OPERATOR ALGEBRAS AND THEIR REPRESENTATIONS

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Contents

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1. Algebras and Representations

By a ***-algebra**, we shall mean an algebra $\mathcal{A}$ over the complex number field $\mathbb{C}$ which is furnished with a distinguished conjugate linear transformation $* : \mathcal{A} \to \mathcal{A}$ (called a *-operation) satisfying

$$(a^*)^* = a \quad (ab)^* = b^*a^*, \quad a, b \in \mathcal{A}.$$

Exercise 1. If $\mathcal{A}$ has a unit $1_\mathcal{A}$ (i.e., $\mathcal{A}$ is **unital**), then $1^* = 1$.

An element a in a *-algebra $\mathcal{A}$ is said to be **hermitian** if $a^* = a$. A hermitian element p is called a **projection** if $p^2 = p$. When $\mathcal{A}$ has a unit 1, a is said to be **unitary** if $aa^* = a^*a = 1$.

A *-algebra is said to be **unitary**[1] if it is generated by unitaries.

Example 1.1. Given a *-algebra $\mathcal{A}$, the $n \times n$ matrix algebra $M_n(\mathcal{A})$ with entries in $\mathcal{A}$ is a *-algebra.

Example 1.2. Let $\mathbb{C}[X]$ be the polynomial algebra of indeterminate X and make it into a *-algebra by $(\sum_{n\geq 0} a_n X^n)^* = \sum_{n\geq 0} \overline{a_n} X^n$. Then 0 and 1 are all the projections and constant polynomials of modulus 1 are all the unitaries.

[1]Warning: This is not a common usage of terminology.

Example 1.3. Given a group G, the free vector space $\mathbb{C}G = \sum_{g\in G} \mathbb{C}g$ generated by elements in G is a *-algebra (called a group algebra) by extending the group product to the algebra multipliction and defining the *-operation so that elements in G are unitary. The group algebra $\mathbb{C}G$ is unitary.

Exercise 2. Let $\mathcal{A}$ be the vector space of functions of finite support on a group G and make it into a *-algebra (convolution algebra) by

$$(ab)(g) = \sum_{g'g''=g} a(g')b(g''), \quad a^*(g) = \overline{a(g^{-1})}.$$

The convolution algebra $\mathcal{A}$ of G is naturally isomorphic to the group algebra $\mathbb{C}G$.

Given *-algebras $\mathcal{A}$ and $\mathcal{B}$, their direct sum $\mathcal{A}\oplus\mathcal{B}$ and tensor product $\mathcal{A}\otimes\mathcal{B}$ are again *-algebras in an obvious manner.

Exercise 3. The matrix algebra $M_n(\mathcal{A})$ is naturally identified with the tensor product $M_n(\mathbb{C})\otimes\mathcal{A}$.

Let $\mathcal{H}$ be a pre-Hilbert space; $\mathcal{H}$ is a complex vector space with a positive definite inner product $(\ |\ )$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is called the **adjoint** of a linear operator $S : \mathcal{H} \to \mathcal{H}$ (and denoted by S^*) if it satisfies $(\xi|S\eta) = (T\xi|\eta)$ (for $\xi, \eta \in \mathcal{H}$). A linear operator S on $\mathcal{H}$ is sadi to be bounded (on the unit ball) if $\|S\| = \sup\{\|S\xi\|; \xi \in \mathcal{H}, \|\xi\| \leq 1\}$ is finite. Let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$ having adjoints, which is a unital *-algebra in an obvious way. The subset $\mathcal{B}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ consisting of bounded operators is a *-subalgebra. When $\mathcal{H}$ is complete, a linear operator on $\mathcal{H}$ has an adjoint if and only if it is bounded thanks to the closed graph theorem and the Riesz lemma, whence $\mathcal{L}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$.

By a ***-representation** of a *-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$, we shall mean an algebra-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^* = \pi(a^*)$ for $a \in \mathcal{A}$. When $\|\pi(a)\| < \infty$ for every $a \in \mathcal{A}$, π is said to be **bounded**. If $\mathcal{A}$ is unitary, any *-representation is automatically bounded, i.e., $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$. Two *-representations $\pi_i : \mathcal{A} \to \mathcal{L}(\mathcal{H}_i)$ $(i = 1, 2)$ are said to be **unitarily equivalent** if we can find a unitary map $T : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $T\pi_1(a) = \pi_2(a)T$ $(a \in \mathcal{A})$.

It is often convenient to regard the representation space $\mathcal{H}$ as a left $\mathcal{A}$-module by $a\xi = \pi(a)\xi$. A right $\mathcal{A}$-module structure then corresponds to a *-antirepresentation, i.e., an algebra-antihomomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ satisfying $\pi(a)^* = \pi(a^*)$, by the relation $\xi a = \pi(a)\xi$. A pre-Hilbert space $\mathcal{H}$ is called an $\mathcal{A}$-$\mathcal{B}$ bimodule ($\mathcal{B}$ being another *-algebra) if we are given a *-representation $\lambda : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a

*-antirepresentation $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ satisfying $\lambda(a)\rho(b) = \rho(b)\lambda(a)$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, i.e., $(a\xi)b = a(\xi b)$ in the module notation. An $\mathcal{A}$-$\mathcal{A}$ bimodule $\mathcal{H}$ is called a ***-bimodule** if we are given an antiunitary [2] involution ξ^* on $\mathcal{H}$ satisfying $(a\xi b)^* = b^*\xi^*a^*$ for $a, b \in \mathcal{A}$ and $\xi \in \mathcal{H}$.

A linear functional φ on a *-algebra $\mathcal{A}$ is defined to be **positive** if $\varphi(a^*a) \geq 0$ for $a \in \mathcal{A}$. A positive linear functional φ on a unital *-algebra $\mathcal{A}$ is called a **state** if $\varphi(1_\mathcal{A}) = 1$ ($1_\mathcal{A}$ being the unit element of $\mathcal{A}$). A linear functional τ on an algebra $\mathcal{A}$ is called a **trace** or said to be **tracial** if $\tau(ab) = \tau(ba)$ for $a, b \in \mathcal{A}$.

Example 1.4. Given a *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a vector $\xi \in \mathcal{H}$, $\varphi(a) = (\xi|\pi(a)\xi)$ gives a positive linear functional. When ξ is a unit vector, φ is a state. This kind of state is called a vector state.

Exercise 4. Given a bounded *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a sequence $\{\xi_n\}_{n\geq 1}$ of vectors in $\mathcal{H}$ satisfying $\sum_{n\geq 1}(\xi_n|\xi_n) = 1$, observe that

$$\varphi(a) = \sum_{n=1}^{\infty}(\xi_n|\pi(a)\xi_n)$$

defines a state on $\mathcal{A}$. By specializing to $\mathcal{A} = M_2(\mathbb{C})$, recognize the difference between states of this form and vector states.

Example 1.5. Let $\mathcal{C}_0(\mathcal{H})$ be the set of finite rank operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{C}_0(\mathcal{H})$ is a *-ideal of $\mathcal{B}(\mathcal{H})$ and the ordinary trace defines a positive tracial functional tr on $\mathcal{C}_0(\mathcal{H})$.

Example 1.6. Every probability measure μ on the real line of finite moments defines a state on the polynomial algebra $\mathbb{C}[X]$ by

$$\varphi\left(\sum_n a_n X^n\right) = \sum_n a_n \int_\mathbb{R} t^n \mu(dt).$$

Conversely, any state arises in this way (the existence part of the Hamburger moment problem). See §X.1 in Reed-Simon for more information.

Example 1.7. In the group algebra $\mathbb{C}G$, positive linear functionals φ are one-to-one correspondence with positive definite functions on G by restriction and linear extension. The state associated to the positive definition function

$$\delta(g) = \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{otherwise} \end{cases}$$

[2] A conjugate-linear operator J on a pre-Hilbert space $\mathcal{H}$ is called an antiunitary if it satisfies $(J\xi|J\eta) = (\eta|\xi)$ and $J\mathcal{H} = \mathcal{H}$.

is called the standard trace.

Exercise 5. The standard trace δ has the trace property: $\delta(ab) = \delta(ba)$ for $a, b \in \mathbb{C}G$.

Given a positive linear functional φ on a *-algebra $\mathcal{A}$, we define a *-representation as follows: The inner product $(a|b) = \varphi(a^*b)$ on $\mathcal{A}$ is positive semidefinite and the representation space is given by the associated pre-Hilbert space $\mathcal{H}$, i.e., $\mathcal{H}$ is the quotient vector space relative to the kernel of $(\ |\ )$. The non-degenerate inner product on the quotient space is also denoted by $(\ |\ )$, whereas the quotient vector of $x \in \mathcal{A}$ in $\mathcal{H}$ is denoted by $x\varphi^{1/2}$. The inner product then looks like $(x\varphi^{1/2}|y\varphi^{1/2}) = \varphi(x^*y)$ and we introduce a representation π by $\pi(a)(x\varphi^{1/2}) = (ax)\varphi^{1/2}$, which is well-defined in view of the Schwarz inequality

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \quad a, b \in \mathcal{A}.$$

In fact, if x is in the kernel of the inner product,

$$\varphi(x^*a^*ax) \leq \varphi(x^*a^*aa^*ax)^{1/2}\varphi(x^*x)^{1/2} = 0$$

shows that ax is in the kernel as well. Moreover, the relation

$$(x\varphi^{1/2}|\pi(a)y\varphi^{1/2}) = \varphi(x^*ay) = (\pi(a^*)x\varphi^{1/2}|y\varphi^{1/2})$$

shows that $\pi(a)^* = \pi(a^*)$, whence π is a *-representation.

The representation obtained in this way is referred to as the **GNS-representation** or its process as the GNS-construction. When $\mathcal{A}$ is unital, we have a distinguished vector $\varphi^{1/2} = 1_{\mathcal{A}}\varphi^{1/2}$ in the representation space, which is **cyclic** with respect to π in the sense that $\mathcal{H} = \pi(\mathcal{A})\varphi^{1/2}$.

Conversely, if we are given a *-representation $(\pi, \mathcal{H})$ of a *-algebra $\mathcal{A}$ and a cyclic vector $\xi \in \mathcal{H}$ for π, the formula $\varphi(a) = (\xi|\pi(a)\xi)$ defines a positive linear functional and the associated GNS-representation is unitarily equivalent to the initial one by the unitary map $a\varphi^{1/2} \mapsto \pi(a)\xi$ $(a \in \mathcal{A})$.

Remark 1. GNS is named after I.M. Gelfand, M.A. Naimark and I.E. Segal.

As a simple application of GNS-representation, we can show that tensor products of positive functionals are again positive: Let $\mathcal{A}$, $\mathcal{B}$ be

*-algebras and $\varphi : \mathcal{A} \to \mathbb{C}$, $\psi : \mathcal{B} \to \mathbb{C}$ be positive. Then

$$\begin{aligned}(\varphi \otimes \psi)(\sum_j a_j \otimes b_j)^*(\sum_k a_k \otimes b_k)) &= \sum_{j,k} \varphi(a_j^* a_k)\psi(b_j^* b_k) \\ &= (\sum_j a_j \varphi^{1/2} \otimes b_j \psi^{1/2} | \sum_k a_k \varphi^{1/2} \otimes b_k \psi^{1/2}) \geq 0.\end{aligned}$$

This can be also seen from the fact that entry-wise multiplications of positive matrices are again positive.

Example 1.8. Given a positive trace τ on a *-algebra $\mathcal{A}$, the associated GNS-representation space $\mathcal{A}\tau^{1/2}$ is made into a *-bimodule by $(a\tau^{1/2})^* = a^*\tau^{1/2}$ $(a \in \mathcal{A})$.

Proposition 1.9. Let ω be a positive functional on a unitary algebra $\mathcal{A}$ with $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ the associated GNS-representation. Then the following formula gives a one-to-one correspondence between positive functionls ω_T on $\mathcal{A}$ majorized by ω and positive operators T in the commutant $\pi(\mathcal{A})' = \{T \in \mathcal{B}(\mathcal{H}); T\pi(a) = \pi(a)T,\ \forall a \in \mathcal{A}\}$ majorized by the identity operator 1_H.

$$\omega_T(a) = (T\omega^{1/2}|\pi(a)\omega^{1/2}), \quad a \in \mathcal{A}.$$

Proof. Let φ be majorized by ω, i.e., $\varphi(a^*a) \leq \omega(a^*a)$ for $a \in \mathcal{A}$. Then by Schwarz inequality

$$|\varphi(x^*y)| \leq \varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} \leq \omega(x^*x)^{1/2}\omega(y^*y)^{1/2} = \|x\omega^{1/2}\|\,\|y\omega^{1/2}\|,$$

we see that $x\omega^{1/2} \times y\omega^{1/2} \mapsto \varphi(x^*y)$ gives a bounded sesquilinear form on the completed Hilbert space $\mathcal{H}$, whence we can find a bounded linear operator T on $\mathcal{H}$ satisfying

$$\varphi(x^*y) = (x\omega^{1/2}|T(y\omega^{1/2})) \quad x, y \in \mathcal{A}.$$

By equating $\varphi(x^*(ay))$ and $\varphi((a^*x)^*y)$, we have $T \in \pi(\mathcal{A})'$. Furthermore, the condition $0 \leq \varphi(a^*a) \leq \omega(a^*a)$ means the operator inequality $0 \leq T \leq 1_{\mathcal{H}}$.

The converse implication is immediate and the proof is left to the reader. □

Definition 1.10. Given a vector η in a Hilbert space $\mathcal{H}$, the linear functional $\eta^* : \mathcal{H} \to \mathbb{C}$ is defined by $\eta^*(\xi) = (\eta|\xi)$ for $\xi \in \mathcal{H}$. By Riesz lemma, the dual space $\mathcal{H}^*$ of $\mathcal{H}$ is of the form $\mathcal{H}^* = \{\eta^*; \eta \in \mathcal{H}\}$ and it is a Hilbert space by the inner product $(\xi^*|\eta^*) = (\eta|\xi)$. The *-algebra $\mathcal{B}(\mathcal{H})$ then naturally acts on $\mathcal{H}^*$ from the right by $\eta^* a = (a^*\eta)^*$. For

$\xi, \eta \in \mathcal{H}$, define a rank one operator $\xi\eta^* \in \mathcal{C}_0(\mathcal{H})$ by[3]

$$(\xi\eta^*)\zeta = (\eta|\zeta)\xi, \quad \zeta \in \mathcal{H}.$$

The notation is compatible with the multiplications by elements in $\mathcal{B}(\mathcal{H})$: $a(\xi\eta^*)b = (a\xi)(\eta^* b)$.

Example 1.11. Let tr be the ordinary trace on the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$. Then the correspondence $\xi\eta^*\mathrm{tr}^{1/2} \mapsto \xi \otimes \eta^*$ gives rise to a unitary map from the GNS-representation space $\overline{\mathcal{C}_0(\mathcal{H})\mathrm{tr}^{1/2}}$ onto $\mathcal{H} \otimes \mathcal{H}^*$.

Example 1.12. The GNS-representation associated to the state on $\mathbb{C}[X]$ realized by a probability measure μ on $\mathbb{R}$ is identified with the multiplication operator by polynomial functions on the Hilbert space $L^2(\mathbb{R}, \mu)$.

Example 1.13. The GNS-representation of the standard trace of a group algebra $\mathbb{C}G$ is identified with the regular representation of G:

$$(a\delta^{1/2}|b\delta^{1/2}) = \delta(a^*b) = \sum_{g\in G} \overline{a_g} b_g \quad \text{for } a = \sum_{g\in G} a_g g,\ b = \sum_{g\in G} b_g g.$$

By the trace property of δ, the representation space $\ell^2(G)$ is a *-bimodule of $\mathbb{C}G$.

When G is commutative, $\ell^2(G)$ is unitarily mapped onto $L^2(\widehat{G})$ ($\widehat{G}$ being the Pontryagin dual of G) with the representation of $\mathbb{C}G$ unitarily transformed into the multiplication operator on $L^2(\widehat{G})$ given by the function

$$\widehat{G} \ni \omega \mapsto \sum_{g\in G} a_g \langle g, \omega\rangle \quad \text{for } a = \sum_{g\in G} a_g g \in \mathbb{C}G.$$

Exercise 6. For $G = \mathbb{Z}$, identify $\widehat{\mathbb{Z}}$ with $\mathbb{T}$ and the unitary map $\ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ with the Fourier expansion.

On a *-algebra $\mathcal{A}$, introduce a seminorm $\|\cdot\|_{C^*}$ by

$$\|a\|_{C^*} = \sup\{\|\pi(a)\|; \pi \text{ is a bounded *-representation}\},$$

which satisfies

$$\|ab\|_{C^*} \leq \|a\|_{C^*}\, \|b\|_{C^*}, \quad \|a^*a\|_{C^*} = \|a\|_{C^*}^2.$$

Exercise 7. Check the following: $\|a^*\|_{C^*} = \|a\|_{C^*}$ for $a \in \mathcal{A}$ and $\{a \in \mathcal{A}; \|a\|_{C^*} = 0\}$ is a *-ideal of $\mathcal{A}$.

Example 1.14. If G is commutative, we shall see later that $\|a\|_{C^*} = \|\pi(a)\|$ for $a \in \mathbb{C}G$, where π is the regular representation of G. The condition is known to be equivalent to the so-called amenability of G.

[3] According to Dirac, $\xi\eta^*$ is often denoted by $|\xi)(\eta|$.

2. Gelfand Theory

Definition 2.1. An algebra A is called a **Banach algebra** if it is furnished with a complete norm satisfying $\|ab\| \leq \|a\|\,\|b\|$ for $a, b \in A$. A Banach algebra A is called a Banach *-algebra if it is further equipped with a *-operation satisfying $\|a^*\| = \|a\|$ for $a \in A$. A Banach *-algebra A is called a **C*-algebra** if the norm satisfies $\|a^*a\| = \|a\|^2$ for $a \in A$.

The completion of the quotient *-algebra $\mathcal{A}/\mathcal{I}$ relative to $\|\cdot\|_{C^*}$ ($\mathcal{I} = \{a \in \mathcal{A}; \|a\|_{C^*}\}$) is a C*-algebra. For a group algebra $\mathcal{A} = \mathbb{C}G$, $\mathcal{I} = \{0\}$ and the associated C*-algebra is called the group C*-algebra and denoted by $C^*(G)$.

Example 2.2. If we introduce a norm on the group algebra $\mathbb{C}G$ by

$$\|\sum_{g\in G} f(g)g\|_1 = \sum_{g\in G} |f(g)|,$$

then it satisfies $\|ab\|_1 \leq \|a\|_1\,\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$, whence the norm completion $\ell^1(G)$ of $\mathbb{C}G$ is a Banach *-algebra.

Example 2.3. The closure of the finite rank operator algebra $\mathcal{C}_0(\mathcal{H})$ in the operator topology on $\mathcal{B}(\mathcal{H})$ is a C*-algebra as a norm-closed *-ideal of $\mathcal{B}(\mathcal{H})$, which is referred to as the **compact operator**[4] algebra and denoted by $\mathcal{C}(\mathcal{H})$.

Exercise 8. If $\mathcal{H}_1$ is compact in the norm topology, then $\dim \mathcal{H} < \infty$.

The norm $\|a\|_2 = \|a\mathrm{tr}^{1/2}\| = \sqrt{\mathrm{tr}(a^*a)}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **Hilbert-Schmidt norm** and satisfies

$$\|ab\|_2 \leq \|a\|\|b\|_2, \quad \|b^*\|_2 = \|b\|_2 \geq \|b\|, \quad a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H}).$$

Thus the completion of $\mathcal{C}_0(\mathcal{H})$ relative to the Hilbert-Schmidt norm, which is included in $\mathcal{C}(\mathcal{H})$ and denoted by $\mathcal{C}_2(\mathcal{H})$, is a Banach *-algebra and realized as a *-ideal of $\mathcal{B}(\mathcal{H})$.

The norm $\|a\|_1 = \sup\{|\mathrm{tr}(ab)|; b \in \mathcal{C}_0(\mathcal{H}), \|b\| \leq 1\}$ on $\mathcal{C}_0(\mathcal{H})$ is known to be the **trace norm** and satisfies

$$\|ab\|_1 \leq \|a\|\|b\|_1, \quad \|b^*\|_1 = \|b\|_1 \geq \|b\|_2, \quad |\mathrm{tr}(b)| \leq \|b\|_1$$

for $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{C}_0(\mathcal{H})$. Thus the completion of $\mathcal{C}_0(\mathcal{H})$ relative to the trace norm, which is included in $\mathcal{C}_2(\mathcal{H})$ and denoted by $\mathcal{C}_1(\mathcal{H})$, is a Banach *-algebra and realized as a *-ideal of $\mathcal{B}(\mathcal{H})$ with the trace functional extended to $\mathcal{C}_1(\mathcal{H})$ by continuity.

[4] The terminology is based on the fact that an element T in $\mathcal{C}(\mathcal{H})$ is characterized by the property that the norm closure of $T(\mathcal{H}_1)$ ($\mathcal{H}_1$ being the unit ball in $\mathcal{H}$) is compact.

Exercise 9. Check the inequalities for the Hilbert-Schmidt and the trace norms.

Exercise 10. Show that, for a positive operator $a \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{tr}(a) = \sum_j (\xi_j | a\xi_j)$$

does not depend on the choice of an orthonormal basis $\{\xi_j\}$ in $\mathcal{H}$.

Exercise 11. Show that $a \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{C}_1(\mathcal{H})$ if and only if $\operatorname{tr}(|a|) < \infty$. Here $|a| = \sqrt{a^*a}$. If this is the case, $\|a\|_1 = \operatorname{tr}(|a|)$.

Exercise 12. For $a \in \mathcal{C}_1(\mathcal{H})$, $\|a\|_1 = \|a\|_2$ if and only if a is a rank-one operator.

Exercise 13. Show that $\mathcal{C}_1(\mathcal{H}) = \mathcal{C}_2(\mathcal{H})\mathcal{C}_2(\mathcal{H})$ and deduce the inequality $\|ab\|_1 \leq \|a\|_2\|b\|_2$ from $|\operatorname{tr}(ab)| \leq \|a\|_2\|b\|_2$ (the Cauchy-Schwarz inequality).

Example 2.4. Given a compact (Hausdorff) space K, the set $C(K)$ of complex-valued continuous functions on K is a commutative C*-algebra by point-wise operations. If $F \subset K$ is a closed subset,

$$A = \{f : K \to \mathbb{C}; f \text{ is continuous and vanishing on } F\}$$

is a closed *-ideal of $C(K)$, which is unital if and only if F is open in K.

If Ω is a locally compact but non-compact space with $K = \Omega \cup \{\infty\}$ the one-point compactification and $F = \{\infty\}$, we write $A = C_0(\Omega)$. A positive functional φ on $C_0(\Omega)$ is one-to-one correspondence with a Radon measure μ on Ω of finite total mass by the formula

$$\varphi(a) = \int_\Omega a(\omega)\, \mu(d\omega).$$

The functional norm is then calculated by $\|\varphi\| = \mu(\Omega)$.

The associated GNS-representation π on $\mathcal{H}$ is identified with the multiplication operators on the Hilbert space $L^2(\Omega, \mu)$ through a natural unitary map $\mathcal{H} \to L^2(\Omega, \mu)$ specified by

$$A\varphi^{1/2} \ni a\varphi^{1/2} \mapsto a \in C_0(\Omega) \subset L^2(\Omega, \mu).$$

Exercise 14. The C*-algebra $C(K)$ has a non-trivial projection if and only if K is not connected.

Exercise 15. Let $[\mu] \subset \Omega$ be the support of μ. Then

$$\|\pi(a)\| = \sup\{|a(\omega)|; \omega \in [\mu]\}.$$

Theorem 2.5.] With respect to the non-degenerate bilinear map $\mathcal{B}(\mathcal{H}) \times \mathcal{C}_1(\mathcal{H}) \ni (b, c) \mapsto \operatorname{tr}(bc) = \operatorname{tr}(cb) \in \mathbb{C}$, we have the duality relations of $\mathcal{C}(\mathcal{H})$: $\mathcal{C}(\mathcal{H})^* = \mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$.

Proof. The equality $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{C}(\mathcal{H})^*$ is by definition and the inclusion $\mathcal{B}(\mathcal{H}) \subset \mathcal{C}_1(\mathcal{H})^*$ is a consequence of

$$\|b\| \geq \sup\{|\operatorname{tr}(bc)|; c \in \mathcal{C}_1(\mathcal{H}), \|c\|_1 \leq 1\} \geq \sup\{|(\xi|b\eta)|; \xi, \eta \in \mathcal{H}_1\} = \|b\|.$$

In view of $\|\xi\eta^*\|_1 = \|\xi\| \, \|\eta\| = \|\xi\eta^*\|$ for $\xi, \eta \in \mathcal{H}$, any $\varphi \in \mathcal{C}_1(\mathcal{H})^*$ defines a bounded sesquilinear form by $\varphi(\xi\eta^*)$ and hence $b \in \mathcal{B}(\mathcal{H})$ by the relation $\varphi(\xi\eta^*) = (\eta|b\xi)$. By rewriting $(\eta|b\xi) = \operatorname{tr}(b\xi\eta^*)$, we see that $\varphi(c) = \operatorname{tr}(bc)$ for $c \in \mathcal{C}_0(\mathcal{H})$ and then, for $c \in \mathcal{C}_1(\mathcal{H})$, by the density of $\mathcal{C}_0$ and the continuity of trace. □

Exercise 16. A φ on $\mathcal{C}(\mathcal{H})$ described by a trace class operator $\rho \in \mathcal{C}_1(\mathcal{H})$ is positive of the form if and only if $\rho \geq 0$.

