{1/2}C_0(V)}.$$

Proposition 11.5. Let P and Q be Fock projections such that $P - Q$ is in the Hilbert-Schmidt class. Then pure states $(\varphi_P)_0$ and $(\varphi_Q)_0$ of $C_0(V)$ are equivalent if and only if $\mathrm{index}(P, Q) = (-1)^{\dim[P\wedge(1-Q)]} = 1$.

Proof. According to the parity of $\mathrm{index}(P, Q)$, $\varphi_Q^{1/2}$ belongs to $\overline{C_0(V)\varphi_P^{1/2}C_0(V)}$ or $\overline{C_1(V)\varphi_P^{1/2}C_1(V)}$ by Lemma parity. □

Remark 6. A covariance operator C satisfys the condition that we can find an extremal covariance operator D (thus φ_D is either a Fock state or a pseud-Fock state) such that $\sqrt{C} - \sqrt{D}$ is in the Hilbert-Schmidt class if and only if $C(I - C)$ is in the trace class. In fact,the condition is sufficient if we choose a spectral operator of C as D. To see the reverse implication, note that $D(I-D)$ is an at most rank one operator for an extremal D, whence

$$\sqrt{C(I-C)} = (\sqrt{C}-\sqrt{D})\sqrt{C} + \sqrt{D}(\sqrt{C}-\sqrt{D}) + \sqrt{D(I-D)}$$

is in the Hilbert-Schmidt class.

Note that D is a projection or not according to the parity of $\dim\ker(C-1/2)$.

Appendix A. Angle Operators

Let P, Q be projections in a Hilbert space $\mathcal{H}$ and set $C=(P-Q)^2$.

Since $-Q \leq P-Q \leq P$, $0 \leq (P-Q)^2 \leq 1$ and

$$P(P-Q)^2 = P(1-Q)(P-Q) = P(1-Q)P = (P-Q)^2P,$$
$$QC = Q(P-Q)^2 = Q(1-P)Q$$

show that C commutes with both of P and Q.

Note that

$$\ker(P-Q)^2 = (P\wedge Q + (1-P)\wedge(1-Q))\mathcal{H},$$
$$\ker(1-(P-Q)^2) = (P\wedge(1-Q) + (1-P)\wedge Q)\mathcal{H}.$$

We think of these subspaces being trivial and focus on operators reduced by the spectral projection $E=[C(1-C)]$ of $C=(P-Q)^2$, which is the orthogonal complement to

$$P\wedge Q + (1-P)\wedge(1-Q) + P\wedge(1-Q) + (1-P)\wedge Q.$$

In other words, subtracting these trivial subspaces out of $\mathcal{H}$, we assume that $E=1$ in what follows.

Consider the polar decomposition of $(1-Q)PQ$. From the relation

$$QP(1-Q)PQ = QPCQ = QPQC = QC(1-C),$$

if we denote the phase part by U, it satisfies

$$(1-Q)PQ = U\sqrt{QC(1-C)} \quad \text{and} \quad U^*U = [QC(1-C)] = Q.$$

Here in the last equality, we have used $[C(1-C)]=1$. From a similar computation

$$(1-Q)PQP(1-Q) = (1-Q)P(1-C)(1-Q) = (1-Q)P(1-Q)(1-C) = (1-Q)C(1-C),$$

we have $UU^* = [(1-Q)C(1-C)] = 1-Q$, which is used to get $U^2 = 0 = U^*U^*$ and $(U-U^*)^2 = -1$.

If we set $iH = (U-U^*)\arcsin\sqrt{C}$, $e^{iH} = \sqrt{1-C} + \sqrt{C}(U-U^*)$ is a unitary and $e^{iH}Qe^{-iH} = P$.

$$\begin{aligned}
e^{iH}Qe^{-iH} &= (1-C)Q + \sqrt{C(1-C)}U + \sqrt{C(1-C)}U^* + C(1-Q)\\
&= (1-(P-Q)^2)Q + (1-Q)PQ + QP(1-Q) + (P-Q)^2(1-Q)\\
&= Q - (1-P)Q - (1-Q)PQ\\
&\quad + (1-Q)PQ + QP(1-Q) + P(1-Q) - QP(1-Q)\\
&= P.
\end{aligned}$$

Remark 7. A model for computation is the following: Let $0 < \theta < \pi/2$.

$$P = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$C = (P-Q)^2 = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad U^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now le P and Q be Fock projections on a *-Hilbert space V. In view of $\overline{C} = C$, we observe

$$\overline{U}\sqrt{C(1-C)} = Q(1-P)(1-Q) = -QP(1-Q) = -U^*\sqrt{C(1-C)},$$

which gives $\overline{U} = -U^*$ by the uniqueness of polar decomposition, whence e^{iH} is an orthogonal transformation on EV.

Since $\overline{P \wedge (1-Q)} = (1-P)\wedge Q$, we can find a partial isometry u from $P\wedge(1-Q)$ to $(1-P)\wedge Q$ and

$$T = P\wedge Q + (1-P)\wedge(1-Q) + u + \overline{u} + \sqrt{E-C} + \sqrt{C}(U-U^*)$$

is an orthogonal transformation on V satisfying $TPT^* = Q$.

The obstruction to writing T of the form $T = e^{iH}$ is the parity of $n = \mathrm{rank}(P\wedge(1-Q)) = \mathrm{rank}((1-P)\wedge Q$. In fact, choose an orthonormal basis $\{\xi_j\}_{1\le j\le n}$ of $[P\wedge(1-Q)]$ and set $v_j = (\xi_j + \overline{\xi_j})/\sqrt{2}$, $w_j = (\xi_j - \overline{\xi_j})/\sqrt{2}i$, which constitute a real basis of $[P\wedge(1-Q) + (1-P)\wedge Q]$ and, if a real partial isometry u is defined by

$$u\xi_j = \overline{\xi_j}, \quad u\overline{\xi_j} = \xi_j,$$

it has a matrix representation of the form

$$\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$$

with respect to the basis $\{v_j, w_j\}$. Thus, if the rank n is finite and odd, it is impossible to express $u + \overline{u}$ in the form e^{ih} with $h^* = h = -\overline{h}$, while for the even rank, we can find such an operator h of spectrum $\{0, \pm\pi\}$, and $T = e^{iH}$ if we set

$$iH = (U-U^*)\arcsin\sqrt{C} + ih,$$

which is in the Hilbert-Schmidt class, i.e., in $o_2(V)$. For example, we may choose

$$ih = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & 0 & \pi & 0 \end{pmatrix}$$

in the case of $n = 2$. Note that, in the last matrix representation, the conjugation is given by the ordinary component-wise complex conjugation, which guarantees $\overline{ih} = ih$.

Remark 8. The previous model does not serve Fock projections.

In the text, we are faced with the situation where $P-Q$ is in the Hilbert-Schmidt class. In that case, the spectrum $\sigma(C)$ is discrete except for the value 0 and, for an eigenvalue $0 < c \leq 1$, $\ker(C-c)$ is finite-dimensional. Note that Q leaves invariant these eigenspaces.

Let $\xi \in V$ satisfy $C\xi = c\xi$ and $Q\xi = \xi$. Then $U\xi \in (1-Q)V$ is orthogonal to ξ^* from

$$(U\xi|\xi^*) = (\xi|U^*\xi^*) = -(\xi|\overline{U}\xi^*) = -(\xi|(U\xi)^*) = -(U\xi|\xi^*).$$

In this way, we have an orthonormal system $\{\xi, \eta, \xi^*, \eta^*\}$ of $\ker(C-c)$ satisfying

$$Q\xi = \xi, \quad Q\eta = \eta, \quad U\xi = \eta^*, \quad U\eta = -\xi^*.$$

If this is not a basis for $\ker(C-c)$, we can repeat the construction to conclude that we can find a Fock system $\{a_j^*, b_j^*, a_j, b_j\}$ constiting of eigenvectors of $C(1-C)$ of non-zero eigenvalues such that $a_j^*, b_j^* \in QV$ and