Any algebra $\mathcal{A}$ is enhanced to have a unit:

$$\widetilde{\mathcal{A}} = \begin{cases} \mathcal{A} & \text{if } 1 \in \mathcal{A}, \\ \mathbb{C}1 + \mathcal{A} & \text{otherwise.} \end{cases}$$

$$0 \longrightarrow \mathcal{A} \longrightarrow \widetilde{\mathcal{A}} \longrightarrow \mathbb{C} \longrightarrow 0$$

If $\mathcal{A}$ is a *-algebra, so is $\widetilde{\mathcal{A}}$ by setting $(\lambda 1 + a)^* = \overline{\lambda} 1 + a^*$. For a Banach *-algebra A without unit, the unital *-algebra $\widetilde{A}$ is a Banach *-algebra by the norm $\|\lambda 1 + a\| = |\lambda| + \|a\|$. When A is a C*-algebra, the unital *-algebra $\widetilde{A}$ is again a C*-algebra and it is in a unique way. Existence: $\widetilde{A}$ is a C*-algebra by the norm

$$\|\lambda 1 + a\| = \sup\{\|\lambda b + ab\|; b \in A, \|b\| \leq 1\}.$$

The uniqueness will be established later.

Exercise 17. Check all these other than the uniqueness.

Definition 2.6. Let a be an element in a Banach algebra A and a power series $f(z) = \sum_{n \geq 0} f_n z^n$ ($f_n \in \mathbb{C}$) be absolutely convergent at $z = \|a\|$. Then $f(a) \in \widetilde{A}$ is defined by

$$f(a) = \sum_{n=1}^{\infty} f_n a^n.$$

Note that, if $f_0 = 0$, then $f(a) \in A$.

Example 2.7.

(i) If $ab = ba$, $e^a e^b = e^{a+b}$.

(ii) If $\|a\| < |\lambda|$ with $\lambda \in \mathbb{C}$, then

$$\frac{1}{\lambda - a} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

is the inverse of $\lambda 1 - a$ in $\widetilde{A}$.

Definition 2.8. Define the **spectrum** of an element a of an algebra $\mathcal{A}$ by

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C}; \lambda 1 - a \text{ is not invertible in } \widetilde{\mathcal{A}}\}.$$

Exercise 18. If $a \in \mathcal{B}(\mathcal{H})$ is a finite rank operator, $\sigma_{\mathcal{B}(\mathcal{H})}(a) = \{\lambda \in \mathbb{C}; a\xi = \lambda\xi$ for some $0 \neq \xi \in \mathcal{H}\}$

Exercise 19. If $1 \notin \mathcal{A}$, then $0 \in \sigma_{\mathcal{A}}(a)$ for any $a \in A$.

Exercise 20. Assume that elements a and b in a Banach algebra A with unit 1 satisfy $ab - ba = 1$. Show $e^a b e^{-a} = b + 1$, $\sigma(b) = 1 + \sigma(b)$ and derive a contradiction.

Lemma 2.9. In a unital Banach algebra A, the set G of invertible elements is open in A and the map $G \ni g \mapsto g^{-1} \in G$ is analytic.

Proof. For $a \in G$, $g = a - (a - g) = a(1 - a^{-1}(a - g))$ is invertible if $\|a^{-1}(a - g)\| \leq \|a^{-1}\| \, \|a - g\| < 1$ with the inverse given by the power series

$$g^{-1} = a^{-1} \sum_{n=0}^{\infty} (a^{-1}(a - g))^n.$$

□

Theorem 2.10 (Gelfand)**.** In a Banach algebra A, $\sigma_A(a)$ is non-empty compact subset of $\mathbb{C}$ for any $a \in A$.

Proof. If $z1 - a$ is invertible for any $z \in \mathbb{C}$, $(z1 - a)^{-1}$ is a bounded holomorphic function, a contradiction. □

Corollary 2.11. If a Banach algebra A is a field, then $A = \mathbb{C}1$.

Exercise 21. Spell out the details of the proof.

Exercise 22 (Analytic Functional Calculus)**.** Let $\mathcal{A}$ be the algebra of functions which are holomorphic in a neighborhood of $\sigma_A(a)$. If we set

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(\lambda)}{\lambda - a} \, d\lambda$$

for $f \in \mathcal{A}$, where the contour integral is performed by surrounding $\sigma_A(a)$, then $\mathcal{A} \ni f \mapsto f(a) \in \widetilde{A}$ gives an algebra-homomorphism.

Theorem 2.12 (Spectral Radius Formula)**.** For an element a of a Banach algebra A,

$$\rho(a) = \sup\{|\lambda|; \lambda \in \sigma_A(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. From the geometric series formula

$$(\lambda 1 - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}, \quad |\lambda| > \|a\|,$$

we have $\rho(a) \leq \|a\|$, which is combined with $\sigma(a)^n \subset \sigma(a^n)$ to get $\rho(a) \leq \|a^n\|^{1/n}$;

$$\rho(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n}.$$

Let $r > \|a\|$ and apply the Cauchy integral formula:

$$a^n = \frac{1}{2\pi i} \int_{|\lambda|=r} \lambda^n (\lambda 1 - a)^{-1}\, d\lambda.$$

The radius r is then decreased to $r > \rho(a)$ by the Caychy inttegral theorem and we obtain the estimate

$$\|a^n\| \leq r^{n+1} \max\{\|(\lambda 1 - a)^{-1}\|; |\lambda| = r\}.$$

Consequently

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r$$

for $r > \rho(a)$. □

Proposition 2.13. If A is a C*-algebra and $aa^* = a^*a$, then

$$\|a\| = \sup\{|\lambda|; \lambda \in \sigma_A(a)\}.$$

Proof. If $aa^* = a^*a$,

$$\|a^2\| = \|(aa)^* aa\|^{1/2} = \|a^* aa^*\|^{1/2} = \|a^* a\| = \|a\|^2,$$

which is used repeatedly to get $\|a^{2^n}\| = \|a\|^{2^n}$. □

Corollary 2.14.

(i) The C*-norm of a C*-algebra A is unique: $\|a\| = \sup\{|\lambda|; \lambda \in \sigma_A(a^*a)\}$ with the right hand side determined by the *-algebra structure of A.

(ii) Let $\phi : A \to B$ be a *-homomorphism from a Banach *-algebra A into a C*-algebra B. Then $\|\phi(a)\| \leq \|a\|$ for $a \in A$.

Proposition 2.15. In a Banach *-algebra, we have the following.

(i) $\sigma_A(u) \subset \mathbb{T}$ for a unitary u.

(ii) $\sigma_A(h) \subset \mathbb{R}$ for a hermitian h.

Proof. By the spectral radius formula, $\sigma(u)$ is included in the unit disk but $\sigma(u^{-1}) = \sigma(u)^{-1}$ shows that $\lambda \in \sigma(u)$ implies $|\lambda^{-1}| \leq 1$.

For $\lambda \in \mathbb{C}$, $e^{i\lambda} - e^{ih} = (\lambda - h)a = a(\lambda - h)$ with

$$a = i \sum_{n=1}^{\infty} \frac{(i\lambda)^{n-1} + \cdots + (ih)^{n-1}}{n!}$$

shows that, if $e^{i\lambda} - e^{ih}$ is invertible, then so is $\lambda - h$ because it admits left and right inverses. Thus $\lambda \in \sigma(h)$ implies $e^{i\lambda} \in \sigma(e^{ih})$, whence $|e^{i\lambda}| = 1$, i.e., $\lambda \in \mathbb{R}$. $\square$

Theorem 2.16. If A is a closed *-subalgebra of a Banach *-algebra B. Then $\sigma_A(a) = \sigma_B(a)$ for $a \in A$.

Proof. Since $\widetilde{A}$ is naturally regarded as a subalgebra of $\widetilde{B}$ with a common unit, we may assume that $\widetilde{A} = A$ and $\widetilde{B} = B$ with the trivial inclusion $\sigma_B(a) \subset \sigma_A(a)$ from the outset.

First assume that $a^* = a$. Since open sets $\mathbb{C} \setminus \sigma_A(a)$ and $\mathbb{C} \setminus \sigma_B(a)$ in $\mathbb{C}$ are connected thanks to $\sigma_A(a) \subset \mathbb{R}$, it suffices for the equality $\sigma_A(a) = \sigma_B(a)$ to show that $\mathbb{C} \setminus \sigma_A(a)$ is a closed subset of $\mathbb{C} \setminus \sigma_B(a)$.

Let $\lambda_n \notin \sigma_A(a)$ converge to $\lambda \notin \sigma_B(a)$. Then $\lambda_n - a$ converges in B to an invertible element $\lambda - a$, whence $(\lambda_n - a)^{-1}$ converges to $(\lambda - a)^{-1}$ in B but $(\lambda_n - a)^{-1} \in A$ implies $(\lambda - a)^{-1} \in A$, i.e., $\lambda \notin \sigma_A(a)$.

Since the non-trivial inclusion $\sigma_A(a) \subset \sigma_B(a)$ is equivalently stated that an element $c \in A$ is invertible in A if it is invertible in B, assume that $bc = cb = 1$ for some $b \in B$. Then $bb^*c^*c = bc = 1$ shows that a hermitian c^*c is invertible in B and hence it is invertible in A as well: we can find $a \in A$ so that $ac^*c = 1$. Now $b = ac^*cb = ac^* \in A$. $\square$

The above theorem allows us to simply write $\sigma(a)$ to stand for $\sigma_A(a)$ when a is an element in a Banach *-algebra A.

Definition 2.17. The **spectrum** σ_A of a commutative Banach algebra A is the set of non-zero homomorphism $A \to \mathbb{C}$.

Example 2.18. If $A = C(K)$, $\sigma_A = \{\delta_x; x \in K\}$. Here $\delta_x(f) = f(x)$.

Each $\omega \in \sigma_A$ satisfies $\|\omega\| \leq 1$. To see this, we observe that $\widetilde{\omega}(\lambda 1 + a) = \lambda + \omega(a)$ defines a homomorphism $\widetilde{A} \to \mathbb{C}$, which gives us $\omega(a) \in \sigma(a)$ and then $|\omega(a)| \leq \|a\|$ for $a \in A$ by the spectral radius formula.

Thus $\Omega = \sigma_A \cup \{0\}$ is a w*-compact subset of A_1^* and σ_A is locally compact if furnished with the relative w*-topology. Write

$$C_0(\sigma_A) = \{f : \Omega \to \mathbb{C}; f \text{ is continuous and } f(0) = 0\}.$$

Note that σ_A is compact if and only if A is unital.

For $a \in A$, the function $\sigma_A \ni \omega \mapsto \omega(a)$ is continuous and denoted by $\widehat{a}$. The correspondance $A \ni a \mapsto \widehat{a} \in C_0(\sigma_A)$ is referred to as the **Gelfand transform** on A. From definition, the Gelfand transform is multiplicative.

Let A be unital and we shall show that, given a proper ideal I of A, there exits a non-zero homomorphism $\omega : A \to \mathbb{C}$ satisfying $\omega(I) = 0$.

In fact let B be the open unit ball of A at 0. Then every element in $1 + B$ is invertible, which means that $J \cap (1 + B) = \emptyset$ if J is a proper ideal of A. Thus $\overline{J} \cap (1 + B) = \emptyset$ and any maximal ideal $J \supset I$ is closed. Since the quotient Banach algebra A/J is a field, $A/J \cong \mathbb{C}$ and a non-zero homomorphism is obtained as a composition $A \to A/J \cong \mathbb{C}$.

Theorem 2.19. Let a be an element in a Banach algebra A.

(i) If A is unital, $\sigma_A(a) = \{\omega(a); \omega \in \sigma_A\}$.
(ii) If A is non-unital, $\sigma_A(a) = \{\omega(a); \omega \in \sigma_A\} \cup \{0\}$.

Proof. Since $\omega : A \to \mathbb{C}$ is an algebra-homomorphism, $\omega(a) \in \sigma_A(a)$. If $1 \in A$ and $\lambda \in \sigma(a)$, then $A(\lambda - a)$ is a proper ideal of A, we can find an $\omega \in \sigma_A$ vanishing on $A(\lambda - a)$, whence $\lambda = \omega(a)$. □

Proposition 2.20. For a commutative Banach *-algebra A, the Gelfand transform is a *-homomorphism.

Proof. In fact, for $h = h^* \in A$ and $\omega \in \sigma_A$, $\omega(h) \in \sigma(h) \subset \mathbb{R}$. □

Corollary 2.21. If a unital Banach *-algebra A is generated by $\{1, a\}$ with $a \in A$, then $\sigma_A \ni \omega \mapsto \omega(a) \in \sigma_A(a)$ is a homeomorphism.

Proof. The compact (Hausdorff) space σ_A is continuously mapped onto $\sigma_A(a)$, which is injective because $\{1, a\}$ generates A. □

Theorem 2.22 (Gelfand)**.** If A is a commutative C*-algebra, the Gelfand transform is an isomorphism of C*-algebras between A and $C_0(\sigma_A)$.

Proof. The Gelfand transform is isometric because of

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda|; \lambda \in \sigma(a^*a)\} = \sup\{|\widehat{a}(\omega)|^2; \omega \in \sigma_A\} = \|\widehat{a}\|^2.$$

Since $\{\widehat{a}; a \in A\}$ is a *-subalgebra of $C_0(\sigma_A)$ and it separates points in σ_A, it coincides with the whole $C_0(\sigma_A)$ thanks to the Stone-Weierstrass theorem. □

Example 2.23.

(i) Let $A = C_0(\Omega)$ with Ω a locally compact space. Then $\Omega \ni x \mapsto \delta_x \in \sigma_A$ is a homeomorphism.

(ii) Let K is a compact subset of $\mathbb{R}$, $h \in C(K)$ be a continuous function defined by $h(t) = t$ ($t \in X$) and A be the C*-subalgebra of $C(K)$ generated by h. Then σ_A is naturally identified with $K \setminus \{0\}$, whereas $\sigma(h) = K$.

Exercise 23. For a commutative C*-algebra A, σ_A is compact if and only if A has a unit.

In a unital C*-algebra A, let $a \in A$ generate a commutative *-subalgebra; $aa^* = a^*a$ and C be the C*-subalgebra of A generated by $\{1, a\}$. Then the spectrum of C is identified with $\sigma(a)$ and the Gelfand theorem enables us to define the element $f(a)$ in C for a continuous function f on $\sigma(a)$. This is referred to as the continuous **functional calculus** of a.

Exercise 24. If g is a continuous function on $\sigma(f(a)) = \{f(\lambda); \lambda \in \sigma(a)\}$, then we have $g(f(a)) = (g \circ f)(a)$.

Example 2.24. A unital C*-algebra is unitary: If $h = h^*$ with $\|h\| \leq 1$, then $u = h + i\sqrt{1 - h^2}$ is unitary and $h = (u + u^*)/2$.

Let $\mathcal{A}$ be a commutative *-algebra with A the associated C*-algebra. Any $\omega \in \sigma_A$ restricts to a non-zero *-homomorphism $\mathcal{A} \to \mathbb{C}$. Conversely, any non-zero *-homomorphism of $\mathcal{A}$ into $\mathbb{C}$ defines a bounded representation of $\mathcal{A}$, whence it is continuous with respect to the C*-seminorm and lifted to a *-homomorphism of A into $\mathbb{C}$. Thus σ_A is identified with the set of non-zero *-homomorphisms of $\mathcal{A}$ into $\mathbb{C}$. If one applies this to a commutative group algebra $\mathbb{C}G$, then σ_A is identified with the set $\widehat{G}$ of group homomorphisms of G into $\mathbb{T}$ and $C^*(G)$ with $C(\widehat{G})$. Let $\ell^2(G) \to L^2(\widehat{G})$ be the Fourier transform. Then it intertwines the left regular representation of $C^*(G)$ and the multiplication operator of the Gelfand transform of $C^*(G)$, which shows that the regular representation is norm-preserving on $C^*(G)$.

Example 2.25. The spectrum of the group C*-algebra $C^*(\mathbb{Z})$ is identified with $\widehat{\mathbb{Z}} = \mathbb{T}$ and $C^*(\mathbb{Z})$ itself with $C(\mathbb{T})$.

Exercise 25. Let G be an abelian group and $a = \sum_{g \in G} a_g g \in \mathbb{C}G$ with $a_g \geq 0$. Then the C*-norm of a is simply calculated by $\|a\| = \sum_{g \in G} a_g$.

The Gelfand theorem establishes a categorical duality between the category of compact spaces and the category of unital commutative C*-algebras. Here morphisms are continuous maps for compact spaces, unit-preserving *-homomorphisms for C*-algebras, which are in a contravariant relation. Thus a conitnuous map $f : X \to Y$ corresponds to a *-homomorphism $\phi : C(Y) \to C(X)$ by $\phi(b) = b \circ f$. Note that ϕ

is injective (resp. surjective) if and only if f is surjective (resp. injective). As an immediate application of this observation, we see that ϕ is isometric if it is injective.

Theorem 2.26. Let $\phi : A \to B$ be an injective *-homomorphism between C*-algebras. Then ϕ preserves the C*-norms.

Proof. Let $\mathbb{C} \times A$ be the C*-algebra obtained by adding an external unit to A. Thus $\mathbb{C} \times A = \widetilde{A}$ if A is not unital and $\mathbb{C} \times A \cong \mathbb{C} \oplus A$ if A is unital. By extending ϕ to an injective *-homomorphism $\mathbb{C} \times A \ni (\lambda, a) \mapsto (\lambda, \phi(a)) \in \mathbb{C} \times B$, the problem is reduced to the unital case. Let $a \in A$ and we shall show that $\|\phi(a)\| = \|a\|$. By passing to the commutative C*-subalgebras $C^*(1 + a^*a)$ and $C^*(1 + \phi(a)^*\phi(a))$, the problem is further reduced to the case of commutative C*-algebras and we are done. $\square$

Related to this, the following reveals a kind of algebraic rigidity in C*-algebras. For the proof, we need the positivity of elements of the form a^*a and it will be postponed until the end of the next section.

Theorem 2.27. Any closed ideal I of a C*-algebra is a *-ideal and the quotient *-algebra A/I is a C*-algebra with the C*-norm given by the quotient norm.

Corollary 2.28. Let $\phi : A \to B$ be a *-homomorphism between C*-algebras. Then the image $\phi(A)$ is closed in B.

3. Positivity in C*-algebras

A hermitian element h in a C*-algebra A is said to be **positive** and denoted by $h \geq 0$ if $\sigma_A(h) \subset [0, \infty)$. Let A_+ be the set of positive elements in A, which is invariant under the scalar multiplication of positive reals.

Lemma 3.1. For a hermitian element h in a C*-algebra, the following conditions are equivalent.

(i) h is positive.
(ii) $\|r1 - h\| \leq r$ for some $r \geq 0$.
(iii) $\|r1 - h\| \leq r$ for any $r \geq 0$.

Consequently, the positive part A_+ is a closed subset of A.

Proof. Realize h as a continuous function on a compact subset of $\mathbb{R}$. $\square$

Corollary 3.2. If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$. Thus A_+ is a convex cone.

Proof. From the positivity, $\| \|a\| - a\| \leq \|a\|$ and $\| \|b\| - b\| \leq \|b\|$, which are used to get
$$\Big\| \|a\| + \|b\| - a - b \Big\| \leq \Big\| \|a\| - a \Big\| + \Big\| \|b\| - b \Big\| \leq \|a\| + \|b\|.$$
$\square$

Exercise 26. If an element a in a C*-algebra satisfies $\|r1 - a\| \leq r$ for any $r \geq r_0$ with r_0 some positive real number, then a is hermitian and therefore positive.

Theorem 3.3 (Kelley-Vaught)**.** For any element a in a C*-algebra, $a^*a \geq 0$.

Proof. Let $a^*a = b - c$ with b, c positive and $bc = cb = 0$. Then $ca^*ac = -c^3 \leq 0$. Thus the problem is reduced to showing that $x^*x \leq 0$ implies $x^*x = 0$. Let $x = h + ik$. Then
$$xx^* = 2h^2 + 2k^2 - x^*x \geq 0$$
and $\sigma(xx^*) \subset [0, \infty)$, which is combined with the next lemma to get $\sigma(x^*x) = \{0\}$. $\square$

Lemma 3.4. In a Banach algebra A,
$$\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$$
for $a, b \in A$.

Proof. Formally
$$\begin{aligned}(\lambda 1 - ab)^{-1} &= \frac{1}{\lambda} + \frac{ab}{\lambda^2} + \frac{abab}{\lambda^3} + \cdots = \frac{1}{\lambda} + \frac{1}{\lambda^2} a \left(1 + \frac{ba}{\lambda} + \frac{baba}{\lambda^2} + \dots \right) b \\ &= \frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - ba)^{-1} b\end{aligned}$$
for $\lambda \neq 0$ but the conclusion is true because of
$$(\lambda - ab) \left(\frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - ba)^{-1} b \right) = 1 - \frac{ab}{\lambda} + \frac{a}{\lambda} (\lambda - ba)(\lambda - ba)^{-1} b = 1$$
and
$$\left(\frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - ba)^{-1} b \right) (\lambda - ab) = 1 - \frac{ab}{\lambda} + \frac{a}{\lambda} (\lambda - ba)^{-1} (\lambda - ba) b = 1.$$
$\square$

Definition 3.5. An order relation in the set of hermitian elements is introduced by $a \leq b \iff b - a \in A_+$.

Exercise 27. For a hermitian element h and an arbitrary element a,
$$-\|h\| a^*a \leq a^*ha \leq \|h\| a^*a.$$

Proposition 3.6. Any positive linear functional φ on a C*-algebra A is continuous and

$$\|\varphi\| = \sup\{\varphi(a); a \geq 0, \|a\| \leq 1\}.$$

Proof. Let $M = \sup\{\varphi(a); a \geq 0, \|a\| \leq 1\}$ and assume that $M = \infty$. Then we can find a sequence $\{a_n\}$ of positive elements in the unit ball such that $\varphi(a_n) \geq n$. From $\|a_n\| \leq 1$, $a = \sum_{n=1}^{\infty} \frac{1}{n^2} a_n$ defines a positive element in A and $\sum_{k=1}^{n} a_k/k^2 \leq a$ implies

$$\varphi(a) \geq \sum_{k=1}^{n} \frac{1}{k^2}\varphi(a_k) \geq \sum_{k=1}^{n} \frac{1}{k} \to \infty \ (n \to \infty),$$

a contradiction. Thus $M < \infty$.

For a hermitian $h \in A$, $-|h| \leq h \leq |h|$ imples $|\varphi(h)| \leq \varphi(|h|) \leq M\|h\|$ and, for an arbitrary a,

$$|\varphi(a)| \leq |\varphi(\frac{a + a*}{2})| + |\varphi(\frac{a - a^*}{2i})| \leq M\|\frac{a + a^*}{2}\| + M\|\frac{a - a^*}{2i}\| \leq 2M\|a\|.$$

Now, for $x, y \in A$,

$$|\varphi(y^*x)|^2 \leq \varphi(y^*y)\varphi(x^*x) \leq M^2\|y^*y\|\,\|x^*x\|.$$

If we choose $y = \frac{xx^*}{t+xx^*}$ with $t > 0$, then

$$(x - y^*x)(x - y^*x)^* = t^2\frac{xx^*}{(t + xx^*)^2}$$

implies $\|x - y^*x\|^2 \leq t \to 0$ $(t \to +0)$ and we obtain the inequality

$$|\varphi(x)|^2 \leq M^2\|x\|^2,$$

showing $\|\varphi\| \leq M$. □

Definition 3.7. An increasing net $\{u_\alpha\}_{\alpha\in I}$ of positive elements in a unit ball of a C*-algebra A is called an **approximate unit** if $a = \lim_{\alpha\to\infty} au_\alpha$ for any $a \in A$.

Proposition 3.8. An approximate unit exists.

Proof. As an index set, choose the directed set of finite subsets of A_+ and, for $\alpha = \{a_1, \dots, a_n\}$, set

$$u_\alpha = \frac{n(a_1 + \cdots + a_n)}{1 + n(a_1 + \cdots + a_n)}.$$

Then, for $\alpha \ni a^*a$,

$$\begin{aligned}(a - au_\alpha)^*(a - au_\alpha) &= \frac{1}{1 + n(a_1 + \cdots + a_n)} a^*a \frac{1}{1 + n(a_1 + \cdots + a_n)} \\ &\leq \frac{a_1 + \cdots + a_n}{(1 + n(a_1 + \cdots + a_n))^2} \\ &\leq \sup\{\frac{t}{(1 + nt)^2}; t \geq 0\} = \frac{1}{4n}\end{aligned}$$

reveals that $\lim_{\alpha \to \infty} au_\alpha = a$. □

Theorem 3.9. For a linear functional φ on a unital C*-algebra A, the following conditions are equivalent.

(i) φ is positive.
(ii) $\|\varphi\| = \varphi(1)$.

Proof. (i) $\implies$ (ii) is a consequence of the previous proposition in view of $a \leq \|a\|1$ for $a \in A_+$.

(ii) $\implies$ (i): We may assume that $\|\varphi\| = \varphi(1) = 1$. Let $h = h^*$ and $\varphi(h) = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$. Then

$$|\lambda + i(\mu + t)| = |\varphi(h + it)| \leq \|h + it\| = \sqrt{\|h\|^2 + t^2},$$

i.e., $\lambda^2 + (\mu + t)^2 \leq \|h\|^2 + t^2$ for $t \in \mathbb{R}$, which implies $\mu = 0$.

Now let $0 \leq h \leq 1$. Then

$$|1 - \varphi(h)| = |\varphi(1 - h)| \leq \|1 - h\| \leq 1.$$

Since $\varphi(h)$ is real, the above inequality requires $\varphi(h) \geq 0$. □

Corollary 3.10. Let φ be a positive linear functional on a non-unital C*-algebra A and $\psi : \widetilde{A} \to \mathbb{C}$ be an extension of φ. Then ψ is positive if and only if $\psi(1) \geq \|\varphi\|$.

Proof. Assume that ψ is positive and let $a \in A_+$.

$$0 \leq \psi((t + a)^2) = \psi(1)t^2 + 2\varphi(a)t + \varphi(a^2)$$

for any $t \in \mathbb{R}$, whence

$$\varphi(a)^2 \leq \psi(1)\varphi(a^2).$$

If we restrict $\|a\| \leq 1$, this implies $\varphi(a)^2 \leq \psi(1)\|\varphi\|$ and then $\|\varphi\|^2 = \sup\{\varphi(a)^2; 0 \leq a \leq 1\} \leq \psi(1)\|\varphi\|$.