$$Ua_j^* = b_j, \quad Ub_j^* = -a_j.$$

If we use $\theta = \arcsin c$ $(0 < \theta < \pi/2)$ as a parameter, on the reducing subspace $\langle a_j^*, a_j^*, a_j, b_j\rangle$, we have the following matrix representation with respect to a Fock basis $\{a_j^*, b_j^*, a_j, b_j\}$:

$$U - U^* = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad C = \sin^2\theta_j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$iH = \begin{pmatrix} 0 & J\theta_j \\ J\theta_j & 0 \end{pmatrix}, \quad e^{iH} = \begin{pmatrix} I\cos\theta_j & J\sin\theta_j \\ J\sin\theta_j & I\cos\theta_j \end{pmatrix},$$

$$Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} I\cos^2\theta_j & -J\cos\theta\sin\theta_j \\ J\cos\theta\sin\theta_j & I\sin^2\theta_j \end{pmatrix}$$

with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, the Hilbert-Schmidt class operator iH is given by

$$iH = \sum_j \theta_j \Big(|b_j)(a_j^*| - |a_j)(b_j^*| + |b_j^*)(a| - |a_j^*)(b_j| \Big).$$

Since $\overline{P}$ is represented by

$$\begin{pmatrix} I\sin\theta_j \\ -J\cos\theta_j \end{pmatrix} \begin{pmatrix} I\sin\theta_j & J\cos\theta_j \end{pmatrix} = \begin{pmatrix} \sin\theta_j & 0 \\ 0 & \sin\theta_j \\ 0 & \cos\theta_j \\ -\cos\theta_j & 0 \end{pmatrix} \begin{pmatrix} \sin\theta_j & 0 & 0 & -\cos\theta_j \\ 0 & \sin\theta_j & \cos\theta_j & 0 \end{pmatrix},$$

we have

$$\overline{P}\langle a_j^*, b_j^*, a_j, b_j\rangle = \langle a_j^*\sin\theta_j - b_j\cos\theta_j, b_j^*\sin\theta_j + a_j\cos\theta_j\rangle.$$

Appendix B. Determinant

Let A be a linear operator on a vector space V. If V is finite-dimensional and A is diagonalized with respect to an eigenbasis (v_j) so that $Av_j = a_j v_j$, then

$$\wedge^k A = A \wedge \cdots \wedge A$$

is diagonalized relative to the basis (v_F), where F is a finite subset of $\{j\}$ such that $|F| = k$ and $v_F = \wedge_{j \in F} v_j$. Therefore $(\wedge^k A)v_F = (\prod_{j \in F} a_j) v_F$ and

$$\mathrm{trace}(\wedge^k A) = \sum_{|F|=k} \prod_{j \in F} a_j.$$

With these trace relations, we have

$$\det(\lambda I - A) = \prod_j (\lambda - a_j) = \lambda^n - \lambda^{n-1}\mathrm{trace}(A) + \cdots + (-1)^n \mathrm{trace}(\wedge^n A)$$

and then, by replacing λ with $-1/z$,

$$\det(I + zA) = 1 + \mathrm{trace}(A)z + \cdots + \mathrm{trace}(\wedge^n A)z^n.$$

Now assume that V is a Hilbert space and A is in the trace class. Recall that for a positive operator A, the value

$$\mathrm{trace}(A) = \sum_j (\xi_j | A\xi_j)$$

is independent of the choice of an orthonormal basis (ξ_j). A bounded operator A is said to be trace-class if $\|A\|_1 = \mathrm{trace}(|A|) < \infty$. For a trace-class operator, the value (called the trace of A)

$$\sum_j (\xi_j | A\xi_j)$$

is again independent of the choice of an orthonormal basis (ξ_j) and denoted by $\mathrm{trace}(A)$.

A bounded operator A is called a Hilbert-Schmidt operator if A^*A is in the trace class. The Hilbert-Schmidt norm of a Hilbert-Schmidt operator A is then defined by $\|A\|_{HS} = \sqrt{\mathrm{trace}(A^*A)}$.

Let A and B be hermitian operators. Then

$$\|A + iB\|_p \geq \max\{\|A\|_p, \|B\|_p\}$$

for $1 \leq p \leq +\infty$.