Conversely assume that $\psi(1) \geq \|\varphi\|$. Then $\psi(\lambda + a) = (\psi(1) - \|\varphi\|)\lambda + \widetilde{\varphi}(\lambda + a)$ and the positivity of ψ is reduced to that of $\widetilde{\varphi}$. Clearly $\|\widetilde{\varphi}\| \geq \|\varphi\|$ and the problem is further reduced to showing that

$\|\widetilde{\varphi}\| \leq \|\varphi\|$. From $0 \leq u_\alpha a u_\alpha \leq u_\alpha^2$ for $0 \leq a \leq 1$ and $u_\alpha a u_\alpha \to a$ $(\alpha \to \infty)$, we have

$$\varphi(a) \leq \liminf \varphi(u_\alpha^2) \leq \limsup \varphi(u_\alpha^2) \leq \|\varphi\|$$

and then, by taking supremum on $0 \leq a \leq 1$, $\|\varphi\| = \lim_{\alpha\to\infty} \varphi(u_\alpha^2)$, which is used to conclude that

$$|\widetilde{\varphi}(\lambda + x)| = \lim_\alpha |\lambda\varphi(u_\alpha^2) + \varphi(xu_\alpha^2)| \leq \limsup_\alpha \|\varphi\| \, \|\lambda u_\alpha^2 + xu_\alpha^2\| \leq \|\varphi\| \, \|\lambda + x\|.$$

□

Exercise 28. For a bounded linear functional φ of a C*-algebra A and an approximate unit $\{u_\alpha\}$ in A, φ is positive if and only if

$$\|\varphi\| = \lim_{\alpha\to\infty} \varphi(u_\alpha).$$

Theorem 3.11. Let A be a closed *-subalgebra of a C*-algebra B. Given a positive linear functional φ on A, we can find a positive linear functional ψ on B so that $\varphi(a) = \psi(a)$ for $a \in A$ and $\|\varphi\| = \|\psi\|$.

Proof. By adding unit, we may assume that B has a unit 1. If $1 \notin A$, we first extend φ to a positive linear functional $\widetilde{\varphi}$ by putting $\widetilde{\varphi}(\lambda 1 + a) = \|\varphi\| + \varphi(a)$ (the above corollary). Thus the problem is reduced to the case $1 \in A \subset B$.

Now ψ be a Hahn-Banach extension of φ: ψ is a linear functional on B satisfying $\varphi(a) = \psi(a)$ for $a \in A$ and $\|\varphi\| = \|\psi\|$. Then $\psi(1) = \varphi(1) = \|\varphi\| = \|\psi\|$ by Theorem 3.9, which guarantees the positivity of ψ again by the same theorem. □

Corollary 3.12. For any $a \in A$, we can find a positive linear functional φ on A so that $\|\varphi\| = 1$ and $\varphi(a^*a) = \|a\|^2$.

Let $\{\varphi_i\}_{i\in I}$ be a family of positive linear functionals on a C*-algebra and assume that, for each $0 \neq a \in A$, $\varphi_i(a^*a) > 0$ for some $i \in I$. Then the direct sum $\pi = \bigoplus_{i\in I} \pi_i$ of GNS representations is faithful.

In fact, if $\pi(a) = 0$, $\varphi_i(b^*a^*ab) = 0$ for $b \in A$ and $i \in I$, and then, by letting $b \to 1$, $\varphi_i(a^*a) = 0$ for all $i \in I$, whence $a = 0$.

Theorem 3.13 (Gelfand-Naimark)**.** Any C*-algebra is *-isomorphic to a closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proof of Theorem 2.27

Proof. Let $a \in I$. Then $\frac{a^*a}{t+a^*a} \in I$ for $t > 0$ and a^* is approximated by $\frac{a^*a}{t+a^*a}a^* \in I$.

Let $\{u_\alpha\}$ be an approximate unit for I. We claim that

$$\|a + I\| = \lim_\alpha \|a - au_\alpha\| = \lim_\alpha \|a - u_\alpha a\|.$$

For $x \in I$,

$$\|a+I\| \leq \|a-au_\alpha\| \leq \|(1-u_\alpha)(a+x)\|+\|x-xu_\alpha\| \leq \|a+x\|+\|x-xu_\alpha\|$$

implies

$$\|a+I\| \leq \liminf \|a-au_\alpha\| \leq \limsup \|a-au_\alpha\| \leq \|a+x\|.$$

Let $a, b \in A$. For $x, y \in I$,

$$\|ab+I\| \leq \|(a+x)(b+y)\| \leq \|a+x\| \, \|b+y\|$$

implies $\|ab+I\| \leq \|a+I\| \, \|b+I\|$.

$$\begin{aligned}\|a+I\|^2 &= \lim \|a-au_\alpha\|^2 = \lim \|(1-u_\alpha)a^*a(1-u_\alpha)\| \\ &\leq \lim \|a^*a(1-u_\alpha)\| \\ &= \|a^*a+I\| \leq \|a^*+I\| \, \|a+I\|.\end{aligned}$$

shows that $\|a^*+I\| = \|a+I\|$ and then the equality $\|a+I\|^2 = \|a^*a+I\|$. □

The *-algebraic operations of a C*-algebra A is now transferred into the dual Banach space A^*: For $\varphi \in A^*$ and $a \in A$, $a\varphi$, φa and φ^* are defined in A^* by

$$(a\varphi)(x) = \varphi(xa), \quad (\varphi a)(x) = \varphi(ax), \quad \varphi^*(x) = \overline{\varphi(x^*)}, \quad \text{for } x \in A$$

with the following relations

$$(a\varphi)b = a(\varphi b), \quad (a\varphi b)^* = b^*\varphi^*a^*.$$

A linear functional $\varphi \in A^*$ is said to be **hermitian** if $\varphi^* = \varphi$. Let A^*_+ be the set of positive functionals on A. Then $aA^*_+a^* \subset A^*_+$ for $a \in A$.

Lemma 3.14. Let H be the set of hermitian elements in A, which is a closed real-linear subspace of A. Then the real Banach space H^* of bounded linear functional on H is identified with the set of hermitian functionals on A.

The following is an analogue of Jordan decomposition in measure theory.

Theorem 3.15 (Grothendieck)**.** Any hermitian functional $\theta \in A^*$ is expressed by $\theta = \varphi - \psi$ with $\varphi, \psi \in A^*_+$ satisfying $\|\theta\| = \|\varphi\| + \|\psi\|$ and such an expression is unique.

Proof. The positive unit ball $A^*_{+,1}$ is compact in the weak* topology and so is the convex hull C of $A^*_{+,1} \cup (-A^*_{+,1})$. Clearly $C \subset H^*_1$ and, if $f \in H^*_1 \setminus C$, we can find $h \in H$ such that $f(h) > \sup\{g(h); g \in C\}$

by Hahn-Banach theorem (H being identified with the weak* dual of H^*).

The Gelfand transform of the C*-algebra $C^*(h)$ generated by $h \in A$ enables us to find $\omega \in \sigma_{C^*(h)}$ satisfying $|\omega(h)| = \|h\|$. Let $\varphi \in A^*_{+,1}$ be an extension of ω. Then $\pm\varphi \in C$ implies $|f(h)| \geq f(h) > |\varphi(h)| = \|h\|$, which contradicts with $\|f\| \leq 1$.

Now we express $\theta/\|\theta\| \in H^*_1$ as an element in C: $\theta/\|\theta\| = t\varphi_1 - (1-t)\psi_1$ with $0 \leq t \leq 1$ and $\varphi_1, \psi_1 \in A^*_{+,1}$. Then the choice $\varphi = t\|\theta\|\varphi_1$ and $\psi = (1-t)\|\theta\|\psi_1$ does the job for the existence part because of

$$\|\theta\| \leq \|\varphi\|+\|\psi\| = t\|\theta\|\|\varphi_1\|+(1-t)\|\theta\|\|\psi\|_1 \leq t\|\theta\|+(1-t)\|\theta\| = \|\theta\|.$$

The uniqueness will be established later on as a consequence of polar decomposition for linear functionals (see Pedersen §3.2 for a direct proof). □

4. Representations and W*-algebras

In connection with representations, the notion of W*-algebra arises in two ways: as the space of intertwiners or as a weaker notion of equivalence of representations. For the analysis of these, the norm topology turns out to be not much adequate.

As an example, consider the Schur's criterion on irreducible representations, which can be achieved by appealing to the spectral decomposition theorem. To get spectral projections starting with a hermitian operator h, a weaker notion of convergence comes into, although the process of recovering h as a limit of linear combinations of projections can be norm-convergent. Cumbersomeness here is that lots of related notions of weaker topologies arise on sets of operators.

Definition 4.1. Too many topologies on operators. Write $\ell^2 = \ell^2(\mathbb{N})$.

(i) The **weak operator topology** is the one described by seminorms $|(\xi|a\eta)|$ $(\xi, \eta \in \mathcal{H})$.
(ii) The **strong operator topology** is the one described by seminorms $\|a\xi\|$ $(\xi \in \mathcal{H})$.
(iii) The ***strong operator topology** is the one described by seminorms $\|a\xi\|$, $\|a^*\xi\|$ $(\xi \in \mathcal{H})$.
(iv) The **$\boldsymbol{\sigma}$-weak topology** is the one described by seminorms $|(\xi|(a \otimes 1)\eta)|$ $(\xi, \eta \in \mathcal{H} \otimes \ell^2)$.
(v) The **$\boldsymbol{\sigma}$-strong topology** is the one described by seminorms $\|(a \otimes 1)\xi\|$ $(\xi \in \mathcal{H} \otimes \ell^2)$.
(vi) The **$\boldsymbol{\sigma}$-*strong topology** is the one described by seminorms $\|(a \otimes 1)\xi\|$, $\|(a^* \otimes 1)\xi\|$ $(\xi \in \mathcal{H} \otimes \ell^2)$.

Exercise 29. Given a vector $\xi \in \mathcal{H} \otimes \mathcal{H}^*$, we can find a (at most) countable orthonormal system $\{\eta_n\}$ in $\mathcal{H}$ such that $\xi \in \overline{\sum_n \mathcal{H} \otimes \eta_n^*}$.

Proposition 4.2. Regard $\mathcal{H} \otimes \mathcal{H}^*$ as a $\mathcal{B}(\mathcal{H})$-bimodule.

(i) The σ-weak topology is given by seminorms $|(\xi|a\eta)|$ $(\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*)$.
(ii) The σ-strong topology is the one described by seminorms $\|a\xi\|$ $(\xi \in \mathcal{H} \otimes \mathcal{H}^*)$.
(iii) The σ-*strong topology is the one described by seminorms $\|a\xi\|$, $\|\xi a\|$ $(\xi \in \mathcal{H} \otimes \mathcal{H}^*)$.

Proof. If $\dim \mathcal{H} = \infty$, the topologies are controlled by vectors of countable decomposition, whereas all these collapse to the single euclidean topology for a finite-dimensional $\mathcal{H}$. In the assertion (iii), note that $\|\xi a\| = \|(\xi a)^*\| = \|a^* \xi^*\|$ covers the adjoint part. $\square$

Proposition 4.3. The operator multiplication is separately continuous in any of these topologies. The star operation is continuous in each of weak, σ-weak and σ-*strong topologies. Given a bounded subset B of $\mathcal{H}$, the operator multiplication on $B \times \mathcal{B}(\mathcal{H})$ is jointly continuous in strong operator topology.

Exercise 30. Let $S : \ell^2 \to \ell^2$ be the unilateral shift operator. Then $(S^*)^n \to 0$ in the strong operator topology but not for its adjoint $\{S^n\}$.

Exercise 31. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral shift operator. Then $T^n \to 0$ $(n \to \infty)$ in the weak operator topology, but not in the strong operator topology.

Proposition 4.4.

(i) On a bounded subset B of $\mathcal{B}(\mathcal{H})$, the weak (resp. strong or *strong) operator topology is equivalent to the σ-weak (resp. σ-strong or σ-*strong) topology.
(ii) On the set $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, all of these six topologies are equivalent and $\mathcal{U}(\mathcal{H})$ is a topological group with respect to this common topology.

Example 4.5. Let $\mathcal{H}$ be a separable Hilbert space and choose a countable dense set $\{\xi_n\}$ in the unit sphere of $\mathcal{H}$. For $x \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_w = \sum_{m,n \geq 1} \frac{1}{2^{m+n}} |(\xi_m | x \xi_n)|$$

defines a norm weaker than the operator norm. Then the topology induced from the distance function $\|x - y\|_w$ coincides with the weak operator topology when restricted to the unit ball B of $\mathcal{B}(\mathcal{H})$. Moreover B is complete with respect to this metric.

Exercise 32. If a sequence $\{T_n\}$ of bounded operators converges to a bounded operator T in the weak operator topology, then $\{\|T_n\|\}$ is bounded and therefore $T_n \to T$ in the σ-weak topology. (Use Banach-Steinhauss theorem twice.)

Exercise 33. Let $\{\xi_n\}$ be a countable dense subset of the unit sphere of a Hilbert space $\mathcal{H}$. Consider the directed set of pairs $\alpha = (e, \epsilon)$ ($\epsilon > 0$ and e being a finite rank projection) with $(e, \epsilon) \prec (e', \epsilon')$ if and only if $e \leq e'$ and $\epsilon \geq \epsilon'$. Let $T_\alpha = \epsilon e + m^2(1-e)|\xi_m)(\xi_m|(1-e)$, where $m = \min\{n \geq 1; \|e\xi_n\|^2 \leq \epsilon\}$.

Show that $T_\alpha \to 0$ in the weak operator topology, whereas

$$\sum_n \frac{1}{n^2}|(\xi_n|T_\alpha\xi_n)| \geq \frac{1}{m^2}|(\xi_m|T_\alpha\xi_m)| \geq (1-\epsilon)^2$$

for any $\alpha = (e, \epsilon)$.

By a conjugation on a Hilbert space $\mathcal{H}$, we shall mean a conjugate-linear involution $\Gamma : \mathcal{H} \to \mathcal{H}$ satisfying $(\Gamma\xi|\Gamma\eta) = (\eta|\xi)$ for $\xi, \eta \in \mathcal{H}$. A conjugation enables us to identify $\mathcal{H}^*$ with $\mathcal{H}$ itself: $\xi^* \mapsto \Gamma\xi$ gives a unitary map between $\mathcal{H}^*$ and $\mathcal{H}$.

As an example, the Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$ is self-dual through the natural conjugation defined by $(\xi \otimes \eta^*)^* = \eta \otimes \xi^*$. In other words, the linear functional $(\xi \otimes \eta^*)^*$ on $\mathcal{H} \otimes \mathcal{H}^*$ is identified with the vector $\eta \otimes \xi^*$ in $\mathcal{H} \otimes \mathcal{H}^*$.

Proposition 4.6. The following conditions on a linear functional $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ are equivalent.

(i) φ is of the form $\varphi(a) = \mathrm{tr}(Ta)$ with $T \in \mathcal{B}(\mathcal{H})$ in the trace class.
(ii) φ is σ-weakly continuous.
(iii) φ is σ-*strongly continuous.
(iv) There exist vectors $\xi, \eta \in \mathcal{H} \otimes \ell^2$ such that $\varphi(x) = (\xi|(x \otimes 1)\eta)$ for $x \in \mathcal{B}(\mathcal{H})$.

Proof. (i) $\iff$ (ii): Write $T = x^*y$ with $x, y \in \mathcal{C}_2$ and let $\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*$ be the vector representatives of x, y. Then $\mathrm{tr}(Ta) = (\xi|a\eta)$ is σ-weakly continuous.

(ii) $\implies$ (iii) is obvious. Assume (iii). Then there exists $\xi \in \mathcal{H} \otimes \mathcal{H}^*$ such that

$$|\varphi(a)| \leq \|a\xi \oplus \xi a\|$$

whence $a\xi \oplus \xi a \mapsto \varphi(a)$ defines a bounded linear functional on $\{a\xi \oplus \xi a; a \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H} \otimes \mathcal{H}^* \oplus \mathcal{H} \otimes \mathcal{H}^*$ and, by the Riesz lemma, we can find $\eta, \zeta \in \mathcal{H} \otimes \mathcal{H}^*$ so that

$$\varphi(a) = (\eta \oplus \zeta|a\xi \oplus \xi a) = (\eta|a\xi) + (\zeta|\xi a) = (\eta|a\xi) + (\xi^*|a\zeta^*),$$

which reveals the σ-weak continuity of φ.

(i) $\Longleftrightarrow$ (iv): If we express T as a product of two operators in the Hilbert-Schmidt class, then we see that $\varphi(x) = (\xi|(\xi \otimes 1)\eta)$ with $\xi, \eta \in \mathcal{H} \otimes \mathcal{H}^*$. Let $\{\zeta^*_{i\in I}\}$ be an orthonormal basis in $\mathcal{H}^*$, then ξ and η are supported by a countable subset $\{i_1, i_2, \dots\}$, whence ξ and η are identified with vectors in $\mathcal{H} \otimes \ell^2$. □

Exercise 34. If a linear functional $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is continuous relative to the *strong operator topology, then it is continuous relative to the weak operator topology.

Proposition 4.7. An operator $a \in \mathcal{B}(\mathcal{H})$ is in the σ-strong closure of a subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ if and only if $(a\otimes 1)\xi \in \overline{(\mathcal{S} \otimes 1)\xi}$ for any $\xi \in \mathcal{H}\otimes \ell^2$. Here $\overline{(\mathcal{S} \otimes 1)\xi}$ denotes the norm closure of $\{(s \otimes 1)\xi; s \in \mathcal{S}\}$ in $\mathcal{H} \otimes \ell^2$.

Proof. Controls at finitely many families $\{\xi_i = \oplus_j \xi_{ij}\}_{1\leq i\leq n}$ are managed by a single vector $\xi = \xi_1 \oplus \cdots \oplus \xi_n$, which is identified with a vector in $\mathcal{H} \otimes \ell^2$ through any bijection $\mathbb{N}^n \cong \mathbb{N}$. □

The following is an analogue of the monotone convergence theorem in Lebesgue integration, which describes a completely different feature from the norm topology.

Proposition 4.8. A bounded increasing net $\{a_\alpha\}$ of positive elements in $\mathcal{B}(\mathcal{H})$ converges to a positive element in the σ-strong topology.

Proof. By the polar identity, the sesquilinear form $(\xi|a_\alpha\eta)$ converges point-wise: there is a positive element $a \in \mathcal{B}(\mathcal{H})$ such that

$$(\xi|a\eta) = \lim_{\alpha\to\infty} (\xi|a_\alpha\eta), \quad \xi, \eta \in \mathcal{H}$$

and then

$$\begin{aligned} \||(a - a_\alpha)\xi|\|^2 &\leq \|(a - a_\alpha)^{1/2}\| \, \|(a - a_\alpha)^{1/2}\xi\|^2 \\ &= \|a - a_\alpha\|^{1/2} \, (\xi|(a - a_\alpha)\xi) \\ &\leq \|a\|^{1/2}(\xi|(a - a_\alpha)\xi) \to 0 \ (\alpha \to \infty). \end{aligned}$$

We now apply this convergence to the net $\{a_\alpha \otimes 1\}$ in $\mathcal{B}(\mathcal{H} \otimes \ell^2)$ to find a positive operator $\widehat{a}$ on $\mathcal{H}\otimes\ell^2$. Since $a_\alpha\otimes 1$ commutes with $1\otimes p_j p_k^*$ $(j, k \geq 1)$ and the operator multiplication is seperately continuous in the strong operator topology, we see that $\widetilde{a}$ commutes with $1 \otimes p_j p_k^*$ as well, whence $\widetilde{a}$ is of the form $a \otimes 1$ with $a \in \mathcal{B}(\mathcal{H})$. □

Definition 4.9. A *-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is said to be non-degenerate if $\mathcal{A}\mathcal{H}$ is dense in $\mathcal{H}$.

A **W*-algebra** on a Hilbert space $\mathcal{H}$ is a σ-weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition 4.10. The **commutant** $\mathcal{S}'$ of a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is defined to be

$$\mathcal{S}' = \{c \in \mathcal{B}(\mathcal{H}); cs = sc \text{ for every } s \in \mathcal{S}\}.$$

Note that by the separate continuity of multiplication, $\mathcal{S}' = \overline{\mathcal{S}}'$, where $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ in the weak operator topology.

Exercise 35. Show that $\mathcal{S}' = \mathcal{S}'''$.

Example 4.11. Given a subset $\mathcal{S} = \mathcal{S}^* \subset \mathcal{B}(\mathcal{H})$, its commutant $\mathcal{S}'$ is a W*-algebra. As a special case of this, given a *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, the space of self intertwiners is a W*-algebra.

Example 4.12. For the left and right regular representations λ, ρ of a group G on $\ell^2(G)$, $\lambda(G)' = \rho(G)''$.

Recal that that $\ell^2(G)$ is a *-bimodule of $\mathbb{C}G$. Let $r \in \lambda(G)'$ and set $\eta = r(\delta^{1/2}) \in \ell^2(G)$. Notice $r(a\delta^{1/2}) = ar(\delta^{1/2}) = a\eta$ ($a \in \mathbb{C}G$ shows that r is given by a right multiplication of η and one may expect that it can be approximated by cutting the support of ξ down to finite sets. Real life is, however, not so easy but still simple enough:

Let $l \in \rho(G)'$ and express it as a left multiplication of $\xi \in \ell^2(G)$: $l(a\delta^{1/2}) = \xi a$ for $a \in \mathbb{C}G$. Then l^* is given by the left multiplication of ξ^* and r^* by the right multiplication of η^*, which are used to see

$$\begin{aligned}(a\delta^{1/2}|lr(b\delta^{1/2})) &= (l^*(a\delta^{1/2})|r(b\delta^{1/2})) = (\xi^* a|b\eta) = (\eta^* b^*|a^*\xi)\\ &= (a\eta^*|\xi b) = (r^*(a\delta^{1/2})|l(b\delta^{1/2})) = (a\delta^{1/2}|rl(b\delta^{1/2})).\end{aligned}$$

Exercise 36. Let two subsets $A, B \subset \mathcal{B}(\mathcal{H})$ commute with each other. Then $A' = B''$ if and only if A' commutes with B'.

Lemma 4.13. Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}(\mathcal{H})$ and $p_j : \mathcal{H} \otimes \ell^2 \to \mathcal{H}$ be the projection to the j-th component.

(i) Then an element $C \in \mathcal{B}(\mathcal{H} \otimes \ell^2)$ belongs to $(\mathcal{A} \otimes 1)'$ if and only if $p_j C p_k^* \in \mathcal{A}'$ for any $j, k \geq 1$.
(ii) We have $(\mathcal{A} \otimes 1)'' = \mathcal{A}'' \otimes 1$.
(iii) A projection $e \in \mathcal{B}(\mathcal{H})$ belongs to $\mathcal{A}'$ if and only if $e\mathcal{H}$ is invariant under $\mathcal{A}$.

Proof. (i) is a consequence of $p_j(a \otimes 1) = ap_j$ for $a \in \mathcal{A}$ and $j \geq 1$.

The inclusion $\mathcal{A}'' \otimes 1 \subset (\mathcal{A} \otimes 1)''$ follows from (i). To get the reverse inclusion, assume that $C \in \mathcal{B}(\mathcal{H} \otimes \ell^2)$ is in the commutant of $(\mathcal{A} \otimes 1)'$. Then C commutes with $p_j^* p_k$ for any $j, k \geq 1$, whence $p_j C p_k^* = \delta_{j,k} c$ with $c \in \mathcal{B}(\mathcal{H})$, i.e., $C = c \otimes 1$. Since C commutes with $p_j^* a' p_j$ ($a' \in \mathcal{A}'$) as well, we see that c is in the commutant of $\mathcal{A}'$.

Non-trivial is the if part: since $\mathcal{A}$ is a *-subalgebra, the invariance of $e\mathcal{H}$ under $\mathcal{A}$ implies the invariance of the orthogonal complement

$(1-e)\mathcal{H}$ as well and we see that $ae\xi = a\xi = ea\xi$ for $\xi \in e\mathcal{H}$, while $ae\xi = 0 = ea\xi$ for $\xi \in (1-e)\mathcal{H}$. □

Lemma 4.14. If a *-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is non-degenerate, then $\xi \in \overline{\mathcal{A}\xi}$ for any $\xi \in \mathcal{H}$.

Proof. Passing to the norm closure, we may assume that $\mathcal{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and let $\{u_\alpha\}$ be an approximate unit in $\mathcal{A}$. Then, on a dense subspace $\mathcal{A}\mathcal{H}$,

$$\lim_{\alpha\to\infty} u_\alpha \sum_{j=1}^n a_j\xi_j = \sum_{j=1}^n \lim_{\alpha\to\infty} u_\alpha a_j \xi_j = \sum_{j=1}^n a_j\xi_j$$

shows that $\lim_{\alpha\to\infty} u_\alpha = 1$ in the strong operator topology. Particularly, $\xi = \lim_\alpha u_\alpha \xi$ belongs to the closure of $\mathcal{A}\xi$. □

Theorem 4.15 (von Neumann's Bicommutant Theorem)**.** Let $\mathcal{A}$ be a non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the bicommutant $\mathcal{A}''$ is the σ-strong closure of $\mathcal{A}$.

Proof. Since $\mathcal{A}''$ is closed in the weak operator topology, we need to show that any $a'' \in \mathcal{A}''$ is in the σ-strong colosure of $\mathcal{A}$. In fact, in view of Proposition 4.7, let $\xi \in \mathcal{H}\otimes\ell^2$ and P be the projection to $\overline{(\mathcal{A}\otimes 1)\xi}$. Then $P \in (\mathcal{A}\otimes 1)'$ commuets with $a''\otimes 1$ by Lemma 4.13 (ii), (iii) and

$$P(a''\otimes 1)\xi = (a''\otimes 1)P\xi = (a''\otimes 1)\xi.$$

Here the non-degeneracy of $\mathcal{A}\otimes 1$ on $\mathcal{H}\otimes\ell^2$ is used to ensure $P\xi = \xi$. □

Corollary 4.16. For a non-degenerate *-subalgebra M of $\mathcal{B}(\mathcal{H})$, the following conditions are equivalent.

(i) $M = M''$.
(ii) M is closed in the weak operator topology.
(iii) M is a W*-algebra.
(iv) M is closed in the σ-*strong topology.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are trivial. Assume that M is closed in the σ-*strong topology. Then it is closed in the σ-strong topology by Hahn-Banach theorem in view of Proposition 4.6, whence $M = M''$. □

Example 4.17. Let M be a W*-algebra on $\mathcal{H}$ and $a = v|a|$ be the polar decomposition of $a \in M$. Then $|a|$ and v belong to M. In fact, if u is a unitary in M', then $a = uau^* = uvu^*u|a|u^*$ and the uniqueness of polar decomposition shows that $uvu^* = v$ and $u|a|u^* = |a|$.

By a similar argument, for a hermitian $h \in M$, the spectral projections of h belong to M.

Lemma 4.18. Let M be a W*-algebra on a Hilbert space $\mathcal{H}$ and $e \in \mathcal{B}(\mathcal{H})$ be a projection. Then $e \in M$ if and only if $e\mathcal{H}$ is invariant under M'.

Corollary 4.19. The set of projections in a W*-algebra M is a complete lattice.

Proof. This is because projections in $\mathcal{B}(\mathcal{H})$ are in one-to-one correspondence with M'-invariant closed subspaces of $\mathcal{H}$. □

Exercise 37. Any σ-weakly closed left ideal of a W*-algebra M is of the form Mp with $p \in M$ a projection. If it is an ideal, then p belongs to the center $M \cap M'$ of M.

Example 4.20. The *-subalgebra of finite rank operators is dense in $\mathcal{B}(\mathcal{H})$ with respect to the σ-*strong topology.

Example 4.21. Let μ be a σ-finite measure on a Borel space Ω and regard $L^\infty(\Omega,\mu)$ as a *-subalgebra of multiplication operators on the Hilbert space $L^2(\Omega,\mu)$. Then $(L^\infty(\Omega,\mu))' = L^\infty(\Omega,\mu)$: $L^\infty(\Omega,\mu)$ is a commutative W*-algebra on $L^2(\Omega,\mu)$, which is maximally commutative in $\mathcal{B}(L^2(\Omega,\mu))$.

In fact, if $b \in \mathcal{B}(L^2(\Omega,\mu))$ is in the commutant of $L^\infty(\Omega,\mu)$ and $b_E \in L^2(\Omega,\mu)$ is defined by $b(1_E\mu^{1/2}) = b_E\mu^{1/2}$ for a Borel subset $E \subset \Omega$ satisfying $\mu(E) < \infty$, then these are patched together to a μ-measurable function b_Ω on Ω satisfying $1_E b_\Omega = b_E$ (the σ-finiteness of μ is used here). Then, for $a \in L^\infty(\Omega,\mu)$ and for E with $\mu(E) < \infty$,

$$b(a1_E\mu^{1/2}) = ab(1_E\mu^{1/2}) = b_E a1_E\mu^{1/2} = b_\Omega a1_E\mu^{1/2}.$$

Since $\cup_E L^\infty(\Omega,\mu)1_E\mu^{1/2}$ is dense in $L^2(\Omega,\mu)$, the boundedness of b compels b_Ω to be bounded and we see that b is the multiplication operator by b_Ω.

We now construct the universal representation of a commutative C*-algebra $A = C_0(\Omega)$. Consider the free A-module over the set of formal symbols $\{\varphi^{1/2}; \varphi \in A_+^*\}$ on which we introduce a positive sesquilinear form by

$$\left(\sum_\varphi x_\varphi \varphi^{1/2} \middle| \sum_\varphi y_\varphi \varphi^{1/2} \right) = \sum_{\varphi,\psi} \int_\Omega \overline{x_\varphi(\omega)} y_\psi(\omega) \sqrt{\varphi(d\omega)}\sqrt{\psi(d\omega)}.$$

Here φ (resp. ψ) is identified with the Radon measure $\varphi(d\omega)$ (resp. $\psi(d\omega)$) on Ω and the Hellinger integral is defined by

$$\int_\Omega f(\omega)\sqrt{\varphi(d\omega)}\sqrt{\psi(d\omega)} = \int_\Omega f(\omega)\sqrt{\frac{d\varphi}{d\mu}(\omega)\frac{d\psi}{d\mu}(\omega)}\,\mu(d\omega)$$

with μ any auxillirary measure satisfying $\varphi \prec \mu$ and $\psi \prec \mu$. Note that the positivity as well as the boundedness of left multiplication by elements in A follows from this expression:

$$\left(\sum_{\varphi} ax_{\varphi}\varphi^{1/2}\middle|\sum_{\varphi} ax_{\varphi}\varphi^{1/2}\right) = \int_{\Omega} |a(\omega)|^2 \left|\sum_{\varphi} x_{\varphi}(\omega)\sqrt{\frac{d\varphi}{d\mu}(\omega)}\right|^2 \mu(d\omega).$$

The associated Hilbert space is denoted by $L^2(A)$ on which A is represented by multiplication. Moreover $L^2(A)$ is a *-bimodule by the involution $(\sum_{\varphi} x_{\varphi}\varphi^{1/2})^* = \sum_{\varphi} x_{\varphi}^*\varphi^{1/2}$.

We shall later generalize the construction to non-commutative C*-algebras in a far-reaching way.

Theorem 4.22. A W*-algebra M is order-complete in the sense that every norm-bounded increasing net $\{a_\imath\}$ of positive elements in M admits a least upper bound a in M and the net $\{a_\imath\}$ converges to a in the σ-strong topology.

Proof. Let M be on $\mathcal{H}$. Then $\{a_\imath\}$ converges to a positive element $a \in sB(\mathcal{H})$ in the σ-strong topology by Proposition 4.8, whence it converges in the weak operator topology. Since M is closed in the weak operator topology by Corollary 3.13, we have $a \in M$. Let b be an upper bound of $\{a_\imath\}$ in M. Then

$$(\xi|a\xi) = \lim(\xi|a_\imath\xi) \le (\xi|b\xi) \quad \text{for } \xi \in \mathcal{H}$$

shows that $a \le b$. Thus a is the least upper bound of $\{a_\imath\}$ in M. $\square$

Theorem 4.23 (Kaplansky Density Theorem)**.** Let $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}(\mathcal{H})$ and $a \in \mathcal{B}(\mathcal{H})$ be in the closure of $\mathcal{A}$ with repsect to the weak operator topology. Then we can find a net $\{a_\imath\}$ of elements in $\mathcal{A}$ such that $\|a_\imath\| \le \|a\|$ and $a = \lim_\imath a_\imath$ in the *strong topology.

Proof. Since $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ is in the weak operator closure of $M_2(\mathcal{A})$, the problem is reduced to approximating $a = a^*$ by $a_\imath = a_\imath^*$ in the strong operator topology.

Let $f(t) = 2\|a\|t/(1+t^2)$ be a $\mathbb{R}$-valued function in $C_0(\mathbb{R})$, which gives a homeomorphism between $[-1, 1]$ and $[-\|a\|, \|a\|]$ when restricted to the interval $[-1, 1]$.

Given a closed subset S of $\mathbb{R}$, let $\mathcal{B}_S(\mathcal{H})$ be the set of hermitian elements, say h, in $\mathcal{B}(\mathcal{H})$ satisfying $\sigma(h) \subset S$. Then $f(\mathcal{B}_{\mathbb{R}}(\mathcal{H})) \subset \mathcal{B}_{[-\|a|,\|a\|]}(\mathcal{H})$ and the functional calculus by f gives a bijection between $\mathcal{B}_{[-1,1]}(\mathcal{H})$ and $\mathcal{B}_{[-\|a\|,\|a\|]}(\mathcal{H})$ with the inverse map given by $g(h)$ ($h \in \mathcal{B}_{[-\|a|,\|a\|]}(\mathcal{H})$), where $g : [-\|a\|, \|a\|] \to [-1, 1]$ is the inverse function of $f|_{[-1,1]}$.

Now choose $b \in \mathcal{B}_{[-1,1]}(\mathcal{H})$ so that $a = f(b)$. Then, for a unitary $u \in \mathcal{A}'$, $ubu^* = f^{-1}(uau^*) = b$ shows that $b \in \mathcal{A}'$ and we can find a net $\{b_\imath = b_\imath^*\}_{\imath \in I}$ in $\mathcal{A}''$ so that $b = \lim_{\imath \to \infty} b_\imath$ in the strong operator topology. From the expression $f(t) = \frac{\|a\|}{i+t} + \frac{\|a\|}{-i+t}$,

$$f(b_\imath) - f(b) = \frac{\|a\|}{i + b_\imath}\Big(b - b_\imath\Big)\frac{1}{i+b} + \frac{\|a\|}{-i + b_\imath}\Big(b - b_\imath\Big)\frac{1}{-i+b}$$

reveals that $f(b_\imath) \to f(b)$ in the strong operator topology. □

Example 4.24. Let $\mathcal{H}$ be a separable Hilbert space and $\|x-y\|_w$ be the complete metric on the unit ball B of $\mathcal{B}(\mathcal{H})$ discussed in Example 4.5. The complete metric space B is then separable. In fact, each operator in B is σ-weakly approximated by finite rank operators in B, which in turn are approximated in norm by finite rank operators of rational entries.

At the closing of this section, we introduce standard terminologies on *-representations. To simplify the description, it is convenient to view representations as left A-modules; $\pi(a)\xi$ is simply denoted by $a\xi$. Let ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ be two *-representations of A. A bounded linear map $T : \mathcal{H} \to \mathcal{K}$ is called an **intertwiner** between them if it satisfies $T(a\xi) = aT(\xi)$ for $a \in A$ and $\xi \in \mathcal{H}$. Regarding the space $\mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{K})$ of such intertwiners as a hom-set, we have the category ${}_A\mathcal{M}od$ of *-representations of A.

Clearly $\mathrm{End}({}_A\mathcal{H}) = \mathrm{Hom}({}_A\mathcal{H}, {}_A\mathcal{H})$ is a W*-algebra on $\mathcal{H}$ and each hom-vector space as an off-diagonal part of the block presentation of the W*-algebra

$$\mathrm{End}({}_A(\mathcal{H} \oplus \mathcal{K})) = \begin{pmatrix} \mathrm{End}(\mathcal{H}) & \mathrm{Hom}(\mathcal{K}, \mathcal{H}) \\ \mathrm{Hom}(\mathcal{H}, \mathcal{K}) & \mathrm{End}(\mathcal{K}) \end{pmatrix}.$$

A closed A-submodule ${}_A\mathcal{H}'$ of ${}_A\mathcal{H}$ is called a **subrepresentation** of ${}_A\mathcal{H}$. Let e be the projection to $\mathcal{H}' \subset \mathcal{H}$. Then $e \in \mathrm{End}({}_A\mathcal{H})$ and there is a one-to-one correspondence between subrepresentations of ${}_A\mathcal{H}$ and projections in $\mathrm{End}({}_A\mathcal{H})$. To make the commutativity with the left action of A visible, let M be the oppositive W*-algebra[5] of $\mathrm{End}({}_A\mathcal{H})$, which acts on $\mathcal{H}$ from the right: $\xi a^\circ = a\xi$ for $a \in \mathrm{End}({}_A\mathcal{H})$ and $\xi \in \mathcal{H}$. Then two subrepresentations ${}_A\mathcal{H}e$ and ${}_A\mathcal{H}f$ with e, f projections in M are unitarily equivalent if and only if e, f are **equivalent** in M in the sense that there exists a partial isometry $v \in M$ such that $v^*v = e$ and $vv^* = f$.

[5]If N is a W*-algebra on a Hilbert space $\mathcal{H}$, then the opposite algebra N° is a W*-algebra on $\mathcal{H}^*$: $a^\circ\xi^* = (a^*\xi)^*$ for $a \in N$ and $\xi \in \mathcal{H}$.

Lemma 4.25 (Bernstein)**.** Assume that e is equivalent to a subprojection f' of f and conversely f is equivalent to a subprojection e' of e. Then e and f are equivalent in M.

Proof. We just imitate the set-theoretical proof: Let u and v be partial isometries satisfying $u^*u = e$, $v^*v = f$, $uu^* \leq f$ and $vv^* \leq e$.

Then

$$u, vu, uvu, vuvu, \cdots \quad \text{and} \quad v, uv, vuv, uvuv, \cdots$$

are sequence of partial isometries with their initial projections satisfying

$$\begin{aligned} e &\geq u^*u \geq u^*v^*vu \geq u^*v^*u^*uvu \geq \cdots , \\ f &\geq v^*v \geq v^*u^*uv \geq v^*u^*v^*vuv \geq \cdots . \end{aligned}$$

Let $e = e_0 + e_1 + \cdots + e_\infty$ and $f = f_0 + f_1 + \cdots + f_\infty$ be the acompanied decomposition, where $e_0 = e - u^*u$, $e_1 = u^*u - u^*v^*vu$ and so on. Since $v^*e_0v = f_1$, $u^*f_0u = e_1$, $v^*e_1v = f_2$, $u^*f_1u = e_2$ and so on, partial isometries defined by

$$u_0 = u(e_0 + e_2 + \cdots), \quad v_0 = v(f_0 + f_2 + \cdots), \quad u_\infty = ue_\infty$$

satisfy

$$\begin{aligned} u_\infty^* u_\infty &= e_\infty, \quad u_\infty u_\infty^* = f_\infty, \\ u_0^* u_0 &= e_0 + e_2 + \cdots , \quad u_0 u_0^* = f_1 + f_3 + \cdots , \\ v_0^* v_0 &= f_0 + f_2 + \cdots , \quad v_0 v_0^* = e_1 + e_3 + \cdots . \end{aligned}$$

Thus the partial isometry $w = u_\infty + u_0 + v_0^*$ gives an equivalence between e and f: $w^*w = e$ and $ww^* = f$. $\square$

A *-representation ${}_A\mathcal{H}$ is said to be **irreducible** if $\mathrm{End}({}_A\mathcal{H}) = \mathbb{C}1_{\mathcal{H}}$. A positive functional is said to be **pure** if the associated GNS-representation is irreducible. A family $\{{}_A\mathcal{H}_j\}$ of *-representations of A is said to be **disjoint** if $\mathrm{Hom}({}_A\mathcal{H}_j, {}_A\mathcal{H}_k) = \{0\}$ for $j \neq k$. Two *-representations of A, π on $\mathcal{H}$ and σ on $\mathcal{K}$, are said to be **quasi-equivalent** if the correspondance $\pi(a) \mapsto \sigma(a)$ $(a \in A)$ extends to a *-isomorphism of $\pi(A)''$ onto $\sigma(A)''$. Two positive functionals φ and ψ of A are said to be **disjoint** (resp. **quasi-equivalent**) if the associated (left) GNS representations are disjoint (resp. quasi-equivalent).

Theorem 4.26.

(i) A positive functional φ is pure if and only if any positive functional ψ satisfying $\psi \leq \varphi$ is proportional to φ.

(ii) A *-representation ${}_A\mathcal{H}$ is irreducible if and only if $\overline{A\xi} = \mathcal{H}$ for any $0 \neq \xi \in \mathcal{H}$.

(iii) Two *-representations ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are not disjoint if and only if we can find non-zero subrepresentations ${}_A\mathcal{H}' \subset {}_A\mathcal{H}$ and ${}_A\mathcal{K}' \subset {}_A\mathcal{K}$ such that ${}_A\mathcal{H}'$ and ${}_A\mathcal{K}'$ are unitarily equivalent.
(iv) Two *-representations ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are quasi-equivalent if and only if there are sets X, Y such that ${}_A\mathcal{H}\otimes\ell^2(X)$ and ${}_A\mathcal{K}\otimes\ell^2(Y)$ are unitarily equivalent.

Corollary 4.27. The set of pure states of a C*-algebra A is invariant under *-automorphisms of A.

Exercise 38. The following conditions on a family $\{\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)\}$ of *-representations of a *-algebra $\mathcal{A}$ are equivalent.
(i) The representations π_i $(i \in I)$ are mutually disjoint.
(ii) $(\bigoplus_i \pi_i)(\mathcal{A})' = \bigoplus_i \pi_i(\mathcal{A})'$.
(iii) $(\bigoplus_i \pi_i)(\mathcal{A})'' = \bigoplus_i \pi_i(\mathcal{A})''$.

5. Linear Functionals on W*-algebras

The L^p-duality is usually established via Radon-Nikodym derivatives and in a measure theoretical way (see Rudin's Real and Complex Analysis Chapter 6 for example.) Try to give a functional analytic proof based on the Riesz lemma, i.e., by modifying the von Neumann's proof of the Radon-Nikodym theorem.

A positive linear functional φ of a W*-algebra M is said to be completely additive if $\varphi(\sum_{\imath\in I} e_\imath) = \sum_{\imath\in I} \varphi(e_\imath)$ for any family $\{e_\imath\}_{\imath\in I}$ of pair-wise orthogonal projections in M. It is said to be **normal**[6] if $\varphi(\sup_{\imath\in I} a_\imath) = \sup_{\imath\in I} \varphi(a_\imath)$ for any bounded increasing net $\{a_\imath\}_{\imath\in I}$ of positive elements in M, where $\sup_{\imath\in I} a_\imath$ denotes the least upper bound of $\{a_\imath\}_{\imath\in I}$. Clearly complete additivity follows from normality.

Theorem 5.1 (Dixmier)**.** The following conditions on a positive functional φ of a W*-algebra are equivalent.
(i) φ is σ-weakly continuous.
(ii) φ is normal.
(iii) φ is completely additive.

Proof. The implication (i) $\Longrightarrow$ (ii) is a consequence of σ-weak convergence of $\sup_\imath a_\imath$ and (ii) $\Longrightarrow$ (iii) is obvious.

We first show that, given any projection $0 \ne p \in M$, we can find a subprojection $0 \ne p' \le p$ such that $M \ni x \mapsto \varphi(xp')$ is σ-strongly continuous. To see this, choose a positive $\psi \in M_*$ so that $\varphi(p) < \psi(p)$. Since both of φ and ψ are completely additive, we can find a

[6] This is a standard but not illuminating terminology; order-continuity would have been much better.

subprojection $p'' \leq p$ which is maximal among subprojections satisfying $\varphi(p'') \geq \psi(p'')$. From $\varphi(p) < \psi(p)$, $p' = p - p'' \neq 0$ and, by maximality, any subprojection e of p' has the property $\varphi(e) \leq \psi(e)$, which implies $\varphi(a) \leq \psi(a)$ for $a \in p'M_+p'$ by spectral decomposition. Now $M \ni x \mapsto \varphi(xp')$ is σ-strongly continuous becuase of

$$|\varphi(xp')| \leq \varphi(1)^{1/2}\varphi(p'x^*xp')^{1/2} \leq \varphi(1)^{1/2}\psi(p'x^*xp')^{1/2}.$$

To complete the implication (iii) $\Longrightarrow$ (i), choose a maximal family $\{p_\imath\}$ of pairwise orthogonal projections satisfying $p_\imath\varphi \in M_*$. Then $\sum_\imath p_\imath = 1$ by the previous step and, for a finite subset $J \subset I$,

$$\left|\varphi(x(1-\sum_{j\in J} p_j))\right| \leq \varphi(xx^*)^{1/2}\varphi(1-\sum_{j\in J} p_j)^{1/2} \leq \|\varphi\|^{1/2}\|x\|\varphi(1-\sum_{j\in J} p_j)^{1/2}$$

shows that

$$\|\sum_{j\in J} p_j\varphi - \varphi\| \leq \|\varphi\|^{1/2}\varphi(1-\sum_{j\in J} p_j)^{1/2} \to 0 \ (J \nearrow I).$$

$\square$

Proposition 5.2. Let M_* be the set of σ-weakly continuous linear functionals on a W*-algebra M, which is referred to as the **predual** of M.

(i) Aa a subset of M^*, M_* is a norm-closed *-subbimodule of M.
(ii) The Banach space M is the dual of M_* and the σ-weak topology on M is equal to the weak* topology as the dual of M_*.

Proof. Since $M \subset \mathcal{B}(\mathcal{H})$ is σ-weakly closed, this follows from $\mathcal{C}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ and the polar relations. $\square$

Example 5.3. The predual of $\mathcal{B}(\mathcal{H})$ is naturally identified with the Banach space $\mathcal{C}_1(\mathcal{H})$ of trace class operators (cf. Theorem 2.5). A normal state on $\mathcal{B}(\mathcal{H})$ is then represented by a positive trace class operator ρ satisfying $\mathrm{tr}(\rho) = 1$, which is referred to as a **density operator**.

Since any trace class operator is a linear combination of four density operators, any normal functiona of a W*-algebra M is a linear combination of four positive normal states.

Exercise 39. Let S be the unitary shift on $\ell^2(\mathbb{Z})$. Then $\lim_{n\to\infty} S^n = 0$ in the weak* topology of $M = \mathcal{B}(\ell^2(\mathbb{Z}))$ but $\|\phi S^n\| = \|\phi\|$ for $0 \neq \phi \in M_*$ does not converge to 0.

Definition 5.4. Let $M \subset \mathcal{B}(\mathcal{H})$ and $N \subset \mathcal{B}(\mathcal{K})$ be W*-algebras. A *-homomorphism $\phi : M \to N$ is said to be **normal** if it is order-continuous: if $\{a_\imath\}$ is a norm-bounded increasing net of positive elements in M, then

$$\phi\left(\sup_\imath a_\imath\right) = \sup_\imath \phi(a_\imath).$$

A *-representation π of M on a Hilbert space $\mathcal{H}$ is said to be normal if $\pi : M \to \mathcal{B}(\mathcal{H})$ is normal.

Proposition 5.5. The following conditions on a *-homomorphism between W*-algebras are equivalent.

(i) ϕ is normal.
(ii) If a positive functional $\omega : M \to \mathbb{C}$ is normal, then so is $\omega \circ \phi$.
(iii) $N_* \circ \phi \subset M_*$.

Proof. (i) $\Longrightarrow$ (ii) is trivial.
(ii) $\Longrightarrow$ (iii): Any $\psi \in N_*$ is a difference of two normal functionals.
(iii) $\Longrightarrow$ (i) follows from the σ-weak convergence of suprema. □

Corollary 5.6. If $\phi : M \to N$ is a *-isomorphism between W*-algebras, ϕ and its inverse ϕ^{-1} are σ-weakly continuous.

Theorem 5.7. If $\phi : M \to N$ is a normal *-homomorphism of W*-algebras, then $\phi(M)$ is σ-weakly closed in N.

Proof. Since ϕ is continuous with respect to the weak* topologies and the unit ball $B \subset M$ is σ-weakly compact, $\phi(B) \subset N$ is σ-weakly compact and therefore it is σ-weakly closed in N. If b is in the σ-weak closure of $\phi(M)$, then it is in the σ-weak closure of $\|b\|\phi(B)$.

Note that $(1+\epsilon)\phi(B)$ contains the unit ball of the C*-algebra $\phi(M)$ for any $\epsilon > 0$. □

Example 5.8.

(i) An ampliation $M \ni x \mapsto x \otimes 1 \in M \otimes 1$ is an injective normal *-homomorphism.
(ii) A unitary map $\mathcal{H} \to \mathcal{K}$ induces an *-isomorphism of W*-algebras $M \ni x \mapsto UxU^* \in UMU^*$.
(iii) Let $\mathcal{K} = e'\mathcal{H}$ be M-invariant with $e' \in M'$ a projection. Then $M \ni x \mapsto xe' \in Me'$ is a surjective normal *-homomorphism.

These are simple examples of normal *-homomorphisms but they are enough to describe general ones.

Theorem 5.9 (Dixmier)**.** Let M on $\mathcal{H}$ and N on $\mathcal{K}$ be W*-algebras, $\phi : M \to N$ be a normal *-homomorphism and suppose that $N =$

$\phi(M)$. Then we can find an index set I, a projection $e \in (M \otimes 1_{\ell^2(I)})'$ and a unitary map $U : e(\mathcal{H} \otimes \ell^2(I)) \to \mathcal{K}$ so that

$$U(a \otimes 1)U^* = \phi(a) \quad \text{for } a \in M.$$

Proof. Choose vectors $\{\eta_j\}_{j\in J}$ in $\mathcal{K}$ so that $\mathcal{K} = \bigoplus_{j\in J} \overline{N\eta_j}$ and let φ_j be positive normal functionals on M defined by $\varphi_j(a) = (\eta_j|\phi(a)\eta_j)$. Then we can find vectors ξ_j in $\mathcal{H} \otimes \ell^2$ satisfying $\varphi_j(a) = (\xi_j|(a \otimes 1)\xi_j)$ $(a \in M)$, which allows us to introduce unitary maps $U_j : \overline{(M \otimes 1)\xi_j} \to \overline{N\eta_j}$ by $U_j((a \otimes 1)\xi_j) = \phi(a)\eta_j$. Note that $N = \phi(M)$ is used here to guarantee the surjectivity of U_j.

Since $\bigoplus_{j\in J} \overline{(M \otimes 1)\xi_j}$ is identified with an $M \otimes 1$-invariant closed subspace of $\mathcal{H} \otimes \ell^2 \otimes \ell^2(J)$, the projection e to the subspace belongs to $(M \otimes 1)'$.

Now, with the choice $I = \mathbb{N} \times J$ and the identification $\ell^2(I) = \ell^2 \otimes \ell^2(J)$, the unitary map $U : e(\mathcal{H} \otimes \ell^2(I)) \to \mathcal{K}$ defined by $U = \bigoplus_{j\in J} U_j$ meets the desired property. $\square$

As a simple application of this theorem, we shall show that the tensor product of W*-algebras has a space-free meaning.

Let M_j (resp. N_j) be W*-algebras on $\mathcal{H}_j$ (resp. $\mathcal{K}_j$) for $j = 1, 2$ and $\alpha : M_1 \to M_2$, $\beta : N_1 \to N_2$ be *-isomorphisms. Then the correspondence $x \otimes y \mapsto \alpha(x) \otimes \beta(y)$ is (uniquely) extended to a *-isomorphism of $M_1 \otimes N_1$ (acting on $\mathcal{H}_1 \otimes \mathcal{K}_1$) onto $M_2 \otimes N_2$ (acting on $\mathcal{H}_2 \otimes \mathcal{K}_2$).

Exercise 40. Describe the details in the proof of the assertion.

In view of the intrinsic nature of σ-weakly continuous linear functionals, it is natural to expect a Jordan type decomposition for $\varphi = \varphi^* \in M_*$.