Let

$$P_n = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma$$

be the projection to $\wedge^n V \subset V^{\otimes n}$. We set

$$v_1 \wedge \cdots \wedge v_n = \sqrt{n!} P_n(v_1 \otimes \cdots \otimes v_n)$$

so that

$$(v_1 \wedge \cdots \wedge v_n | w_1 \wedge \cdots \wedge w_n) = \det(v_j | w_k).$$

Wedge operation is also applied to operators: Given operators $A_1, \dots, A_n$ on V, set

$$A_1 \wedge \cdots \wedge A_n = P_n(A_1 \otimes \cdots \otimes A_n)P_n = .$$

When $A_j = A$ for $1 \le j \le n$, we write $\wedge^n A$ and remark that

$$(\wedge^n A)(v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n$$

in view of the commutativity of P_n and $A^{\otimes n}$. The following is immediate from the definition.

$$(A_1 \wedge \cdots \wedge A_n)^* = A_1^* \wedge \cdots \wedge A_n^*$$

and

$$(\wedge^n A)(\wedge^n B) = \wedge^n (AB).$$

In particular, if $A = U|A|$ is a polar decomposition, then so is $\wedge^n A = (\wedge^n U)(\wedge^n |A|)$.

If $|A|$ is trace-class with $\{a_j\}$ eigenvalues of $|A|$,

$$k!\mathrm{trace}(\wedge^k |A|) \le (\sum_j a_j)^k = \|A\|_1^k$$

shows that $\| \wedge^k (A)\|_1 \le \|A\|_1^k / k!$. Taking summation on $k \ge 0$, we define the (Fredholm) determinant

$$\det(I + A) = \sum_{k \ge 0} \mathrm{trace}(\wedge^k A)$$

of $I + A$ as an absolutely covergent series, which satisfies

$$|\det(I + A)| \le e^{\|A\|_1}.$$

The estimate for $\wedge^n A$ is now generalized in a multiple way: If A, B and C are trace-class operators on V,

$$\| \overbrace{A \wedge \cdots \wedge A}^{k\text{-times}} \wedge \overbrace{B \wedge \cdots \wedge B}^{l\text{-times}} \wedge \overbrace{C \wedge \cdots \wedge C}^{m\text{-times}} \|_1 \le \frac{1}{k!l!m!} \|A\|_1^k \|B\|_1^l \|C\|_1^m.$$

In fact,

$$\begin{aligned}
&\|P_{k+l+m}(A^{\otimes k} \otimes B^{\otimes l} \otimes C^{\otimes m})P_{k+l+m}\|_1 \\
&\qquad = \|P_{k+l+m}(A^{\otimes k} \otimes B^{\otimes l} \otimes C^{\otimes m})(P_k \otimes P_l \otimes P_m)P_{k+l+m}\|_1 \\
&\qquad \le \|P_{k+l+m}\| \, \|(A^{\otimes k} \otimes B^{\otimes l} \otimes C^{\otimes m})(P_k \otimes P_l \otimes P_m)\|_1 \, \|P_{k+l+m}\| \\
&\qquad = \| \wedge^k A \otimes \wedge^l B \otimes \wedge^m C\|_1 = \| \wedge^k A\|_1 \, \| \wedge^l B\|_1 \, \| \wedge^m C\|_1 \\
&\qquad \le \frac{\|A\|_1^k}{k!} \frac{\|B\|_1^l}{l!} \frac{\|C\|_1^m}{m!}.
\end{aligned}$$

Let A and B be trace-class operators on V. Then

$$|\det(I + A) - \det(I + B)| \le \|A - B\|_1 e^{\|A\|_1 + \|B\|_1}.$$

If the identity

$$A^{\otimes n} - B^{\otimes n} = A^{\otimes n} - A^{\otimes (n-1)} \otimes B + A^{\otimes (n-1)} \otimes B - A^{\otimes (n-2)} \otimes B \otimes B + \cdots + A \otimes B^{\otimes (n-1)} - B^{\otimes n}$$

is reduced by the projection P_n, then we have

$$\wedge^n A - \wedge^n B = \wedge^{n-1} A \wedge (A - B) + \wedge^{n-2} A \wedge (A - B) \wedge B + \cdots + (A - B) \wedge \wedge^{n-1} B$$

and its trace norm is estimated by the previous inequality as

$$\begin{aligned}\| \wedge^n A - \wedge B^n\|_1 &\leq \|A-B\|_1 \sum_{k=0}^{n-1} \| \wedge^{n-k-1} A\|_1 \| \wedge^k B\|_1 \\ &\leq \|A-B\|_1 \sum_{k=0}^{n-1} \frac{1}{(n-k-1)!k!} \|A\|_1^{n-k-1} \|B\|_1^k \\ &= \frac{1}{(n-1)!} \|A-B\|_1 (\|A\|_1 + \|B\|_1)^{n-1}.\end{aligned}$$

Let C_n be a sequence of positive operators and assume that it converges to a positive operator C in the weak operator topology. By a simple manipulation like Fatou's lemma, we see

$$\liminf_{n\to\infty} \operatorname{trace}(C_n) \geq \operatorname{trace}(C).$$

If we further assume that $0 \leq C_n \leq 1$ and $\operatorname{trace}(C) = +\infty$, then

$$\lim_{n\to\infty} \det(I - C_n) = 0.$$

In fact, if $\limsup_{n\to\infty} \det(I - C_n) > 0$, then we can find $\delta > 0$ and a subsequence (n') such that

$$\det(I - C_{n'}) \geq \delta$$

for $n \geq 1$. This particularly implies $C_{n'} \leq (1-\delta)I$ and

$$\det(I + C_{n'} + C_{n'}^2 + \dots) \leq \frac{1}{\delta}$$

and hence

$$\operatorname{trace}(C_{n'}) \leq \operatorname{trace}(C_{n'} + C_{n'}^2 + \dots) \leq \frac{1}{\delta},$$

which contradicts with

$$\liminf_{n\to\infty} \operatorname{trace}(C_n) \geq \operatorname{trace}(C) = \infty.$$

Appendix C. Purification

Consider a measurable family $\{M_t\}_{t\in T}$ of W*-algebras with the integrated W*-algebra

$$M = \int_T^\oplus M_t.$$

Here T is a measure space with a specified measure class. Then the purification representation π of $M \otimes M^\circ$ is disintegrated as

$$\pi(a \otimes b^\circ) = \int_T^\oplus \pi_t(a_t \otimes b_t^\circ)$$

for

$$a = \int_T^\oplus a_t, \qquad b = \int_T^\oplus b_t$$

in M. Let

$$\varphi = \int_T \varphi_t\, \mu(dt)$$

be an integrated positive normal functional of M. Note that the measure μ is the pullback of φ by $L^\infty(T) \subset M$ and the measurable family $\{\varphi_t\}$ is supported by μ. Then the purification Φ of φ is also decomposed as

$$\Phi(a \otimes b^\circ) = \int_T \Phi_t(a_t \otimes b_t^\circ)\,\mu(dt).$$

If each M_t is factorial, then Φ can be described in terms of a measurable family of density operators (rank-one operators in this context) as

$$\Phi(\cdot) = \int_T \mu(dt)\,\mathrm{tr}\Big(|\varphi_t^{1/2})(\varphi_t^{1/2}|\pi_t(\cdot)\Big)$$

and then $\Phi^{1/2}$ corresponds to

$$\sqrt{\mu(dt)}\frac{1}{\varphi_t(1)^{1/2}}|\varphi_t^{1/2})(\varphi_t^{1/2}|\mathrm{tr}^{1/2}.$$

Thus, if Ψ is a purification of ψ with the pullbacked measure ν,

$$(\Phi^{1/2}|\Psi^{1/2}) = \int_T \sqrt{\mu(dt)\nu(dt)}\,\frac{(\varphi_t^{1/2}|\psi_t^{1/2})^2}{\sqrt{\varphi_t(1)\psi_t(1)}}.$$