Definition 5.10. Given a normal linear functional φ on a W*-algebra M, its left and right supports $[\varphi]_l$ and $[\varphi]_r$ are the largest projection e and f in M satisfying $e\varphi = \varphi$ and $\varphi f = \varphi$ respectively.

Note that, if $\varphi^* = \varphi$, left and right supports coincide and are denoted by $[\varphi]$. If φ is positive, $\|\varphi\| = \varphi(1) = \varphi([\varphi])$ and φ is faithful on $[\varphi]M[\varphi]$ in the sense that $\varphi(a) = 0$ implies $a = 0$ for any positive $a \in [\varphi]M[\varphi]$ (apply φ to the spectral projections of a).

Example 5.11. Let $M = \mathcal{B}(\mathcal{H})$ and represent φ in terms of a trace class operator operator T on $\mathcal{H}$. Then left and right supports of φ are those of T.

Proposition 5.12. A projection e in a W*-algebra M is of the form $e = [\varphi]$ for some $\varphi \in M_*^+$ if and only if it is ***σ*-finite** in the sense that an orthogonal decomposition $\sum_{i\in I} e_i$ of e in M is possible only for a countable index set I.

Theorem 5.13 (Sakai)**.** Each normal linear functional φ on a W*-algebra M is of the form $u\,|\varphi|$, where $|\varphi|$ is a positive normal linear functional and u is a partial isometry in M satisfying $u^*u = [\varphi]_r = [|\varphi|]$ and the pair $(u, |\varphi|)$ is uniquely determined by these properties. Moreover we have $uu^* = [\varphi]_l$.

Proof. By replacing φ with $\varphi/\|\varphi\|$, we may assume that $\|\varphi\| = 1$. Let $e = [\varphi]_r$ and $[\varphi]_l = f$.

We first check the uniqueness. Let (u, ω) be such a pair. From $\|\varphi\| \leq \|u\|\,\|\omega\| = \|\omega\|$ and $\|\omega\| \leq \|u^*\|\,\|\varphi\| = \|\varphi\|$, we see $\|\omega\| = \|\varphi\| = 1$, whence $1 = \omega(1) = \varphi(u^*)$.

Assume that $\varphi(a) = \|\varphi\| = 1$ with a in the unit ball of eMf. Then $1 = \omega(b)$ with $b = au$ in the unit ball of eMe and

$$1 = \omega(b) \leq \sqrt{\omega(1)}\sqrt{\omega(b^*b)} = \sqrt{\omega(b^*b)} \leq 1$$

shows that $\omega(b^*b) = 1 = \omega(1) = \omega(e)$. Since b^*b is in the unit ball of eMe and ω is faithul on eMe, $\omega(e - b^*b) = 0$ implies $e = b^*b$. Likewise, starting with $\omega(b^*) = 1$, we obtain $e = bb^*$. Thus b is a unitary in the unital C*-algebra eMe and $te + (1-t)b$ $(0 < t < 1)$ can be identified with the continuous function $t + (1-t)z$ of $z \in \sigma(b) \subset \mathbb{T}$. Let μ be the probability measure in $\sigma(b)$ associated with the state ω. Then

$$\begin{aligned} 1 = \omega(te + (1-t)b) &= \int_{\sigma(b)} (t + (1-t)z)\,\mu(dz) \\ &\leq \int_{\sigma(b)} |t + (1-t)z|\,\mu(dz) \\ &\leq \int_{\sigma(b)} (t + (1-t)|z|)\,\mu(dz) = 1. \end{aligned}$$

Since $|t + (1-t)z| < 1$ for $1 \neq z \in \sigma(b)$, μ must be supported by a single point $\{1\}$ and we have

$$\omega((e-b)^*(e-b)) = \int_{\sigma(b)} |1 - z|^2\,\mu(dz) = 0,$$

which means $b = e$ by the faithfulness of ω on eMe, i.e., $a = u^*$. Thus the partial isometry part $u \in fMe$ is uniquely determined by $\varphi(u^*) = 1$ and so is $\omega = u^*\varphi$.

Now we establish the existence. Since the unit ball B of M is σ-weakly compact, the function $B \ni x \mapsto |\varphi(x)|$ is σ-weakly continous, and since $1 = \|\varphi\| = \sup\{|\varphi(x)|; x \in B\}$, we can find $x \in B$ such that

$|\varphi(x)| = 1$ and then $a = x/\varphi(x) \in B$ satisfies $\varphi(a) = 1$. Replacing a with eaf, we may further assume that a belongs to the unit ball of eMf. Let $a^* = uh$ be the polar decomposition of a^* with $0 \leq h \leq 1$ and $p \leq e$ be the support projection of h. Then the linear functional $\omega = u^*\varphi$ satisfies $\|\omega\| = 1$ and $\omega(h) = \varphi(a) = 1$.

We claim that ω is positive. In fact, $\|h + e^{i\theta}(1-h)\| \leq 1$ and, for the choise of $\theta \in \mathbb{R}$ satisfying $e^{i\theta}\omega(1-h) \geq 0$,

$$1 \leq 1 + e^{i\theta}\omega(1-h) = \omega(h + e^{i\theta}(1-h)) \leq \|h + e^{i\theta}(1-h)\| \leq 1$$

$\omega(1-h) = 0$, i.e., $\omega(1) = \omega(h) = 1$ with $\|\omega\| = 1$, whence ω is positive.

Now the existence is proved by checking $p = e$, i.e., $\varphi p = \varphi$. If not, $\varphi(1-p) \neq 0$ and we can find $b \in B$ satisfying $(1-p)b = b$ and $\varphi(b) > 0$. Then $a^*b = a^*p(1-p)b = 0$ and, for $t > 0$,

$$\|a + tb\|^2 = \|(a+tb)^*(a+tb)\| = \|a^*a + t^2b^*b\| \leq 1 + t^2$$

and therefore

$$1 + t\varphi(b) = \varphi(a + tb) \leq \|a + tb\| \leq \sqrt{1 + t^2},$$

which is impossible for $\varphi(b) > 0$. □

6. Tomita-Takesaki Theory

Mainly we follow the presentation in [Bratteli-Robinson 1].

A positive normal functional φ on a W*-algebra M is said to be **faithful** if $\varphi(a) = 0$ for a positive $a \in M$ implies $a = 0$. A positive normal functional φ is faithful if and only if $[\varphi] = 1$ and any positive normal functional φ is faithful when restricted to $[\varphi]M[\varphi]$.

Example 6.1. Let $M = \mathcal{B}(\mathcal{H})$ and represent φ in terms of a positive trace class operator operator ρ on $\mathcal{H}$. Then φ is faithful if and only if $\ker \rho = \{0\}$.

Exercise 41. A W*-algebra M admits a faithful positive normal functional if and only if it is *sigma*-finite in the sense that any mutually orthogonal family $\{e_i\}_{i \in I}$ of non-zero projections in M has counatble cardinarity in the index set I.

Let M_*^+ be the set of positive normal functionals. Then M_*^+ and M_+ are in the polarity relation: $M_+ = \{a \in M; \omega(a) \in \mathbb{R}_+ \ \forall \omega \in M_*^+\}$ and, if $a \in M_+$ satisfies $\omega(a) = 0$ for all $\omega \in M_*^+$, then $a = 0$.

Given a faithful $\varphi \in M_*^+$, we shall identify the left and right GNS representation spaces $\overline{M\varphi^{1/2}}$ and $\overline{\varphi^{1/2}M}$ in such a way that the identified Hilbert space $L^2(M)$ allows a *-bimodule structure satisfying $(\varphi^{1/2})^* = \varphi^{1/2}$ and the closed convex hull of $\{a\varphi^{1/2}a^*; a \in M\}$ gives a positive cone in $L^2(M)$, i.e., $(\varphi^{1/2}a|a\varphi^{1/2}) = (\varphi^{1/2}|a\varphi^{1/2}a^*) \geq 0$ for any $a \in M$ and $\varphi \in M_*^+$.

To get a hint for the construction, think of a (possibly unbounded) operator $\Delta^{1/2}$ formally defined by $\Delta^{1/2}(a\varphi^{1/2}) = \varphi^{1/2}a$, which is positive by the positive cone assumption, and introduce the notation J to stand for the conjugate-linear isometric involution $\xi \mapsto \xi^*$ ($\xi \in L^2(M)$). Then the combination $J\Delta^{1/2}$ satisfies

$$J\Delta^{1/2}(a\varphi^{1/2}) = a^*\varphi^{1/2}.$$

In other words, if we introduce a (possibly unbounded) conjugate-linear involution S on the left GNS space $\mathcal{H} = \overline{M\varphi^{1/2}}$ by $S(a\varphi^{1/2}) = a^*\varphi^{1/2}$, then J and $\Delta^{1/2}$ can be captured as parts of the polar decomposition of S. Now we regard $M \subset \mathcal{B}(\mathcal{H})$ and introduce the *-operation as well as the right multiplication of $a \in M$ on $\mathcal{H}$ by $\xi^* = J\xi$ and $\xi a = Ja^*J\xi$ for $\xi \in \mathcal{H}$. At this point, we have $(\varphi^{1/2})^* = \varphi^{1/2}$ because $\varphi^{1/2}$ is invariant under S.

The condition $(a\xi)b = a(\xi b)$ is then equivalent to $JMJ \subset M'$, which turns out to be enough to ensure $(a\xi b)^* = b^*\xi^*a^*$ for $a, b \in M$. We also need to show the inequality $(a^*\varphi^{1/2}|J(a\varphi^{1/2})) \geq 0$ ($a \in M$) to realize the positive cone assumption.

Under these backgrounds, we introduce two conjugate-linear involutions S_0 and F_0 by

$$S_0(a\varphi^{1/2}) = a^*\varphi^{1/2}, \quad F_0(a'\varphi^{1/2}) = (a')^*\varphi^{1/2}, \quad a \in M, a' \in M'$$

Lemma 6.2. Both of S_0 and F_0 are closable with their closure S and F being adjoints of F_0 and S_0 respectively. Moreover, the following conditions on $\xi, \eta \in \mathcal{H}$ are equivalent.

(i) $\xi \in D(F)$ and $\eta = F\xi$.

(ii) There is a closed operator ρ affiliated[7] with M' satisfying $\rho\varphi^{1/2} = \xi$ and $\rho^*\varphi^{1/2} = \eta$.

A similar statement holds for S and M.

Proof. Inclusions $S_0 \subset F_0^*$ and $F_0 \subset S_0^*$ are immediate.

For $\xi \in \mathcal{H}$, a densely defined operator ρ_ξ in $\mathcal{H}$ is set to be $\rho_\xi(a\varphi^{1/2}) = a\xi$ for $a \in M$. If $\xi \in D(S_0^*)$ and $\eta = S_0^*\xi$, i.e., if

$$(a\varphi^{1/2}|\eta) = (\xi|a^*\varphi^{1/2}) \quad \text{for all } a \in M,$$

then

$$(a\varphi^{1/2}|\rho_\xi(b\varphi^{1/2})) = (b^*a\varphi^{1/2}|\xi) = (\eta|a^*b\varphi^{1/2}) = (\rho_\eta(a\varphi^{1/2})|b\varphi^{1/2})$$

[7]If $v|\rho|$ denotes the polar decomposition of ρ, this means that v and spectral projections of $|\rho|$ belong to M'.

shows that $\rho_\xi \subset \rho_\eta^*$ and $\rho_\eta \subset \rho_\xi^*$. Let $\rho = \rho_\xi^{**}$ be the closure of ρ_ξ with $v|\rho|$ the polar decomposition of ρ (cf. Reed-Simon §VIII.9).

If $u \in M$ is a unitary, $\rho_\xi(ua\varphi^{1/2}) = ua\xi = u\rho_\xi(a\varphi^{1/2})$ shows that $u^*\rho_\xi u = \rho_\xi$ and hence $u^*\rho u = \rho$. By the uniqueness of the polar decomposition, $v \in M'$ and the spectral projections of $|\rho|$ belong to M', i.e., ρ is affiliated to M'. Let $e'_n \in M'$ be the one associated to the interval $[0, n]$ and set $\rho_n = v|\rho|e'_n \in M'$. Then, the convergence

$$\rho_n\varphi^{1/2} = ve'_nv^*\rho_\xi\varphi^{1/2} = ve'_nv^*\xi \to \xi,$$
$$\rho_n^*\varphi^{1/2} = e'_n\rho_\xi^*\varphi^{1/2} = e'_n\eta \to \eta$$

shows that $\xi \oplus \eta$ is in the closure of the graph of F_0. □

Let $S = J\Delta^{1/2}$ be the polar decomposition. Recall that $\Delta^{1/2}$ is a positive self-adjoint operator specified by $\|\Delta^{1/2}\xi\|^2 = \|S\xi\|^2$ with $D(\Delta^{1/2}) = D(S)$ and the antiunitary operator J by $J : \Delta^{1/2}\xi \mapsto S\xi$ for $\xi \in D(S)$. Since $S = S^{-1}$ as the closure of $S_0 = S_0^{-1}$, we have $J\Delta^{1/2} = \Delta^{-1/2}J^{-1} = J^{-1}J\Delta^{-1/2}J^{-1}$ and the uniqueness of the polar decomposition gives

$$J^{-1} = J, \quad J\Delta^{1/2}J = \Delta^{-1/2}.$$

Thus the polar decomposition of F is given by $S^* = \Delta^{1/2}J = J\Delta^{-1/2}$ and the positive self-adjoint operator $FS = SF$ is equal to the square Δ of $\Delta^{1/2}$.

Lemma 6.3 (Fundamental Lemma)**.** Let $a \in M$, $\lambda \in \mathbb{C} \setminus [0, \infty)$ and set $\xi = \frac{1}{\lambda - \Delta^{-1}}a\varphi^{1/2}$. Then an element ρ_λ in M' is defined by

$$\rho_\lambda(x\varphi^{1/2}) = x\xi \quad \text{for } x \in M$$

with an estimate

$$\|\rho_\lambda\| \leq \frac{\|a\|}{\sqrt{2|\lambda| - \lambda - \lambda^*}}.$$

Proof. Since $\xi = \frac{1}{\lambda - \Delta^{-1}}a\varphi^{1/2}$ is in the domain $D(S^*) = D(\Delta^{-1/2})$, we can find a closed operator ρ_λ affiliated to M' satisfying

$$\rho_\lambda(x\varphi^{1/2}) = x\xi \quad \text{for } x \in M$$

and the problem is reduced to showing the estimate on $\|\rho_\lambda\|$.

Let $x \in M$. In the expression

$$\|x\xi\|^2 = (x^*x\xi|\xi) = \left(\frac{1}{\lambda^* - \Delta^{-1}}x^*x\xi \middle| a\varphi^{1/2}\right),$$

we use the fact that $\frac{1}{\lambda^* - \Delta^{-1}} x^* x\xi$ is in the domain of F to find a closed operator ρ affiliated to M' and satisfying

$$\rho\varphi^{1/2} = \frac{1}{\lambda^* - \Delta^{-1}} x^* x\xi, \quad \rho^*\varphi^{1/2} = F\left(\frac{1}{\lambda^* - \Delta^{-1}} x^* x\xi\right).$$

Let $v|\rho|$ be the polar decomposition of ρ. Then

$$\begin{aligned}\|x\xi\|^2 = (\rho\varphi^{1/2}|a\varphi^{1/2}) &= \left(|\rho|^{1/2}\varphi^{1/2}\middle|a|\rho|^{1/2}v^*\varphi^{1/2}\right)\\ &\leq \|a\|\,\||\rho|^{1/2}\varphi^{1/2}\|\,\||\rho|^{1/2}v^*\varphi^{1/2}\|.\end{aligned}$$

Since $\||\rho|^{1/2}v^*\varphi^{1/2}\|^2 = (\rho\varphi^{1/2}|v\varphi^{1/2})$ and $\||\rho|^{1/2}v^*\varphi^{1/2}\|^2$ is equal to

$$(v^*\varphi^{1/2}|\rho^*\varphi^{1/2}) = (F(v\varphi^{1/2})|F(\rho\varphi^{1/2})) = (\rho\varphi^{1/2}|\Delta^{-1}v\varphi^{1/2}),$$

we observe that

$$\begin{aligned}\left|\lambda\||\rho|^{1/2}\varphi^{1/2}\|^2 - \||\rho|^{1/2}v^*\varphi^{1/2}\|^2\right| &= \left|((\lambda^* - \Delta^{-1})\rho\varphi^{1/2}|v\varphi^{1/2})\right|\\ &= \left|(x^*x\xi|v\varphi^{1/2})\right| = \left|(x\xi|vx\varphi^{1/2})\right|\\ &\leq \|x\xi\|\,\|x\varphi^{1/2}\|,\end{aligned}$$

which is combined with the quadratic inequality

$$|\lambda s - t|^2 \geq |\lambda s - t|^2 - (|\lambda|s - t)^2 = (2|\lambda| - \lambda - \lambda^*)st$$

for the choice $s = \||\rho|^{1/2}\varphi^{1/2}\|^2$, $t = \||\rho|^{1/2}v^*\varphi^{1/2}\|^2$ to get

$$\begin{aligned}\sqrt{2|\lambda| - \lambda - \lambda^*}\|x\xi\|^2 &\leq \sqrt{2|\lambda| - \lambda - \lambda^*}\|a\|\||\rho|^{1/2}\varphi^{1/2}\|\,\||\rho|^{1/2}v^*\varphi^{1/2}\|\\ &\leq \|a\|\,\|x\xi\|\,\|x\varphi^{1/2}\|.\end{aligned}$$

□

Lemma 6.4 (Fundamental Relation)**.** As a sesquilinear form relation, we have

$$JaJ = \lambda\Delta^{-1/2}\rho_\lambda^*\Delta^{1/2} - \Delta^{1/2}\rho_\lambda^*\Delta^{-1/2}$$

on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$.

Proof. We start with the relation $\rho_\lambda\varphi^{1/2} = (\lambda - \Delta^{-1})^{-1}a\varphi^{1/2}$ in the form

$$(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) = \lambda(x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) - (x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2})$$

for $x', y' \in M'$ and rewrite each of three terms as follows:

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) &= ((y')^*\varphi^{1/2}|a(x')^*\varphi^{1/2}) = (F(y'\varphi^{1/2})|aF(x'\varphi^{1/2}))\\ &= (JaJ\Delta^{-1/2}x'\varphi^{1/2}|\Delta^{-1/2}y'\varphi^{1/2}),\end{aligned}$$

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) &= ((y')^*\varphi^{1/2}|(x')^*\rho_\lambda\varphi^{1/2}) = (F(y'\varphi^{1/2})|F(\rho_\lambda^* x'\varphi^{1/2}))\\ &= (\Delta^{-1/2}\rho_\lambda^* x'\varphi^{1/2}|\Delta^{-1/2}y'\varphi^{1/2})\end{aligned}$$

and

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2}) &= (F(\rho_\lambda\varphi^{1/2})|F(x'(y')^*\varphi^{1/2}))\\ &= (\rho_\lambda^*\varphi^{1/2}|y'(x')^*\varphi^{1/2}) = ((y')^*\rho_\lambda^*\varphi^{1/2}|(x')^*\varphi^{1/2})\\ &= (F(\rho_\lambda y'\varphi^{1/2})|F(x'\varphi^{1/2}))\\ &= (\Delta^{-1/2}x'\varphi^{1/2}|\Delta^{-1/2}\rho_\lambda y'\varphi^{1/2})\end{aligned}$$

Note here that we need to be alert on the domain of undounded operators.

To get rid of these nuisances, we again apply the fundamental lemma. For any $x \in M$, $(1+\Delta^{-1})^{-1}x\varphi^{1/2}$ is of the form $x'\varphi^{1/2}$ with $x' \in M'$ and similarly for y'. With this special choice of x' and y', we obtain the following bounded version:

$$\begin{aligned}(x'(y')^*\varphi^{1/2}|a\varphi^{1/2}) &= \left(\frac{\Delta^{-1/2}}{1+\Delta^{-1}}JaJ\frac{\Delta^{-1/2}}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)\\ (x'(y')^*\varphi^{1/2}|\rho_\lambda\varphi^{1/2}) &= \left(\frac{\Delta^{-1}}{1+\Delta^{-1}}\rho_\lambda^*\frac{1}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)\\ (x'(y')^*\varphi^{1/2}|\Delta^{-1}\rho_\lambda\varphi^{1/2}) &= \left(\frac{1}{1+\Delta^{-1}}\rho_\lambda^*\frac{\Delta^{-1}}{1+\Delta^{-1}}x\varphi^{1/2}\middle|y\varphi^{1/2}\right)\end{aligned}$$

Since $x, y \in M$ are arbitrary, these relations lead us to

$$\frac{\Delta^{-1/2}}{1+\Delta^{-1}}JaJ\frac{\Delta^{-1/2}}{1+\Delta^{-1}} = \lambda\frac{\Delta^{-1}}{1+\Delta^{-1}}\rho_\lambda^*\frac{1}{1+\Delta^{-1}} - \frac{1}{1+\Delta^{-1}}\rho_\lambda^*\frac{\Delta^{-1}}{1+\Delta^{-1}}$$

and, as a sesquilinear form relation, we conclude[8] that

$$JaJ = \lambda\Delta^{-1/2}\rho_\lambda^*\Delta^{1/2} - \Delta^{1/2}\rho_\lambda^*\Delta^{-1/2}$$

on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$. □

Theorem 6.5 (Tomita-Takesaki)**.** We have $\Delta^{it}JMJ\Delta^{-it} = M'$ for $t \in \mathbb{R}$.

Proof. We shall prove $\Delta^{it}JMJ\Delta^{-it} \subset M'$ for $t \in \mathbb{R}$ in a series of arguments below. Then the inclusion for M' gives $\Delta^{-it}JM'J\Delta^{it} \subset M'' = M$ for $t \in \mathbb{R}$ in view of $J' = J$ and $\Delta' = \Delta^{-1}$, whence the equality holds. □

---

[8]Recall that the range of $\frac{1}{\Delta^{1/2}+\Delta^{-1/2}}$ is a core for $\Delta^{1/2}$ and $\Delta^{-1/2}$.

Before giving a proof, we discuss a general fact on one-parameter automorphism group of the form $\mathrm{Ad}\Delta^{it}$.

Let Δ^{it} be a one-parameter group of unitaries on $\mathcal{H}$ and $C \in \mathcal{B}(\mathcal{H})$. Let $\xi, \eta \in \mathcal{H}$ be entirely analytic vectors for Δ^z ($z \in \mathbb{C}$) and consider a holomorphic function of the form $G(z) = g(z)(\Delta^{\overline{z}}\xi | C\Delta^{-z}\eta)$, where $g(z)$ is a holomorphic function in the punctured strip domain $\{|\mathrm{Re}(z)| \leq 1/2\} \setminus \{0\}$ with a residue r at $z = 0$, and try to have the following form of Cauchy's formula:

$$2\pi r(\xi|C\eta) = \int_{-\infty}^{\infty} (G(it + 1/2) - G(it - 1/2))\, dt,$$

where the right hand side is equal to

$$\int_{-\infty}^{\infty} dt\, (\xi|\Delta^{it} \left(g\Big(it + \frac{1}{2}\Big)\Delta^{1/2}C\Delta^{-1/2} - g\Big(it - \frac{1}{2}\Big)\Delta^{-1/2}C\Delta^{1/2} \right) \Delta^{-it}\eta).$$

To tie the integrand to the fundamental relation in Lemma 6.4, we require

$$g\Big(it - \frac{1}{2}\Big) = \lambda g\Big(it + \frac{1}{2}\Big) \quad \text{for } t \in \mathbb{R}.$$

Then, by the choice $\mu = \log(-\lambda)$, the function $f(z) = g(z)e^{\mu z}$ is extended to an entire function satisfying $f(z + 1) = -f(z)$ with simple poles at integer points. Thus it is reasonable to set[9] $g(z)e^{\mu z} = 1/\sin(\pi z)$, i.e., $g(z) = e^{-\mu z}/\sin(\pi z)$.

With this choice, $g(it + s)$ ($t \in \mathbb{R}$, $-1/2 \leq s \leq 1/2$) is rapidly decreasing as $t \to \pm\infty$ if and only if $-\pi < \mathrm{Im}\,\mu < \pi$. Thus, for μ in this range, the above integral formula in fact holds and takes the form

$$2C = e^{-\mu/2} \int_{-\infty}^{\infty} dt\, \frac{e^{-i\mu t}}{\cosh(\pi t)} \Delta^{it} \left(\Delta^{1/2}C\Delta^{-1/2} - \lambda\Delta^{-1/2}C\Delta^{1/2}\right) \Delta^{-it}.$$

as sesquilinear forms on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$.

Now apply the formula just established for the choice $C = JaJ$ to get

$$-2e^{\mu/2}\rho_\lambda^* = \int_{-\infty}^{\infty} dt\, \frac{e^{-i\mu t}}{\cosh(\pi t)} \Delta^{it} JaJ \Delta^{-it}.$$

Note here that μ in the range $|\mathrm{Im}\,\mu| < \pi$ meets the condition $\lambda \notin [0, \infty)$.

Finally, let $b \in M$. Then, for $\xi, \eta \in \mathcal{H}$, we have

$$\int_{-\infty}^{\infty} dt\, \frac{e^{-i\mu t}}{\cosh(\pi t)} (\xi|[\Delta^{it} JaJ \Delta^{-it}, b]\eta) = 0$$

[9]The choice is effectively unique up to scalar multiplication, see Appendix M.

first for $0 < |\mathrm{Im}\,\mu| < \pi$ and then for $\mu \in \mathbb{R}$ by continuity on the parameter μ. Thus $\Delta^{it} JaJ\Delta^{-it}$ commutes with every $b \in M$ and we are done.

Corollary 6.6. We have $JMJ = M'$ and $\Delta^{it} M \Delta^{-it} = M$ for $t \in \mathbb{R}$.

Exercise 42. It is instructive to see what is going on in the case $M = \mathcal{B}(\mathcal{H})$.