Appendix D. Quadrature

Finally consider the case that $\dim V$ is finite and odd. With the notation in §1,

$$\begin{aligned}
\pi(z' \oplus 0) &: x' \oplus y' \mapsto z'x' \oplus z'y' \\
\pi(c \oplus 0) &: x' \oplus y' \mapsto x' \oplus -y' \\
\pi(0 \oplus z') &: x' \oplus y' \mapsto \epsilon(y')z' \oplus \epsilon(x')z' \\
\pi(0 \oplus c) &: x' \oplus y' \mapsto \epsilon(y') \oplus -\epsilon(x').
\end{aligned}$$

The third and the fourth are combined to have

$$\pi(0 \oplus z')\pi(0 \oplus c) : x' \oplus y' \mapsto -x'z' \oplus y'z'.$$

The commutativity with the first operators forces us to take

$$(x', y') \mapsto (x', y')\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

with $a', b', c', d' \in C(V')$. The commutativity with the second operator then requires $b' = c' = 0$, whereas the combined operations show that a' and d' are scalars in view of the factoriality of $C(V')$. Lastly, the commutativity with the fourth operator means $a' = d'$, proving the triviality of $\pi(C(V \oplus V))'$.

Appendix E. Bilinear Forms on Hilbert Spaces

Let $\beta(x, y)$ be a bounded bilinear form on a Hilbert space $\mathcal{H}$. The unitary group $U(\mathcal{H})$ naturally acts on the set of bilinear forms by

$$(\beta T)(x, y) = \beta(Tx, Ty), \qquad x, y \in \mathcal{H}.$$

Choose a real structure on $\mathcal{H}$, i.e., a conjugation $x \mapsto \overline{x}$ on $\mathcal{H}$, and the associated transposed map operation on $T \in \mathcal{H}(\mathcal{H})$ is denoted by ${}^tT$. If we express β by a bounded operator B so that

$$\beta(x, y) = (\overline{x}|By),$$

then βT corresponds to ${}^tTBT$. Note that, we change the real structure to a new one $\Gamma x = V\overline{V^* x} = V^t V\overline{x}$ with $V \in U(\mathcal{H})$, then B is changed to $V^t V B$.

The bilinear form is symmetric or alternating according to ${}^tB = B$ or ${}^tB = -B$, which is assumed in what follows and we try to find a standard form under the action of $U(\mathcal{H})$. By passing to the orthogonal complement of

$$\{x \in \mathcal{H}; \beta(x,y) = \forall y \in \mathcal{H}\} = \{y \in \mathcal{H}; \beta(x,y) = \forall x \in \mathcal{H}\},$$

we may assume that β is non-degenerate for this problem.

Let $B = U|B|$ be the polar decomposition. Note that the positive part $|B|$ is independent of the choice of real structure. Taking transposed, the uniqueness of polar decomposition implies

$${}^tU = \pm U, \qquad |B| = {}^tU^{*t}|B|^tU.$$

The first identity is then used to We introduce an antiunitary operator $C : \mathcal{H} \to \mathcal{H}$ by

$$Cx = \overline{Ux},$$

which satisfies $C^2 = \pm 1$ as a consequence of the first identity ${}^tU = \pm U$. The second identity $\overline{|B|} = U|B|U^*$ is then equivalent to $|B|C = C|B|$ by

$$|B|Cx = |B|\overline{Ux} = \overline{\overline{|B|}Ux} = \overline{U|B|x} = C|B|x.$$

Proposition E.1. Let β be a non-degenerate bilinear form which is symmetric or alternating.

Then we can find a positive operator D and an antiunitary operator C on $\mathcal{H}$ so that C commutes with D, satisfies $C^2 = \pm 1$ according to $\beta(x,y) = \pm\beta(y,x)$, and

$$\beta(x,y) = \pm(Cx|Dy)$$

for $x, y \in \mathcal{H}$.

Conversely, a pair of operators (C, D) satisfying $CD = DC$ and $C^2 = \pm 1$ gives rise to a symmetric or alternating bilinear form β by the above relation, and the pair (C, D) is uniquely determined by η.

We describe the results in the finite-dimensional cases.