Let N be another W*-algebra and $\psi \in N_*^+$ be faithful. In view of the expression

$$\varphi(x)\psi(y) = (\varphi^{1/2} \otimes \psi^{1/2}|(x \otimes y)(\varphi^{1/2} \otimes \psi^{1/2}), \quad x \in M, y \in N$$

$M \otimes N \ni x \otimes y \mapsto \varphi(x)\psi(y)$ defines a normal positive functional $\varphi \otimes \psi$ of $M \otimes N$. Note that $\varphi \otimes \psi$ is faithful because $M' \otimes N' \subset (M \otimes N)'$ and $\overline{(M' \otimes N')(\varphi^{1/2} \otimes \psi^{1/2})}$ is dense in $\overline{M\varphi^{1/2}} \otimes \overline{N\psi^{1/2}}$ in view of $\overline{M'\varphi^{1/2}} = \overline{M\varphi^{1/2}}$ and $\overline{N'\psi^{1/2}} = \overline{N\psi^{1/2}}$. Since the algebraic tensor product of M and N is σ-weakly dense in $M \otimes N$, Kaplansky densitiy theorem 4.23 ensures that $M\varphi^{1/2} \otimes N\psi^{1/2}$ is a core of $S_{\varphi\otimes\psi}$ and we have

$$J_{\varphi\otimes\psi} = J_\varphi \otimes J_\psi, \quad \Delta_{\varphi\otimes\psi} = \Delta_\varphi \otimes \Delta_\psi.$$

Now, on the Hilbert space $\overline{M\varphi^{1/2}} \otimes \overline{N\psi^{1/2}}$, we have

$$(M \otimes N)' = J_{\varphi\otimes\psi}(M \otimes N)J_{\varphi\otimes\psi} = (J_\varphi M J_\varphi) \otimes (J_\psi N J_\psi) = M' \otimes N'$$

Theorem 6.7 (Tomita)**.** Let $M \subset \mathcal{B}(\mathcal{H})$ and $N \subset \mathcal{B}(\mathcal{K})$ be W*-algebras. Then $(M \otimes N)' = M' \otimes N'$ on $\mathcal{H} \otimes \mathcal{K}$.

Proof. We use a general fact that, given a projection $e \in M$, $(eMe)' = M'e$. By the previous discussion, we know

$$(M\otimes N)'(e\otimes f) = ((e\otimes f)(M\otimes N)(e\otimes f))' = (eMe\otimes fNf)' = M'e\otimes N'f$$

if $e = [\varphi]$, $f = [\psi]$ with $\varphi \in M_*^+$, $\psi \in N_*^+$ and the theorem follows from the next lemma. □

Lemma 6.8. For the directed set structure in M_*^+, we have

$$1 = \lim_{\varphi \nearrow \infty} [\varphi]$$

with respect to the σ-weak convergence in M.

Proof. This is a consequence of the fact that, for $a \in M_+$, $\varphi(a) = 0$ for all $\varphi \in M_*^+$ implies $a = 0$, which in turn follows from $M = (M_*)^*$ and $M_* = M_*^+ - M_*^+ + iM_*^+ - iM_*^+$. □

7. Standard Hilbert Spaces

We now identify the left GNS space $\overline{M\varphi^{1/2}}$ and the right GNS Hilbert space $\overline{\varphi^{1/2}M}$ by the unitary map $J(x\varphi^{1/2}) \mapsto \varphi^{1/2}x^*$ with the identified Hilbert space denoted by $L^2(M,\varphi)$ as a non-commutative analogue of $L^2(\Omega,\mu)$. On the Hilbert space $L^2(M,\varphi)$, we have $J(x\varphi^{1/2}) = \varphi^{1/2}x^* = (x\varphi^{1/2})^*$; J is the star operation on $L^2(M,\varphi)$ as expected.

Let $\rho(b)$ be the right multiplication operator by $b \in M$. Then

$$\rho(b)J(x\varphi^{1/2}) = \varphi^{1/2}x^*b = J(b^*x\varphi^{1/2}) = Jb^*JJ(x\varphi^{1/2})$$

shows that $\rho(b) = Jb^*J$ on the left GNS space and the associativity $(a\xi)b = a(\xi)b$ for $a, b \in M$ and $\xi \in L^2(M,\varphi)$ follows from the commutativity $JMJ \subset M'$. The compatibility $(a\xi b)^* = b^*\xi^*a^*$ is also reduced to the commutativity $Ja(Jb^*J) = b^*(JaJ)J$. The positivity assumption (at the beginning of the previous section) is a consequence of the positivity of $\Delta^{1/2}$: $(a^*\varphi^{1/2}|J(a\varphi^{1/2})) = (a\varphi^{1/2}|\Delta^{1/2}a\varphi^{1/2}) \geq 0$. Note that another positivity holds also: For $a, b \in M_+$, $(a\varphi^{1/2}|\varphi^{1/2}b) = (\varphi^{1/2}|aJbJ\varphi^{1/2}) \geq 0$ because $aJbJ$ is a positive operator on $L^2(M,\varphi)$.

Though the J operation is enough to identify left and right GNS Hilbert spaces, the fact $\Delta^{it}M\Delta^{-it} = M$ is also fundamental in further analysis of φ.

When $\varphi \in M_*^+$ is not faithful, we set $L^2(M,\varphi) = L^2([\varphi]M[\varphi],\varphi)$.

Definition 7.1. The **modular automorphism group**[10] $\{\sigma_t\}_{t\in\mathbb{R}}$ of M associated to a faithful $\varphi \in M_*^+$ is defined by $\sigma_t(a) = \Delta^{it}a\Delta^{-it}$ $(a \in M)$.

Let $\mathcal{M}$ be the set of entirely analytic elements for $\{\sigma_t\}$. Then $\mathcal{M}$ is a *-subalgebra of M. In fact, if $f(z) = \sigma_z(x)$ and $g(z) = \sigma_z(y)$ are analytic continuations of $\sigma_t(x)$ and $\sigma_t(y)$ for $x, y \in \mathcal{M}$, then $f(\overline{z})^*$ and $f(z)g(z)$ are analytic continuations of $\sigma_t(x^*)$ and $\sigma_t(xy)$ respectively. Moreover, thanks to the Gaussian regularization and the Kaplansky density theorem 4.23, $\{xx^*; x \in \mathcal{M}, \|x\| \leq 1\}$ is strongly dense in the operator interval $\{a \in M_+; \|a\| \leq 1\}$.

For a natural number $n \geq 2$, let $M_n(M) = M \otimes M_n(\mathbb{C})$ be the matrix ampliation of M. Given a finite family $\{\omega_j\}$ in M_*^+, let $\omega \in M_n(M)_*^+$ be defined by

$$\omega(\{a_{jk}\}) = \sum_{j=1}^{n} \omega_j(a_{jj}).$$

[10]Whereas the natural notation is definitely $\varphi^{it}(\cdot)\varphi^{-it}$, it is customary to use the same symbol σ with the spectrum.

Then $[\omega] = \mathrm{diag}([\omega_1], \cdots, [\omega_n])$ and $[\omega]M_n(M)[\omega]$ is of the form

$$\begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix} \quad \text{with } M_{jk} = [\omega_j]M[\omega_k].$$

The left and right GNS spaces are obviously identified with

$$\overline{[\omega]M_n(M)\omega^{1/2}} = \begin{pmatrix} \overline{M_{11}\omega_1^{1/2}} & \overline{M_{12}\omega_2^{1/2}} & \dots & \overline{M_{1n}\omega_n^{1/2}} \\ \overline{M_{21}\omega_1^{1/2}} & \overline{M_{22}\omega_2^{1/2}} & \dots & \overline{M_{2n}\omega_n^{1/2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{M_{n1}\omega_1^{1/2}} & \overline{M_{n2}\omega_2^{1/2}} & \dots & \overline{M_{nn}\omega_n^{1/2}} \end{pmatrix}$$

and

$$\overline{\omega^{1/2}M_n(M)[\omega]} = \begin{pmatrix} \overline{\omega_1^{1/2}M_{11}} & \overline{\omega_1^{1/2}M_{12}} & \dots & \overline{\omega_1^{1/2}M_{1n}} \\ \overline{\omega_2^{1/2}M_{21}} & \overline{\omega_2^{1/2}M_{22}} & \dots & \overline{\omega_2^{1/2}M_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\omega_n^{1/2}M_{n1}} & \overline{\omega_n^{1/2}M_{n2}} & \dots & \overline{\omega_n^{1/2}M_{nn}} \end{pmatrix}$$

respectively. Here $\omega^{1/2}$ is identified with the diagonal matrix

$$\begin{pmatrix} \omega_1^{1/2} & & O \\ & \ddots & \\ O & & \omega_n^{1/2} \end{pmatrix}$$

to utilize the matrix structure in $L^2(M_n(M), \omega)$.

Since $\overline{[\omega]M_n(M)\omega^{1/2}} = L^2(M_n(M), \omega) = \overline{\omega^{1/2}M_n(M)[\omega]}$, we have a natural identification $\overline{M_{jk}\omega_k^{1/2}} = \overline{\omega_j^{1/2}M_{jk}}$, the explicit procedure of which will be recalled in the present context:

A densely defined conjugate-linear map

$$\overline{M_{jk}\omega_k^{1/2}} \ni a_{jk}\omega_k^{1/2} \mapsto a_{jk}^*\omega_j^{1/2} \in \overline{M_{kj}\omega_j^{1/2}}$$

is closable with its closure S_{jk} satisfying $S_{jk}^{-1} = S_{kj}$. Let $S_{jk} = J_{jk}\Delta_{jk}^{1/2}$ be the polar decomposition[11] with $J_{jk} : \overline{M_{jk}\omega_k^{1/2}} \to \overline{M_{kj}\omega_j^{1/2}}$ antiunitary and the identification of Hilbert spaces is given by

$$\overline{M_{jk}\omega_k^{1/2}} \ni \xi \mapsto (J_{jk}\xi)^* \in \overline{\omega_j^{1/2}M_{jk}}.$$

Note here that the diagonal stuff is associated to the restriction of φ_j on the reduced algebras $[\varphi_j]M[\varphi_j]$.

[11] $\Delta_{j,k}$ are referred to as relative modular operators.

The unitaries Δ_{jk}^{it} on $\overline{M_{jk}\omega_k^{1/2}}$ induces a σ-weakly continuous one-parameter group σ_t^{jk} of isometries on M_{jk} so that

$$\Delta_{jk}^{it} a_{jk}\omega_k^{1/2} = \sigma_t^{jk}(a_{jk})\omega_k^{1/2}$$

and, for entirely analytic elements of σ^{jk}, the identification is also specified by

$$a_{jk}\omega_k^{1/2} = \omega_j^{1/2}\sigma_{i/2}^{jk}(a_{jk})$$

We notice, in particular, that the identification depends not on the whole family $\{\omega_j\}$ but only on the pair (ω_j, ω_k), and the modular automorphism group of ω is realized by

$$\sigma_t \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \sigma_t^{11}(a_{11}) & \sigma_t^{12}(a_{12}) & \dots & \sigma_t^{1n}(a_{1n}) \\ \sigma_t^{21}(a_{21}) & \sigma_t^{22}(a_{22}) & \dots & \sigma_t^{2n}(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{n1}(a_{n1}) & \sigma_t^{n2}(a_{n2}) & \dots & \sigma_t^{nn}(a_{nn}) \end{pmatrix}.$$

Note also that J_{jk} is the restriction of the J for ω to the subspace $\overline{M_{jk}\omega_k^{1/2}}$.

For a pair (φ, ψ) in M_*^+, it is cusomary use the more memorable notation such as $\Delta_{\varphi,\psi}$, $J_{\varphi,\psi}$ and $\sigma_t^{\varphi,\psi}$.

The pair-wise identifications are now patched up to a single Hilbert space. For each $\varphi \in M_*^+$, let $M \otimes \varphi^{1/2} \otimes M$ be a dummy of the algebraic tensor product $M \otimes M$, which is an M-M bimodule in an obvious way with a compatible *-operation defined by the relation $(a \otimes \varphi^{1/2} \otimes b)^* = b^* \otimes \varphi^{1/2} \otimes a^*$. On the algebraic direct sum

$$\bigoplus_{\varphi \in M_*^+} M \otimes \varphi^{1/2} \otimes M$$

of these *-bimodules, introduce a sesquiliear form by

$$\left(\bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \middle| \bigoplus x'_k \otimes \omega_k^{1/2} \otimes y'_k \right) = \sum_{j,k} ([\omega_k](x'_k)^* x_j \omega_j^{1/2} | \omega_k^{1/2} y'_k y_j^* [\omega_j]),$$

which is positive because of

$$\begin{aligned} \sum_{j,k} ([\omega_k] x_k^* x_j \omega_j^{1/2} | \omega_k^{1/2} y_k y_j^* [\omega_j]) &= (X\omega^{1/2} | \omega^{1/2} Y) \\ &= (X^{1/2}\omega^{1/2}Y^{1/2} | X^{1/2}\omega^{1/2}Y^{1/2}) \geq 0. \end{aligned}$$

Here

$$X = [\omega] \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} [\omega] \quad \text{and} \quad Y = [\omega] \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \begin{pmatrix} y_1^* & \dots & y_n^* \end{pmatrix} [\omega]$$

are positive elments in $[\omega]M_n(M)[\omega]$. Recall that $[\omega] = \mathrm{diag}([\omega_1], \dots, [\omega_n])$.

The associated Hilbert space is denoted by $L^2(M)$ and the image of $a \otimes \varphi^{1/2} \otimes b$ in $L^2(M)$ by $a\varphi^{1/2}b$. Here the notation is compatible with the one for $L^2(M, \varphi)$ because

$$[\varphi]M[\varphi] \otimes \varphi^{1/2} \otimes [\varphi]M[\varphi] \ni a \otimes \varphi^{1/2} \otimes b \mapsto a\varphi^{1/2}b \in L^2(M, \varphi)$$

gives an isometric map by the very definition of inner products. Similar remarks are in order for left and right GNS spaces.

The left and right actions of M are compatible with taking quotients and they are bounded on $L^2(M)$: For $a \in M$,

$$\left\| \bigoplus_j ax_j \otimes \omega_j^{1/2} \otimes y_j \right\|^2 = (\omega^{1/2} | ZJYJ\omega^{1/2})$$

with

$$0 \leq Z = [\omega] \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix} a^*a \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} [\omega] \leq \|a\|^2 X.$$

Moreover, these actions give *-representations of M: $(a\xi|\eta) = (\xi|a^*\eta)$ and $(\xi a|\eta) = (\xi|\eta a^*)$ for $\xi, \eta \in L^2(M)$ and $a \in M$, which is immediate from the definition of inner product.

The *-operation on $L^2(M)$ is also compatible with the inner product:

$$\begin{aligned} \left\| \left(\bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \right)^* \right\|^2 &= \left\| \bigoplus_j y_j^* \otimes \omega_j^{1/2} \otimes x_j^* \right\|^2 = (Y\omega^{1/2} | \omega^{1/2} X) \\ &= ((\omega^{1/2}X)^* | (Y\omega^{1/2})^*) = (X\omega^{1/2} | \omega^{1/2} Y) \\ &= \left\| \bigoplus_j x_j \otimes \omega_j^{1/2} \otimes y_j \right\|^2. \end{aligned}$$

So far, we have constructed a *-bimodule $L^2(M)$ of M in such a way that $L^2(M, \varphi) \subset L^2(M)$ for each $\varphi \in M_*^+$ and the closed subspaces $\overline{M\varphi^{1/2}}$, $\overline{\varphi^{1/2}M}$ in $L^2(M)$ are naturally identified with the left and right GNS spaces of φ respectively. Moreover, for $\varphi, \psi \in M_*^+$, we have $[\varphi]\overline{M\psi^{1/2}} = \overline{\varphi^{1/2}M}[\psi]$ in $L^2(M)$, which is just a reflection of the fact that the same identification inside $L^2(M_n(M), \omega)$ is used in the definition of inner product.

Lemma 7.2. Given a countable family $\{a_j\}$ in M and a countable family $\{\varphi_j\}$ in M_*^+, let $\varphi \in M_*^+$ be defined by

$$\varphi = \sum_{j,k\geq 1} \frac{1}{2^{j+k}} \frac{a_j\varphi_k a_j^*}{\varphi_k(a_j^* a_j)}.$$

Then $[\varphi]a_j\varphi_k^{1/2} = a_j\varphi_k^{1/2}$ for $j,k\geq 1$.

Proof. In fact, $0 = \varphi(1-[\varphi]) = \sum_{j,k} \frac{1}{2^{j+k}} \|(1-[\varphi])a_j\varphi_k^{1/2}\|^2/\|a_j\varphi_k\|^2$. □

Proposition 7.3. Let $\varphi \in M_*^+$ and p be the central support[12] of φ. Then we have

$$\overline{\varphi^{1/2}M} = [\varphi]L^2(M), \quad L^2(M,\varphi) = [\varphi]L^2(M)[\varphi], \quad \overline{M\varphi^{1/2}M} = pL^2(M).$$

Proof. The first equality follows from

$$[\varphi]\sum_j a_j\omega_j^{1/2}b_j \in \sum_j [\varphi]\overline{M\omega_j^{1/2}}b_j = \sum \overline{\varphi^{1/2}M}[\omega_j]b_j \subset \overline{\varphi^{1/2}M}.$$

The second equality is a consequence of the first equality by

$$L^2(M,\varphi) = \overline{\varphi^{1/2}[\varphi]M[\varphi]} = \overline{\varphi^{1/2}M}[\varphi].$$

Let $\mathfrak{U}$ be the set of unitaries in M. Then $p = \bigvee_{u\in\mathfrak{U}} u[\varphi]u^*$ and

$$pL^2(M) = \overline{\sum_{u\in\mathfrak{U}} u[\varphi]u^*L^2(M)} = \overline{\sum_{u\in\mathfrak{U}} u[\varphi]L^2(M)} = \overline{\sum_{u\in\mathfrak{U}} u\overline{\varphi^{1/2}M}} \subset \overline{M\varphi^{1/2}M}.$$

□

Corollary 7.4. The algebraic sum $\sum_{\varphi\in M_*^+} M\varphi^{1/2}$ is dense in $L^2(M)$. Consequently the left and right representations of M on $L^2(M)$ are σ-weakly continuous.

Proof. By the previous lemma, $[\varphi] \nearrow 1$ as $\varphi \nearrow \infty$, which is used to have

$$L^2(M) = \lim_{\varphi\nearrow\infty} [\varphi]L^2(M) = \lim_{\varphi\nearrow\infty} \overline{\varphi^{1/2}M}.$$

□

Lemma 7.5. Let $\varphi, \psi \in M_*^+$ satisfy $\varphi \leq \psi$. Then there exists exactly one $a \in M$ satisfying $\varphi^{1/2} = a\psi^{1/2}$ and $a[\psi] = a$.

[12] p is the minimal projection in $M \cap M'$ satisfying $p\varphi = \varphi$.

Proof. If $c\psi^{1/2} = 0$ with $c[\psi] = c$, then $c^*c \in [\psi]M[\psi]$ and $\psi(c^*c) = 0$ imply $c^*c = 0$, showing the uniqueness of a. In particular, a satisfies $[\varphi]a = a$, whence $a \in [\varphi]M[\psi] \subset [\psi]M[\psi]$. Thus, replacing M with $[\psi]M[\psi]$, we may assume that ψ is faithful for the existence.

The map $\psi^{1/2}M \ni \psi^{1/2}x \mapsto \varphi^{1/2}x \in L^2(M)$ is contractive and it gives a bounded linear operator a on $L^2(M)$ by the density of $\psi^{1/2}M$ in $L^2(M)$. Clearly a commutes with the right action of M and therefore it belongs to M. □

Proposition 7.6. Let $\varphi = \sum_{n\geq 1} \varphi_n$ with $\varphi_n \in M_*^+$. Then

$$\overline{M\varphi^{1/2}} = \overline{\sum_{n\geq 1} M\varphi_n^{1/2}}.$$

Moreover $M(\varphi_1 + \cdots + \varphi_n)^{1/2}$ is increasing in $n \geq 1$ and their union is dense in $M\varphi^{1/2}$.

Proof. First note that there is a one-to-one correspondence between closed M-submodules of ${}_M L^2(M)$ and projections in M: Given a projection $e \in M$, $L(M)e$ is a closed submodule and any closed submodule is of this form. Consequently, $L^2(M)(\bigvee_{i\in I} e_i) = \overline{\sum_{i\in I} L^2(M)e_i}$.

Now $[\varphi] = \bigvee_{n\geq 1}[\varphi_n]$ shows that

$$\overline{\sum_{n\geq 1} M\varphi_n^{1/2}} = \overline{\sum_{n\geq 1} L^2(M)[\varphi_n]} = L^2(M)(\bigvee_{n\geq 1}[\varphi_n]) = L^2(M)[\varphi] = \overline{M\varphi^{1/2}}.$$

Finally, by the previous lemma, $\varphi_j^{1/2} = a_j(\varphi_1 + \cdots + \varphi_n)^{1/2}$ for some $a_j \in M$ $(1 \leq j \leq n)$ shows that

$$M\varphi_1^{1/2} \subset M(\varphi_1 + \varphi_2)^{1/2} \subset \cdots \subset M\varphi^{1/2}$$

and

$$\sum_{j=1}^{n} M\varphi_j^{1/2} = \sum_{j=1}^{n} Ma_j(\varphi_1 + \cdots + \varphi_n)^{1/2} \subset M(\varphi_1 + \cdots + \varphi_n)^{1/2},$$

which give the density in question. □

Corollary 7.7. Let $\varphi_1, \ldots, \varphi_n \in M_*^+$. Then

$$\overline{M(\varphi_1^{1/2} + \cdots + \varphi_n^{1/2})} = \overline{M\varphi_1^{1/2} + \cdots + M\varphi_n^{1/2}} = \overline{M(\varphi_1 + \cdots + \varphi_n)^{1/2}}.$$

Proof. We rely on L^4-calculation in the first equality: If $(\varphi_1^{1/2} + \cdots + \varphi_n^{1/2})e = 0$, $e\varphi_1^{1/2}e + \cdots + e\varphi_n^{1/2}e = 0$, which implies $e\varphi_j^{1/2}e = 0$ for $1 \leq j \leq n$. Thus $\varphi_j^{1/4}e = 0$ and hence $\varphi_j e = 0$. □

The projection e associated to $\overline{M(\varphi_1^{1/2} + \cdots + \varphi_n^{1/2})}$ satisfies $[\varphi_j]$ which supports

Lemma 7.8. Let $f \in M$ be a projection of the form $f = [\omega]$ for some $\omega \in M_*^+$. Then each $T \in \mathrm{End}({}_M L^2(M)f)$ is realized by the right multiplication of a uniquely determined element in fMf.

Proof. Let $T \in \mathcal{B}(L^2(M)f)$ commute with the left action of M. Let $\varphi \in M_*^+$ satisfy $[\varphi] \geq f$ and $T_\varphi \in \mathcal{B}([\varphi]L^2(M)[\varphi])$ be defined by

$$T_\varphi(\xi) = [\varphi]T(\xi f) \quad \text{for } \xi \in [\varphi]L^2(M)[\varphi].$$

Clearly T_φ commutes with the left action of $[\varphi]M[\varphi]$ and it is realized by the right multiplication of a uniquely determined element $a_\varphi \in [\varphi]M[\varphi]$. Since the range of T_φ is included in $L^2(M)f$ and T_φ vanishes on $[\varphi]L^2(M)([\varphi]-f)$, a_φ belongs to fMf. Now let $\psi \in M_*^+$ be another functional satisfying $[\psi] \geq f$. Since $\omega^{1/2} \in fL^2(M)f$ is separating for fMf, the equality

$$\omega^{1/2}a_\varphi = T\omega^{1/2} = \omega^{1/2}a_\psi$$

implies $a_\varphi = a_\psi$. Thus, writing a for the common a_φ,

$$[\varphi]T[\varphi]\xi = \xi a \quad \text{for } \xi \in L^2(M)f$$

and then, by taking the limit $\varphi \nearrow \infty$, we conclude that $T\xi = \xi a$ for $\xi \in L^2(M)f$. □

Theorem 7.9. The left and right actions of M on $L^2(M)$ give the commutants of each other.

Proof. Let T commute with the left action of M on $L^2(M)$. For $\varphi \in M_*^+$, let $[\varphi]' \in \mathcal{B}(L^2(M))$ be given by the right multiplication of $[\varphi] \in M$. Then $[\varphi]'T[\varphi]'$ on $L^2(M)[\varphi]$ is realized by $a_\varphi \in [\varphi]M[\varphi]$:

$$[\varphi]'T[\varphi]'\xi = \xi a_\varphi.$$

For $[\psi] \geq [\varphi]$ with $\psi \in M_*^+$, we see $a_\varphi = [\varphi]a_\psi[\varphi]$ by the uniqueness, whence $a \in M$ is well-defined by the relation $a_\varphi = [\varphi]a[\varphi]$ and we have

$$T\xi = \lim_{\varphi \nearrow \infty} \xi a_\varphi = \xi a$$

for $\xi \in L^2(M)$. □

Definition 7.10. Let $L^2_+(M)$ be the closed convex cone generated by the set $\{a\varphi^{1/2}a^*; a \in M, \varphi \in M_*^+\}$.

Lemma 7.11. Let $\varphi, \psi \in M_*^+$ and $a \in M$. Then

$$(\varphi^{1/2}|a\psi^{1/2}a^*) \geq 0.$$

Proof.

$$0 \le \left(\begin{pmatrix} \varphi^{1/2} & 0 \\ 0 & \psi^{1/2} \end{pmatrix} \middle| \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi^{1/2} & 0 \\ 0 & \psi^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a^* & 0 \end{pmatrix} \right) = (\varphi^{1/2}|a\psi^{1/2}a^*).$$

□

For $\xi, \eta \in L^2(M)$, let $\xi\eta \in M_*$ be defined by

$$\langle x, \xi\eta \rangle = (\eta^*|x\xi), \quad x \in M.$$

Clearly $\|\xi\eta\| \le \|\xi\|\,\|\eta\|$ and the bilinear map $\xi \times \eta \mapsto \xi\eta$ is compatibile with the *-bimodule sutructure of $L^2(M)$: $(\xi b)\eta = \xi(b\eta)$, $(a\xi)\eta = a(\xi\eta)$ and $(\xi\eta)^* = \eta^*\xi^*$.