(i) Symmetric Case

We fan find an orthonormal basis $\{\xi_j\}$ with $d_j \geq 0$ so that

$$\beta(\xi_j, \xi_k) = \delta_{j,k} d_j.$$

(ii) Alternating Case

We can find an orthonormal basis $\{\xi_j, \eta_j\} \cup \{\zeta_l\}$ with $d_j > 0$ so that $\{\zeta_l\}$ spans the kernel of β and

$$\beta(\xi_j, \eta_k) = -\beta(\eta_k, \xi_j) = \delta_{j,k} d_j, \qquad \beta(\xi_j, \xi_k) = \beta(\eta_j, \eta_k) = 0.$$

Appendix F. Von Neumann's Theorem

Lemma F.1. Let $0 \leq a \leq 1$ be a bounded operator on a Hilbert space $\mathcal{H}$. Given a finite family of vectors $\{\xi_j\}_{1 \leq j \leq n}$ in $\mathcal{H}$ and $\epsilon > 0$, we can find a finite rank projection p and a finite-rank hermitain operator h such that

$$\|(1-p)\xi_j\| \leq \epsilon \ (1 \leq j \leq n), \quad \|h\|_2 \leq \epsilon, \quad 0 \leq a + h \leq 1, \quad [a + h, p] = 0.$$

Theorem F.2. Let a be a bounded hermitian operator on a seperable Hilbert space $\mathcal{H}$ satisfying $\lambda I \le a \le \mu I$ with $\lambda < \mu$ real numbers. Given any $\epsilon > 0$, we can find a hermitian operator b of pure point spectra satisfying $\lambda I \le b \le \mu I$ and $\|a - b\|_2 \le \epsilon$.

Proof. By a linear functional calculus, we may suppose that $\lambda = 0$ and $\mu = 1$. Choose a countable dense set $\{\xi_j\}_{j\ge 1}$ of vectors in $\mathcal{H}$. Then applying the lemma for $\{\xi_1\}$ and $\epsilon/2$, we can find a finite-rank projection p_1 and a finite-rank hermitian h_1 such that

$$\|(1-p_1)\xi_1\| \le \epsilon/2, \quad \|h_1\|_2 \le \epsilon/2, \quad 0 \le a + h_1 \le 1, \quad [a + h_1, p_1] = 0.$$

Next, apply the lemma to $(1-p_1)(a+h_1)$ on $(1-p_1)\mathcal{H}$, $\{(1-p_1)\xi_1, (1-p_1)\xi_2\}$ and $\epsilon/2^2$, we can find a finite-rank projection $p_2 \le 1 - p_1$ and a finite-rank hermitian $h_2 = h_2(1-p_1)$ such that $0 \le a + h_1 + h_2 \le 1$ and

$$\|(1 - p_1 - p_2)\xi_j\| \le \epsilon/2^2 \ (j = 1, 2), \quad \|h_2\|_2 \le \epsilon/2^2, \quad [a + h_1 + h_2, p_2] = 0.$$

By an obvious repetition of constructions, we can find a sequence of mutually orthogonal finite-rank projections $\{p_n\}$ and a sequence of finite-rank hermitians $\{h_n\}$ such that $h_n = h_n(1 - p_1 - \cdots - p_{n-1})$ and

$$\|(1 - p_1 - \cdots - p_n)\xi_j\| \le \epsilon/2^n \ (1 \le j \le n), \quad \|h_n\|_2 \le \epsilon/2^n,$$
$$0 \le a + h_1 + \cdots + h_n \le 1, \quad [a + h_1 + \cdots + h_n, p_n] = 0$$

for $n \ge 1$.

Now set $b = a + \sum_{n\ge 1} h_n$. Then $0 \le b \le 1$ with $\|a - b\|_2 \le \sum \|h_n\|_2 \le \epsilon$. Since $\lim_n (1 - p_1 - \cdots - p_n)\xi_j = 0$ for each $j \ge 1$, we have $\sum_n p_n = 1$. From $h_k(p_1 + \cdots + p_{k-1}) = 0$, we see

$$[b, p_n] = [a + h_1 + \cdots + h_n, p_n] = 0,$$

whence p_n is a spectral projection of b. □

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