Given $\xi \in L^2(M)$, its left (resp. right) support is the minimal projection e (resp. f) in M satisfying $(1-e)\xi = 0$ (resp. $\xi(1-f) = 0$). When $\xi^* = \xi$, these projections coincide and are denoted by $[\xi]$.

Theorem 7.12 (Polar Decomposition)**.** Each $\xi \in L^2(M)$ has exactly one expression of the form $\xi = v|\xi|$, where $|\xi| \in L^2_+(M)$ and $v \in M$ satisfies $v^*v = [|\xi|]$. Moreover the unique $|\xi|$ is equal to $(\xi^*\xi)^{1/2}$ with $\xi^*\xi \in M_*^+$.

Proof. Let e and f be the left and right supports of ξ. Clearly $\varphi \equiv \xi^*\xi \in M_*^+$ and

$$\varphi^{1/2}x \mapsto \xi x, \quad x \in M$$

defines an isometry of $fL^2(M)$ onto $eL^2(M)$, which commutes with the right action of M and gives rise to a partial isometry $v \in M$ satisfying $v^*v = e$, $vv^* = f$ and $\xi = v\varphi^{1/2}$.

Now assume that $\xi \in L^2_+(M)$ and we shall show that $v = e = f$.

First, from the invariance $\xi^* = \xi$, $e = f$. By replacing M with the reduced W*-algebra eMe, we may assume that φ is faithful and v is a unitary.

Then the densely defined conjugate-linear map $\varphi^{1/2}x \mapsto \xi x^* = v\varphi^{1/2}x^*$ has the closure $vF = v\Delta^{1/2}J$ with its adjoint given by $J\Delta^{1/2}v^*$. Note here that, by the unitarity of v, both of $v\Delta^{1/2}$ and $\Delta^{1/2}v^*$ are closed and adjoints of each other. Now, for $x \in M$,

$$\begin{aligned} v\Delta^{1/2}J(\varphi^{1/2}x) &= v(\varphi^{1/2}x^*) = \varphi^{1/2}v^*x^* = \Delta^{1/2}J(\varphi^{1/2}xv) \\ &= \Delta^{1/2}J(Jv^*J)(\varphi^{1/2}x) = \Delta^{1/2}v^*J(\varphi^{1/2}x), \end{aligned}$$

where we used $v\varphi^{1/2} = (v\varphi^{1/2})^* = \varphi^{1/2}v^*$ at the middle of the first line. Since $J(\varphi^{1/2}M) = M\varphi^{1/2}$ is a core for $v\Delta^{1/2}$ and $\Delta^{1/2}v^*$, we see that $v\Delta^{1/2} = \Delta^{1/2}v^*$ is self-adjoint.

Finally we take the positivity of $\xi \in L^2(M)_+$ into account to have

$$\begin{aligned}0 \leq (\xi|x\varphi^{1/2}x^*) &= (x^*\varphi^{1/2}v^*|\varphi^{1/2}x^*) = (JvJ(x^*\varphi^{1/2})|\varphi^{1/2}x^*)\\ &= (Jv\Delta^{1/2}(x\varphi^{1/2})|\varphi^{1/2}x^*) = (x\varphi^{1/2}|v\Delta^{1/2}(x\varphi^{1/2}))\end{aligned}$$

for $x \in M$. Since $M\varphi^{1/2}$ is a core for the self-adjoint operator $v\Delta^{1/2}$, this implies $v\Delta^{1/2} \geq 0$ and we conclude that $v = 1$ thanks to the uniqueness of polar decomposition. □

Corollary 7.13 (Jordan Decomposition)**.**

(i) Each $\xi \in L^2_+(M)$ is of the form $\varphi^{1/2}$ with a unique $\varphi \in M_*^+$.
(ii) Each $\xi = \xi^*$ in $L^2(M)$ has a unique decomposition $\xi = \xi_+ - \xi_-$, where $\xi_\pm \in L^2_+(M)$ satisfies $(\xi_+|\xi_-) = 0$.

Proof. Let $\xi = v|\xi|$ be the polar decomposition. Then $\xi = \xi^* = |\xi|v^* = v^*(v|\xi|v^*)$ gives another polar decomposition and the uniqueness implies $v = v^*$ and $v|\xi|v^* = |\xi|$. Let $v = p_+ - p_-$ be the spectral decomposition. Since $v|\xi| = |\xi|v$, $v^*|\xi| = |\xi|v^*$ and $p_\pm$ are weak* limits of polynomials of $v = v^*$, we see that $p_\pm$ commute with $|\xi|$ and $\xi_\pm = p_\pm|\xi| = |\xi|p_\pm$ give the decomposition.

Let $\xi = \eta_+ - \eta_-$ be another Jordan decomposition. Then $\xi_+ - \eta_+ = \xi_- - \eta_-$ and the uniqueness follows from

$$\|\xi_+ - \eta_+\|^2 = (\xi_+ - \eta_+|\xi_- - \eta_-) = -(\xi_+|\eta_-) - (\eta_+|\xi_-) \leq 0.$$

□

Example 7.14. Let θ be a *-automorphism or a *-antiautomorphism of a W*-algebra M. Then $\Theta\varphi^{1/2} = (\varphi\circ\theta)^{1/2}$ gives a unitary on $L^2(M)$. Conversely, given a unitary Θ on $L^2(M)$ which preserves $L^2_+(M)$,

Exercise 43. Let $\varphi \in M_*^+$ and $v \in M$ satisfy $v^*v = [\varphi]$. Then $(v\varphi v^*)^{1/2} = v\varphi^{1/2}v^*$.

Exercise 44. For $\xi, \eta \in L^2_+(M)$, the following conditions are equivalent. (i) $(\xi|\eta) = 0$. (ii) $[\xi][\eta] = 0$. (iii) $\xi\eta = 0$. (Hint: the construction and the uniqueness of the Jordan decomposition of $\xi - \eta$.)

As applications of this basic fact, we record here two further results.

Theorem 7.15 (Powers-Störmer-Araki)**.** For $\varphi, \psi \in M_*^+$,

$$\|\varphi^{1/2} - \psi^{1/2}\|^2 \leq \|\varphi - \psi\| \leq \|\varphi^{1/2} + \psi^{1/2}\| \, \|\varphi^{1/2} - \psi^{1/2}\|.$$

Proof. We first remark that

$$\varphi(a) - \psi(a) = \frac{1}{2}\Big((\varphi^{1/2}+\psi^{1/2}|a(\varphi^{1/2}-\psi^{1/2})) + (\varphi^{1/2}-\psi^{1/2}|a(\varphi^{1/2}+\psi^{1/2}))\Big),$$

from which the second inequality follows.

Let $\varphi^{1/2} - \psi^{1/2} = \xi - \eta$ ($\xi, \eta \in L^2_+(M)$, $(\xi|\eta) = 0$) be a Jordan decomposition. Then, for the choice $a = [\xi] - [\eta]$,

$$\begin{aligned}
\|\varphi - \psi\| \geq \varphi(a) - \psi(a) &= \mathrm{Re}(\varphi^{1/2} - \psi^{1/2}|a(\varphi^{1/2} + \psi^{1/2})) \\
&= \mathrm{Re}(\xi - \eta|a(\varphi^{1/2} + \psi^{1/2})) \\
&= (\xi + \eta|\varphi^{1/2} + \psi^{1/2}) \\
&\geq (\xi - \eta|\varphi^{1/2}) - (\xi - \eta|\psi^{1/2}) \\
&= \|\varphi^{1/2} - \psi^{1/2}\|^2.
\end{aligned}$$

□

Theorem 7.16. $L^2_+(M)$ is a self-dual cone in the sense that

$$L^2_+(M) = \{\xi \in L^2(M); (\xi|\eta) \geq 0 \ \forall \eta \in L^2_+(M)\}.$$

Proof. Assume that $\zeta \in L^2(M)$ is evaluated with elements in $L^2_+(M)$ to have positive reals. Then, in terms of the four sum decomposition of $\zeta \in L^2(M)$, $\zeta = \xi_+ - \xi_- + i(\eta_+ - \eta_-)$, $0 = \mathrm{Im}(\eta_\pm|\zeta) = \pm\|\eta_\pm\|^2$ implies $\eta_\pm = 0$ and then $0 \leq (\xi_-|\zeta) = -\|\xi_-\|^2$ shows that $\xi_- = 0$. □

7.1. **More on reduced subspaces.** Let $L^2_+(M, \varphi)$ be the closure of $\{x\varphi^{1/2}x^*; x \in [\varphi]M[\varphi]\}$ in $L^2(M, \varphi)$.

Lemma 7.17. Let $\varphi \in M_*^+$ be faithful. Then we have $\overline{\Delta^{1/4}(M_+\varphi^{1/2})} = L^2_+(M, \varphi) = \overline{\Delta^{-1/4}(M'_+\varphi^{1/2})}$.

Proof. Let $a \in \mathcal{M}$ (= the set of entirely analytic elements). Then the relation $\Delta^{it}(aa^*\varphi^{1/2}) = \sigma_t(a)\sigma_t(a)^*\varphi^{1/2}$ is analytically continued to

$$\begin{aligned}
\Delta^{1/4}(aa^*\varphi^{1/2}) = \sigma_{-i/4}(a)(\sigma_{i/4}(a))^*\varphi^{1/2} &= \sigma_{-i/4}(a)J\Delta^{1/2}(\sigma_{i/4}(a)\varphi^{1/2}) \\
&= \sigma_{-i/4}(a)J(\sigma_{-i/4}(a)\varphi^{1/2}) = \sigma_{-i/4}(a)\varphi^{1/2}\sigma_{-i/4}(a)^*.
\end{aligned}$$

Since $\sigma_{-is}(\mathcal{M}) = \mathcal{M}$ is σ-weakly dense in M, the Kaplansky density theorem 4.23 shows that each $b \in M$ is boundedly approximated by elements in $\mathcal{M}$ in the strong operator topology and we see that $L^2_+(M, \varphi) \subset \overline{\Delta^{1/4}M_+\varphi^{1/2}}$.

Conversely, the Kaplansky's density theorem is again used to approximate $a \in M_+$ in the strong operator topology by a sequence $a_n = b_nb_n^*$ with $b_n \in \mathcal{M}$. Since $J\Delta^{1/2}(a_n\varphi^{1/2}) = a_n\varphi^{1/2} \to a\varphi^{1/2} = J\Delta^{1/2}(a\varphi^{1/2})$,

$$\|\Delta^{1/4}((a_n - a)\varphi^{1/2})\|^2 = ((a_n - a)\varphi^{1/2}|\Delta^{1/2}(a_n - a)\varphi^{1/2}) \to 0.$$

Recall that $M\varphi^{1/2} \subset D(\Delta^{1/2}) \subset D(\Delta^{1/4})$. Thus $\Delta^{1/4}(a\varphi^{1/2})$ is approximated by $\Delta^{1/4}(b_nb_n^*\varphi^{1/2}) = \sigma_{-i/4}(b_n)\varphi^{1/2}\sigma_{-i/4}(b_n)^* \in L^2_+(M, \varphi)$. □

Lemma 7.18. Let $\varphi_j \in M_*^+$ be faithful for $j = 1, 2$. Then $L^2_+(M, \varphi_1) = L^2_+(M, \varphi_2)$.

Proof. Let $\begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}$ be the set of entirely analytic elements of the modular automorphism group σ_t associated to a faithful $\varphi = \mathrm{diag}(\varphi_1, \varphi_2)$ on $M_2(M)$: If we introduce σ-weakly continuous one-parameter groups $\{\sigma_t^{j,k}\}$ of isometries on M by

$$\sigma_t \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \sigma_t^{1,1}(a_{11}) & \sigma_t^{1,2}(a_{12}) \\ \sigma_t^{2,1}(a_{21}) & \sigma_t^{2,2}(a_{22}) \end{pmatrix},$$

$\mathcal{M}_{j,k}$ is the set of entirely analytic elements of M for $\sigma_t^{j,k}$. The modular operator Δ is also split into four parts $\Delta_{j,k}$ so that

$$\Delta = \begin{pmatrix} \Delta_{1,2} & \Delta_{1,2} \\ \Delta_{2,1} & \Delta_{2,2} \end{pmatrix},$$

where positive self-adjoint operators $\Delta_{j,k}$ on $L^2(M)$ are specified by $\Delta_{j,k}^{1/2}(x\varphi_k^{1/2}) = \varphi_j^{1/2}x$ $(x \in M)$.

Let $a \in M_2(M)$ with $a_{j,k} \in \mathcal{M}_{j,k}$. As in the proof of the previous lemma, we have the relation $\Delta^{1/4}(aa^*\varphi^{1/2}) = \sigma_{-i/4}(a)\varphi^{1/2}\sigma_{-i/4}(a)^*$ and then, for the choice $a = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$,

$$\Delta_{1,1}^{1/4}(xx^*\varphi_1^{1/2}) = \sigma_{-i/4}^{1,2}(x)\varphi_2^{1/2}\sigma_{-i/4}^{1,2}(x)^*.$$

Since $\begin{pmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} \end{pmatrix}$ is σ-weakly dense in $M_2(M)$, the Kaplansky density theorem 4.23 shows that each $y \in M$ is boundedly approximated by elements in $\sigma_{-i/4}^{1,2}(\mathcal{M}_{1,2}) = \mathcal{M}_{1,2}$ in the *strong topology and we see that $y\varphi_2^{1/2}y^*$ is in the weak closure of $\Delta_{1,1}^{1/4}(M_+\varphi_1^{1/2}) \subset L^2_+(M, \varphi_1)$. Since $L^2_+(M, \varphi_1)$ is a convex set, this means that $y\varphi_2^{1/2}y^* \in L^2_+(M, \varphi_1)$. □

Recall that $L^2_+(M, \varphi)$ is the norm closure of $\{a\varphi^{1/2}a^*; a \in [\varphi]M[\varphi]$ in $L^2(M)$.

Theorem 7.19.

(i) We have $L^2_+(M, \varphi) = [\varphi]L^2_+(M)[\varphi]$ and the unitary Δ^{it} on $[\varphi]L^2(M)[\varphi]$ leaves $L^2_+(M, \varphi)$ invariant globally.

(ii) Let p be the central support of $\varphi \in M_*^+$. Then the closed convex hull C of $\{a\varphi^{1/2}a^*; a \in M\}$ in $L^2(M)$ is equal to $pL^2_+(M) = L^2_+(M)p$.

Proof. (i) Let $\omega^{1/2} \in [\varphi]L^2_+(M)[\varphi]$. Then $\omega + \epsilon\varphi$ with $\epsilon > 0$ is faithful on $[\varphi]M[\varphi]$ and we see $(\omega + \epsilon\varphi)^{1/2} \in L^2_+(M, \omega + \epsilon\varphi) = L^2_+(M, \varphi)$, whence $\omega^{1/2} = \lim_{\epsilon\to+0}(\omega + \epsilon\varphi)^{1/2} \in L^2_+(M, \varphi)$.

(ii) Let $\omega \in pM_*^+$. Since $p = \bigvee_{u\in\mathcal{U}} u[\varphi]u^*$, we can find a sequence $\{u_n\}$ of unitaries in M such that $[\omega] = \bigvee_{n\geq 1} u_n[\varphi]u_n^*$. Then $[\omega] = [\psi]$ with $\psi^{1/2} = \sum_{n\geq 1} 2^{-n}u_n\varphi^{1/2}u_n^*$ implies $\omega^{1/2} \in L^2_+(M, \psi) \subset C$. □

Corollary 7.20. $L^2_+(M, \varphi)$ is a self-dual cone in $L^2(M, \varphi)$.

Exercise 45. Identify the standard Hilbert space of $M = \mathcal{B}(\mathcal{H})$ with $\mathcal{H} \otimes \mathcal{H}^*$ and describe its positive cone.

8. Universal Representations

Let A be a C*-algebra and we shall construct a *-bimodule $L^2(A)$ of A in such a way that it generalizes the commutative case discussed in §4.

Consider the Hilbert space direct sum $\mathcal{U} = \bigoplus_{\varphi\in A^*_+} \overline{A\varphi^{1/2}}$ of left GNS spaces on which A is represented by left multiplication. Thanks to the Gelfand-Naimark theorem, the representation is faithful and we regard A as a C*-subalgebra of $\mathcal{B}(\mathcal{U})$. Let $M = A''$ be the W*-algebra on $\mathcal{U}$ generated by A. Since A is a weakly dense *-subalgebra of M, M_* is identified with a subspace of A^* by restriction. Note that $A^*_+ \subset M_*^+$ by the way of our construction. Since elements in A^* are linear combinations of positive functionals (Theorem 3.15), the equality $M_* = A^*$ holds as a linear space.

We claim that $M_* = A^*$ as a Banach space. This follows from the Kaplansky density theorem 4.23: Let $\phi \in M_*$, $a \in M$ and choose a net $\{a_\imath\} \subset A$ so that $\|a_\imath\| \leq \|a\|$ and $a = \lim_{\imath\to\infty} a_\imath$ in the σ-weak topology of M. Then

$$|\phi(a)| = \lim_{\imath\to\infty} |\phi(a_\imath)| \leq \limsup_{\imath\to\infty} \|\phi\|\|a_\imath\| \leq \|\phi\|\|a\|$$

shows that the norm of ϕ in M_* is equal to that in A^*. Thus M is identified with the second dual A^{**} of A.

Now we set $L^2(A) = L^2(M)$. For an index set I, let

$${}_AL^2(A)^{\oplus I} = \bigoplus_{i\in I} {}_AL^2(A) = {}_AL^2(A) \otimes \ell^2(I).$$

The opposite algebra of $\mathrm{End}({}_AL^2(A)^{\oplus I})$ is naturally identified with the matrix ampliation $M_I(M)$ of M, which acts on $L^2(A)^{\oplus I}$ from the right. Any *-representation ${}_A\mathcal{H}$ of A is unitarily equivalent to ${}_AL^2(A)^{\oplus I}e$, where e is a projection in $M_I(M)$ and the cardinality of I is specified by the existence of an orthogonal decomposition of the form $\mathcal{H} = \bigoplus_{i\in I} \overline{A\xi_i}$ (see Theorem 5.9).

Denote the associated *-representation of A by π_e and let e' be the projection onto the subspace $L^2(A)^{\oplus I}e \subset L^2(A)^{\oplus I}$ with p the central support of e'. Note that p belongs to the center of $M_I(M) = M \otimes \mathcal{B}(\ell^2(I))$, i.e., $p \in (M \cap M') \otimes 1 \cong M \cap M'$. By this natural identification, p is realized by the right multiplication of $\bigvee_{u\in U_I(M)} ueu^*$ as an element in $(M \cap M') \otimes 1$ and by the left multiplication as an element in $M \cap M'$.

We claim that $\pi_e(A)'' = Me'$ is isomorphic to Mp. In fact, for $x \in M$, $xe' = 0$, i.e., $xL^2(A)^{\oplus I}e = 0$ if and only if $x\sum_{u\in U_I(M)} L^2(A)^{\oplus I}eu = 0$, i.e., $xp = 0$.

Let $f \in M_I(M)$ be another projection with π_f denoting the associated *-representation of A. Then π_e and π_f are quasi-equivalent if and only if e and f have the same central support in $M_I(M)$. From

$$\mathrm{Hom}({}_AL^2(A)^{\oplus I}e, {}_AL^2(A)^{\oplus I}f) = fM_I(M)e,$$

π_e and π_f are disjoint if and only if the central supports of e and f are orthogonal.

In other words, if we denote the central support of ${}_A\mathcal{H}$ by $[{}_A\mathcal{H}] \in A^{**}$, then ${}_A\mathcal{H}$ and ${}_A\mathcal{K}$ are disjoint (resp. quasi-equivalent) if and only if $[{}_A\mathcal{H}][{}_A\mathcal{K}] = 0$ (resp. $[{}_A\mathcal{H}] = [{}_A\mathcal{K}]$).

Theorem 8.1. Let φ and ψ be positive functionals on a C*-algebra A.

(i) φ and ψ are disjoint if and only if $A\varphi^{1/2}A$ and $A\psi^{1/2}A$ are orthogonal. When $\overline{A\varphi^{1/2}} = \overline{\varphi^{1/2}A}$, this is further equivalent to $(\varphi^{1/2}|\psi^{1/2}) = 0$.

(ii) φ and ψ are quasi-equivalent if and only if $\overline{A\varphi^{1/2}A} = \overline{A\psi^{1/2}A}$.

(iii) φ is pure if and only if $\overline{A\varphi^{1/2}} \cap \overline{\varphi^{1/2}A} = \mathbb{C}\varphi^{1/2}$.

Proof. (ii) and the first statement of (i) follow from $[{}_A\overline{A\varphi^{1/2}}]L^2(A) = \overline{A\varphi^{1/2}A}$. From the identity $(\varphi^{1/2}|\psi^{1/2}) = \|\varphi^{1/4}\psi^{1/4}\|^2$, the vanishing of transition probability is equivalent to $\varphi^{1/2}\psi^{1/2} = 0$, i.e., the orthogonality of $A\varphi^{1/2}$ and $A\psi^{1/2}$. When $\overline{A\varphi^{1/2}} = \overline{\varphi^{1/2}A}$, this implies

$L^4(M)$ calculus here!

$$(A\varphi^{1/2}A|A\psi^{1/2}A) \subset (\varphi^{1/2}A|A\psi^{1/2}) \subset (\overline{A\varphi^{1/2}}|A\psi^{1/2}) = \{0\}.$$

Let e be the support of φ in A^{**}. Then the identity

$$\overline{A\varphi^{1/2}} \cap \overline{\varphi^{1/2}A} = L^2(A^{**})e \cap eL^2(A^{**}) = L^2(eA^{**}e)$$

shows that the condition in (iii) is equivalent to $eA^{**}e = \mathbb{C}e$, i.e., the purity of φ. Note that $\mathrm{End}({}_AL^2(A)e) \cong eA^{**}e$. ∎

Given a *-representation π of A on a Hilbert space $\mathcal{H}$, let

$$A^*_\pi = \{\phi \circ \pi; \phi \in \pi(A)''_*\},$$

which is an A-biinvariant closed subspace of A^* and the Hilbert space $L^2_\pi(A) = \overline{\sum_{0\leq\varphi\in A^*_\pi} A\varphi^{1/2}A}$ is naturally isomorphic to $L^2(\pi(A)'')$.

Now let A be commutative and choose a family $\{\varphi_i\}_{i\in I}$ of states in A^*_π so that they have mutually disjoint supports and $L^2_\pi(A) = \bigoplus_{i\in I} \overline{A\varphi_i^{1/2}}$ with $A\varphi_i^{1/2} = \varphi_i^{1/2}A$. Then the Radon measure μ_i on $\Omega = \sigma_A$ associated with φ_i have mutually disjoint supports Ω_i and, if we define a measure μ on Ω by $\mu|_{\Omega_i} = \mu_i$, then $L^2_\pi(A) = L^2(\Omega,\mu)$ on which $\pi(A)''$ is identified with $L^\infty(\Omega,\mu)$. Note that μ is σ-finite if and only if I is a countable set.

Example 8.2. Let $A = C(\Omega)$ be commutative with expressions

$$\varphi(a) = \int_\Omega a(\omega)\,\mu(d\omega), \quad \psi(a) = \int_\Omega a(\omega)\,\nu(d\omega).$$

(i) φ and ψ are disjoint if and only if we can find Borel subsets Ω_μ and Ω_ν such that $\Omega_\mu \cap \Omega_\nu = \emptyset$ and $\mu(\Omega\setminus\Omega_\mu) = 0 = \nu(\Omega\setminus\Omega_\nu)$,

In fact, in the expression

$$(\varphi^{1/2}|\psi^{1/2}) = \int_\Omega \sqrt{\frac{d\mu}{d(\mu+\nu)}(\omega)\frac{d\nu}{d(\mu+\nu)}(\omega)}(\mu+\nu)(d\omega),$$

let Ω_μ and Ω_ν be the (Borel) supports of $d\mu/d(\mu+\nu)$ and $d\nu/d(\mu+\nu)$ respectively. If $(\varphi^{1/2}|\psi^{1/2}) = 0$, Ω_μ and Ω_ν can be chosen disjoint by adjustment up to $(\mu+\nu)$-negligible sets.

(ii) $\varphi^{1/2} \in \overline{A\psi^{1/2}}$ if and only if μ is absolutely continuous relative to ν. Consequently, φ and ψ are quasi-equivalent if and only if μ and ν are equivalent measures.

In fact, if $\mu \prec \nu$, $\mu(d\omega) = f(\omega)\nu(d\omega)$ with $0 \leq f \in L^1(\Omega,\nu)$ (Radon-Nykodym theorem), whence $\varphi^{1/2} = \sqrt{f(\omega)}\sqrt{\nu(d\omega)} \in L^2(\Omega,\nu) = \overline{A\psi^{1/2}}$. Conversely, unless $\mu \prec \nu$, $\mu(\Omega\setminus\Omega_\nu) > 0$ and, if $\mu^{1/2} \in L^2(\Omega,\nu)$, i.e., $\sqrt{\mu(d\omega)} = g(\omega)\sqrt{\nu(d\omega)}$ with $g \in L^2(\Omega,\nu)$, then

$$0 < \int_{\Omega\setminus\Omega_\nu} \mu(d\omega) = \int_{\Omega\setminus\Omega_\nu} |g(\omega)|^2\,\nu(d\omega) = 0,$$

a contradiction.

Exercise 46. There is a one-to-one correspondence between closed A-subbimodules in $L^2(A)$ and closed A-subbimodules in A^*.

Let $\{\tau_t\}_{t\in\mathbb{R}}$ be a one-parameter group of *-automorphisms of a C*-algebra A, which is continuous in the sense that $\mathbb{R} \ni t \mapsto \tau_t(a)$ is norm-continuous for any $a \in A$. When A is a W*-algebra, we assume the weak* continuity in $\mathbb{R} \ni t \mapsto \tau_t(a)$.

A state ω of a C*-algebra A (or a normal state of a W*-algebra) is called a τ-KMS state if it satisfies the **KMS condition**[13] with respect to τ: Given $x, y \in A$, the function $\mathbb{R} \ni t \mapsto \omega(x\tau_t(y))$ is analytically extended to a continuous function on the strip $\{\zeta \in \mathbb{C}; -1 \leq \operatorname{Im}\zeta \leq 0\}$ so that $\omega(x\tau_t(y))|_{t=-i} = \omega(yx)$.

Proposition 8.3.

(i) A τ-KMS state is τ-invariant.

(ii) Let $\mathcal{A}$ be the set of entirely analytic elements in A for τ. Then a state ω on A satisfies the KMS condition if and only if $\omega(x\tau_{-i}(y)) = \omega(yx)$ for $x, y \in \mathcal{A}$.

(iii) For $x, y \in A$, the analytic extension $f(z)$ of $\omega(x\tau_t(y))$ in the KMS condition is estimated by

$$|f(t - is)| \leq (\|\omega^{1/2}x\|\|y\omega^{1/2}\|)^{1-s}(\|\omega^{1/2}y\|\|x\omega^{1/2}\|)^s.$$

Proof. If ω is a KMS state, then the KMS condition for $x, y \in \mathcal{A}$ is reduced to $\omega(x\tau_{-i}(y)) = \omega(yx)$.

Conversely, assume that $\omega(x\tau_{-i}(y)) = \omega(yx)$ for $x, y \in \mathcal{A}$. Since $\omega^{1/2}$ is approximated in norm by $x\omega^{1/2}$ with $x = x^* \in \mathcal{A}$, this implies $\omega(\tau_{-i}(y)) = \omega(y)$ for any $y \in \mathcal{A}$ and the entire function $\omega(\tau_z(y))$ is periodic of period $-i$. Thus it is bounded with a bound $\sup\{\|\tau_{-is}(y)\|; 0 \leq s \leq 1\}$ and must be a constant function, i.e., $\omega(\tau_t(y)) = \omega(y)$ for $y \in \mathcal{A}$ and $t \in \mathbb{R}$. Since $\mathcal{A}$ is dense in A, we see that ω is τ-invariant and one-parameter unitaries $v(t)$ are defined on $\overline{A\varphi^{1/2}}$ by $v(t)(a\varphi^{1/2}) = \tau_t(a)\omega^{1/2}$.

The three line theorem is now applied to a bounded analytic function $\omega(x\tau_z(y))$ on $-1 \leq \operatorname{Im} z \leq 0$ with a bound $\sup\limits_{0\leq s\leq 1}\{\|x\|\|\tau_{-is}(y)\|\}$ to get

$$\begin{aligned}|\omega(x\tau_{t-is}(y))| &\leq (\|\omega^{1/2}x\|\|y\omega^{1/2}\|)^{1-s}(\|\omega^{1/2}y\|\|x\omega^{1/2}\|)^s \\ &\leq \|\omega^{1/2}x\|\|y\omega^{1/2}\| \vee \|\omega^{1/2}y\|\|x\omega^{1/2}\|\end{aligned}$$

for $t \in \mathbb{R}$ and $0 \leq s \leq 1$. Notice here that

$$\omega(x\tau_t(y)) = (x^*\omega^{1/2}|v(t)(y\omega^{1/2})), \quad \omega(\tau_t(y)x) = (v(t)(y^*\omega^{1/2})|x\omega^{1/2}).$$

Thus, if sequences $\{x_n\omega^{1/2}\}$, $\{\omega^{1/2}x_n\}$, $\{y_n\omega^{1/2}\}$ and $\{\omega^{1/2}y_n\}$ with $x_n, y_n \in \mathcal{A}$ are convergent in $L^2(A)$, the sequence of analytic functions

[13] recognized by R. Kubo (1957), P.C. Martin and J. Schwinger (1959) as a characterisitc property of thermally equilibrium states and formulated in the operator algebraic setting by R. Haag, N.M. Hugenholtz and M. Winnink (1967).

$\{\omega(x_n\tau_z(y_n))\}_n$ is uniformly convergent on the region $-1 \leq \operatorname{Im} z \leq 0$ to an analytic function $f(z)$. In view of strong* density of $\mathcal{A}$ in A when these are represented on $\overline{A\omega^{1/2}}$, we can let sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{A}$ converge to arbitrary x and y in A so that the boundary values of the analytic function $f(z)$ are

$$f(t) = \lim(x_n^*\omega^{1/2}|v(t)(y_n\omega^{1/2})) = (x^*\omega^{1/2}|v(t)(y\omega^{1/2})) = \omega(x\tau_t(y)),$$
$$f(t-i) = \lim(v(t)(y_n^*\omega^{1/2})|x_n\omega^{1/2}) = (v(t)y^*\omega^{1/2}|x\omega^{1/2}) = \omega(\tau_t(y)x)$$

with the whole $|f(t-is)|$ estimated by

$$\begin{aligned}\lim|\omega(x_n\tau_{t-is}(y_n)| &\leq \lim(\|\omega^{1/2}x\|\|y\omega^{1/2}\|)^{1-s}(\|\omega^{1/2}y\|\|x\omega^{1/2}\|)^s \\ &\leq (\|\omega^{1/2}x\|\|y\omega^{1/2}\|)^{1-s}(\|\omega^{1/2}y\|\|x\omega^{1/2}\|)^s.\end{aligned}$$

□

Lemma 8.4. If ω satisfies the KMS condition, then $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$.

Proof. We argue as in BR2 Corollary 5.3.9:

By the invariance of ω, a unitary operator $v(t)$ on $\overline{A\omega^{1/2}}$ is defined by $v(t)(x\omega^{1/2}) = \tau_t(x)\omega^{1/2}$, which is continuous in t from the continuity of τ_t, and the estimate in the previous proposition ensures that, for $x, y \in A^{**}$, the function $\mathbb{R} \ni t \mapsto (x^*\omega^{1/2}|v(t)(y\omega^{1/2}))$ is analytically continued to the strip $\{-1 \leq \operatorname{Im}\zeta \leq 0\}$ so that it satisfies the KMS condition:

$$(x\omega^{1/2}|v(t)(y\omega^{1/2}))|_{t=-i} = (\omega^{1/2}x|\omega^{1/2}y) \quad \text{for } x, y \in A^{**}.$$

Let p be the central support of ω in A^{**} and assume that $a \in pA^{**}$ satisfies $a\omega^{1/2} = 0$. Then, $xa\omega^{1/2} = 0$ for $x \in A^{**}$ and therefore $(\omega^{1/2}(xa)|\omega^{1/2}y) = 0$ for any $y \in A^{**}$ by analytic continuation, whence $\omega^{1/2}xa = 0$. Thus $L^2(A^{**})a = L^2(A^{**})pa = \overline{A\omega^{1/2}A}a = 0$ and we have $a = 0$.

In this way, we have proved that $\omega^{1/2}$ is separating for pA^{**}, which finally implies

$$\overline{A\omega^{1/2}} = \overline{A^{**}\omega^{1/2}} = L^2(A^{**}, \omega) = \overline{\omega^{1/2}A^{**}} = \overline{\omega^{1/2}A}.$$

□

Conversely suppose that $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$. Then the support projection $[\omega] \in A^{**}$ of ω is central and the accompanied modular automorphism group $\{\sigma_t\}$ on $M = [\omega]A^{**}$ satisfies the KMS condition: Let $\mathcal{M}$ be the dense *-subalgebra of entirely analytic elements for $\{\sigma_t\}$. Then for $a, b \in \mathcal{M}$

$$\begin{aligned}\omega(a\sigma_{-i}(b)) &= (a^*\omega^{1/2}|\Delta(b\omega^{1/2})) \\ &= (J\Delta^{1/2}(b\omega^{1/2})|J\Delta^{1/2}(a^*\omega^{1/2})) \\ &= (b^*\omega^{1/2}|a\omega^{1/2}) = \omega(ba).\end{aligned}$$

Thus by the previous proposition ω is a normal σ-KMS state on M.

Let $\{u(t)\}$ and $\{v(t)\}$ be one-parameter groups of unitaries on $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$ with the induced automorphism groups of M satisfying the KMS condition for ω.

Let $x \in M$ be entirely analytic for σ_t and $y \in M$ be entirely analytic for τ_t. Then $x\omega^{1/2} \in D(u(-i))$, $y\omega^{1/2} \in D(v(-i))$ and we have

$$(u(-i)(x\omega^{1/2})|y\omega^{1/2}) = (\omega^{1/2}x|\omega^{1/2}y) = (x\omega^{1/2}|v(-i)(y\omega^{1/2})).$$

Since $\{x\omega^{1/2}\}$ and $\{y\omega^{1/2}\}$ are cores for $u(-i)$ and $v(-i)$ respectively by Example A.9, we see that $u(-i)$ and $v(-i)$ are adjoints of each other; $u(-i) = v(-i)$, whence $u(t) = v(t)$ for $t \in \mathbb{R}$.

Theorem 8.5 (Takesaki)**.** If a state ω of a C*-algebra A satisfies $\overline{A\omega^{1/2}} = \overline{\omega^{1/2}A}$, then the support projection $[\omega] \in A^{**}$ of ω is central and ω meets the KMS condition for the modular automorpshim group $\{\sigma_t\}$ of $[\omega]A^{**}$. Moreover, the modular automorphism group for ω is characterized as the one making ω a normal KMS state of $[\omega]A^{**}$.

Returning to the case of C*-algebras, given an automorphic action τ_t and a τ-invariant state ω of a C*-algebra A, let $u(t)$ be the associated one-parameter unitary group on the left GNS space $\overline{A\omega^{1/2}}$. Then, for $a \in A$, $\omega(a^*\tau_t(a)) = (a\omega^{1/2}|u(t)a\omega^{1/2})$ is a positive definition function of $t \in \mathbb{R}$ and we have a positive finite measure μ on $\mathbb{R}$ satisfying $\omega(a^*\tau_t(a)) = \int_{\mathbb{R}} e^{it\tau}\mu(d\tau)$. Likewise, $\omega(\tau_t(a)a^*)$ is the Fourier transform of another positive measure ν.

Let ω satisfy the KMS-condition. For $f \in C_c(\mathbb{R})$, its Fourier transform

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e^{-ist}f(s)\,ds$$

is extended to an entirely holomorphic function which decays rapidly in the real direction with local uniformilty on the imaginary direction.

Then $\widehat{f}(t)\omega(a^*\tau_t(a))$ has an analytic extension to a bounded continuous function on the domain $i[-1,0] + \mathbb{R}$ which decays rapidly in the real direction with uniformity on the imaginary direction. Thanks to the Cauchy integral theorem, we have

$$\int_{-\infty}^{\infty} \widehat{f}(t)\omega(a^*\tau_t(a))\,dt = \int_{-\infty}^{\infty} \widehat{f}(t-i)\omega(\tau_t(a)a^*)\,dt,$$

which is further rephrased by equalities

$$\int_{-\infty}^{\infty} \widehat{f}(t)\omega(a^*\tau_t(a))\,dt = 2\pi \int_{\mathbb{R}} f(\tau)\mu(d\tau),$$
$$\int_{-\infty}^{\infty} \widehat{f}(t-i)\varphi(\tau_t(a)a^*)\,dt = 2\pi \int_{\mathbb{R}} f(\tau)e^{-\tau}\nu(d\tau)$$

to get $\mu(d\tau) = e^{-\tau}\nu(d\tau)$.

Conversely, assume that positve measures associated to correlational positive definite functions are mutually related in this way. Consider the expression

$$F(z) = \int_{\mathbb{R}} e^{i\tau z}\mu(d\tau),$$

which is a well-defined and continuous function of $z = t-ir \in i[-1,0]+\mathbb{R}$ in view of $|e^{i\tau z}| = e^{r\tau} \leq 1 \vee e^{\tau}$ $(\tau \in \mathbb{R})$ and

$$\int_{\mathbb{R}} (1 \vee e^{\tau})\,\mu(d\tau) = \int_{(-\infty,0)} \mu(d\tau) + \int_{[0,\infty)} e^{\tau}\,\mu(d\tau) < \infty.$$

Moreover, it is analytic by

$$\oint_C F(z)\,dz = \int_{\mathbb{R}} \left(\oint_C e^{i\tau z}\,dz \right) \mu(d\tau) = 0$$

with the boundary values

$$F(t) = \omega(a^*\tau_t(a)), \quad F(t-i) = \omega(\tau_t(a)a^*).$$

The KMS-condition now frollows from the polarization identity.

Lemma 8.6 (Araki)**.** A τ-invariant state ω on A satisfies the KMS-condition if and only if, for each $a \in A$, the positive measure μ on $\mathbb{R}$ defined by $\omega(a^*\tau_t(a)) = \displaystyle\int_{\mathbb{R}} e^{it\tau}\,\mu(d\tau)$ satisfies $\displaystyle\int_{\mathbb{R}} e^{\tau}\,\mu(\tau) < \infty$ and $\omega(\tau_t(a)a^*) = \displaystyle\int_{\mathbb{R}} e^{it\tau+\tau}\,\mu(d\tau)$.

9. Modular Algebras

Matrix ampliation technique is simple but very useful as already witnessed in §7.1 and we record here relations concerning modular automorphisms of a W*-algebra M. Recall that, for $\phi, \varphi \in M_*^+$, $\{\sigma_t^{\phi,\varphi}\}$ is a one-parameter group of isometries on $[\phi]M[\varphi]$: $\sigma_s^{\phi,\varphi}\sigma_t^{\phi,\varphi} = \sigma_{s+t}^{\phi,\varphi}$ and $\sigma_{-t}^{\phi,\varphi} = (\sigma_t^{\phi,\varphi})^{-1}$ for $s,t \in \mathbb{R}$.

Lemma 9.1. Let ϕ, φ and ψ in M_*^+. For $a \in [\phi]M[\varphi]$, $b \in [\varphi]M[\psi]$ and $t \in \mathbb{R}$, we have the following relations.

(i) $\varphi_t^{\phi,\varphi}(a)^* = \sigma_t^{\varphi,\phi}(a^*)$.

(ii) $\sigma_t^{\phi,\varphi}(a)\sigma_t^{\varphi,\psi}(b) = \sigma_t^{\phi,\psi}(ab)$.

Proof. Let e_n $(n = 2, 3)$ be a diagonal matrice in $M_n(M)$ of diagonal entries $[\phi], [\varphi]$ for $n = 2$ or $[\phi], [\varphi], [\psi]$ for $n = 3$, with σ_t denoting the modular automorphim group of $e_n M_n(M) e_n$ associated the faithful diagonal functional $\phi \oplus \varphi$ or $\phi \oplus \varphi \oplus \psi$.

To get (i) and (ii), just apply σ_t on $e_2 M_2(M) e_2$ or σ_t on $e_3 M_3(M) e_3$ to

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ a^* & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Corollary 9.2. Assume that $[\varphi] \leq [\psi]$.

(i) $\sigma_t^{\psi}(\sigma_{-t}^{\varphi,\psi}([\varphi])) = \sigma_t^{\psi,\varphi}([\varphi])$.
(ii) $\sigma_t^{\psi,\varphi}([\varphi])\sigma_t^{\varphi}(x) = \sigma_t^{\psi}(x)\sigma_t^{\psi,\varphi}([\varphi])$. Here $x \in [\varphi]M[\varphi]$.
(iii) $\sigma_{s+t}^{\varphi,\psi}([\varphi]) = \sigma_s^{\varphi,\psi}([\varphi])\sigma_s^{\psi}(\sigma_t^{\varphi,\psi}([\varphi]))$.

Proof. (i): $\sigma_{-t}^{\psi,\varphi}(a')\sigma_{-t}^{\varphi,\psi}(a) = \sigma_{-t}^{\psi}(a'a)$.

(ii): If $ax = x'a'$ with $x \in [\varphi]M[\varphi]$ and $x' \in [\phi]M[\phi]$,

$$\sigma_t^{\phi,\varphi}(a)\sigma_t^{\varphi,\varphi}(x) = \sigma_t^{\phi,\varphi}(ax) = \sigma_t^{\phi,\phi}(x')\sigma_t^{\phi,\varphi}(a').$$

(iii): Cocycle condition follows from

$$\sigma_{s+t}^{\varphi,\psi}([\varphi]) = \sigma_s^{\varphi,\psi}([\varphi]\sigma_t^{\varphi,\psi}([\varphi])) = \sigma_s^{\varphi,\psi}([\varphi])\sigma_s^{\psi,\psi}(\sigma_t^{\varphi,\psi}([\varphi])).$$

□

Although these cocycle relations bear certain algebraic structures behind, the customary notation is not self-explanatory in its understanding. To remedy this fault, we introduce dummy symbols such as φ^{it} of which resolve these cocycle relations and make the algebraic structures apparent.

For the moment, we deal with faithful positive normal functionals and we imagine the following situation: various φ^{it}'s together with elements of M constitute a *-algebra $M(i\mathbb{R})$ in such a way that (i) M is a *-subalgebra of $M(i\mathbb{R})$, (ii) $\{\varphi^{it}\}$ is a one-parameter group of unitaries in $M(i\mathbb{R})$ and (iii) φ^{it}'s and $a \in M$ satisfy the following commutation relations

$$\varphi^{it}a = \sigma_t^{\varphi,\psi}(a)\psi^{it}.$$

Obviously the condition (iii) implies

$$\varphi^{it}a = \sigma_t^{\varphi}(a)\varphi^{it}, \quad \varphi^{it} = \sigma_t^{\varphi,\psi}(1)\psi^{it},$$

which, in turn, recover the commutation relations in (iii):

$$\varphi^{it}a = \sigma_t^{\varphi}(a)\varphi^{it} = \sigma_t^{\varphi}(a)\sigma_t^{\varphi,\psi}(1)\psi^{it} = \sigma_t^{\varphi,\varphi}(a)\sigma_t^{\varphi,\psi}(1)\psi^{it} = \sigma_t^{\varphi,\psi}(a)\psi^{it}.$$

Let $M(it) = M\varphi^{it} = \varphi^{it}M$. Since $\varphi^{it}\psi^{-it} \in M$, this does not depend on the choice of a faithful φ and we notice that $M(it)^* = M(-it)$, $M(is)M(it) = M(i(s+t))$ for $s, t \in \mathbb{R}$ and $M(i\mathbb{R}) = \sum_{t\in\mathbb{R}} M(it)$. We claim that $M(0) = M$ and $\sum_{t\in\mathbb{R}} M(it)$ is an algebraic direct sum. To see this, choose a reference functional ω and consider the algebraic crossed product of M by $\{\sigma_t^{\omega}\}$, which is a *-algebra based on a vector space $\bigoplus_{t\in\mathbb{R}} M = \sum_{t\in\mathbb{R}} M\omega^{it}$ with the *-algebra operations defined by

$$(a\omega^{is})(b\omega^{it}) = (a\sigma_s^{\phi}(b))\omega^{i(s+t)}, \quad (a\omega^{it})^* = \omega^{-it}a^* = \sigma_{-t}^{\omega}(a^*)\omega^{-it}.$$

Here the dummy symbol ω^{it} is introduced to indicate the t-component.

In the crossed product algebra, M is obviously identified with a subalgebra $M\omega^{i0}$ and, to each $\varphi^{it} \in M(it)$, we associate an element in $\sum M\omega^{it}$ by $\Phi^{it} = \sigma_t^{\varphi,\omega}(1)\omega^{it}$ so that

$$(\Phi^{it})^* = \sigma_{-t}^{\omega}(\sigma_t^{\omega,\varphi}(1))\omega^{-it} = \sigma_{-t}^{\varphi,\omega}(1)\omega^{-it} = \Phi^{-it}$$

$$\Phi^{is}\Phi^{it} = \sigma_s^{\varphi,\omega}(1)\sigma_s^{\omega}(\sigma_t^{\varphi,\omega}(1))\omega^{i(s+t)} = \sigma_{s+t}^{\varphi,\omega}(1)\omega^{i(s+t)} = \Phi^{i(s+t)}.$$

Moreover, for $a \in M$ and $\Psi^{it} = \sigma_t^{\psi,\omega}(1)\omega^{it}$, we have

$$\Phi^{it}a = \sigma_t^{\varphi,\omega}(1)\omega^{it}a = \sigma_t^{\varphi,\omega}(a)\omega^{it} = \sigma_t^{\varphi,\psi}(a)\sigma_t^{\psi,\omega}(1)\omega^{it} = \sigma_t^{\varphi,\psi}(a)\Psi^{it}.$$

Thus, we have a homomorphism of $M(i\mathbb{R})$ onto the crossed product algebra $\sum_{t\in\mathbb{R}} M\omega^{it}$, which preserves *-operations in view of $(a\varphi^{it})^* = \sigma_{-t}^{\varphi}(a^*)\varphi^{it}$ in $M(i\mathbb{R})$ and

$$(a\Phi^{it})^* = \Phi^{-it}a^* = \sigma_{-t}^{\varphi}(a^*)\Phi^{-it}.$$

Since $M(i\mathbb{R}) = \sum_{t\in\mathbb{R}} M(it)$ and its image in the crossed product is a direct sum, the *-homomorphism is in fact a *-isomorphism. Consequently, $M(i0) = M$ and $M(i\mathbb{R}) = \sum_{t\in\mathbb{R}} M(it)$ is a direct sum.

Now the notation φ^{it} is extended to a not necessarily faithful $\varphi \in M_*^+$: Set

$$\varphi^{it} = \sigma_t^{\varphi,\omega}([\varphi])\omega^{it},$$

which does not depend on the choice of a faithful ω. In fact, for another faithful functional ϕ in M_*^+,

$$\sigma_t^{\varphi,\phi}([\varphi])\phi^{it} = \sigma_t^{\varphi,\omega}([\varphi])\sigma_t^{\omega,\phi}(1)\phi^{it} = \sigma_t^{\varphi,\omega}([\varphi])\omega^{it}.$$

We then notice the following properties.

(i) $\varphi^{is}\varphi^{it} = \varphi^{i(s+t)}$ and $(\varphi^{it})^* = \varphi^{-it}$ with $\varphi^{i0} = [\varphi]$.

(ii) $\varphi^{it} a \psi^{-it} = \sigma_t^{\varphi,\psi}([\varphi] a [\psi])$.

Proof. (i) The equality $\varphi^{i0} = [\varphi]$ is trivial, while

$$(\varphi^{it})^* = \omega^{-it} \sigma_t^{\omega,\varphi}([\varphi]) = \sigma_{-t}^{\omega}(\sigma_t^{\omega,\varphi}([\varphi])\omega^{-it} = \sigma_{-t}^{\varphi,\omega}([\varphi])\omega^{-it} = \varphi^{-it}$$

and

$$\varphi^{is}\varphi^{it} = \sigma_s^{\varphi,\omega}([\varphi])\sigma_s^{\omega}(\sigma_t^{\varphi,\omega}([\varphi]))\omega^{i(s+t)} = \sigma_{s+t}^{\varphi,\omega}([\varphi])\omega^{i(s+t)} = \varphi^{i(s+t)}.$$

(ii) The commutation relations follow from

$$\begin{aligned}\varphi^{it} a \psi^{-it} &= \sigma_t^{\varphi,\omega}([\varphi])\sigma_t^{\omega}(a\sigma_{-t}^{\psi,\omega}([\psi])) = \sigma_t^{\varphi,\omega}([\varphi])\sigma_t^{\omega}(a)\sigma_t^{\omega}(\sigma_{-t}^{\psi,\omega}([\psi]))\\ &= \sigma_t^{\varphi,\omega}([\varphi])\sigma_t^{\omega}(a)\sigma_t^{\omega,\psi}([\psi]) = \sigma_t^{\varphi,\psi}([\varphi]a[\psi]).\end{aligned}$$

□

We say that an element $a \in M$ is **finitely supported** if $a = [\varphi]a[\varphi]$ for some $\varphi \in M_*^+$. Let M_f be the set of finitely supported elements in M.

Lemma 9.3. M_f is a weak* dense *-subalgebra of M and closed under sequential weak* limits in M. Moreover,

$$M_f = \sum_{\varphi \in M_*^+} M[\varphi] = \sum_{\varphi \in M_*^+} [\varphi]M.$$

Proof. Clearly M_f is closed under the *-operation and M_f is a subalgebra in view of $[\varphi] \vee [\psi] \leq [\varphi + \psi]$. The *-subalgebra M_f is then weak*-dense in M in view of $\vee_{\varphi \in M_*^+}[\varphi] = 1$. If $a = a[\varphi]$, $[a\varphi a^*]$ is the left support of a and $[\varphi + a\varphi a^*]a[\varphi a\varphi a^*] = a$. Let a be a weak* limit of $\{a_n\}_{n\geq 1}$ in M_f with $[\varphi_n][a_n][\varphi_n] = a_n$ for $n \geq 1$. Then, for $\varphi = \sum_{n=1}^{\infty} 2^{-n}\varphi_n/\varphi_n(1) \in M_*^+$, $[\varphi]a[\varphi]$. □

Now we relax the existence of faithful functionals in M_*^+ and set

$$M_f(i\mathbb{R}) = \bigcup_{\varphi \in M_*^+} [\varphi]M[\varphi](i\mathbb{R}),$$

where the natural inclusions $[\varphi]M[\varphi](i\mathbb{R}) \subset [\psi]M[\psi](i\mathbb{R})$ for $\varphi, \psi \in M_*^+$ satisfying $[\varphi] \subset [\psi]$ are assumed in the union.

Finally we add formal expressions of the form $\omega^{it} = \sum_{j\in I}\omega_j^{it}$ for families $\{\omega_j \in M_*^+\}_{j\in I}$ of mutually orthogonal supports and allow multiplications of elements in M to get $\{M(it)\}_{t\in\mathbb{R}}$ so that $M_f(it) \subset M(it)$ and $M(0) = M$. In what follows, a formal sum $\omega = \sum_{j\in I}\omega_j$ is referred as a **weight** of M. A weight $\omega = \sum \omega_j$ is said to be **faithful** if $1 = \sum[\omega_j]$ in M. Note that any weight is extended to a faithful one and $\{\omega^{it}\}$ is a one-parameter group of unitaries in $M(i\mathbb{R}) = \oplus M(it)$ if the weight ω is faithful in the sense that $1 = \sum_j [\omega_j]$ and, for another faithful weight $\phi = \sum_{k \in J} \phi_k$ and $a \in M$,

$$\phi^{it} a \omega^{-it} = \sum_{j,k} \phi_k^{it} a \omega_j^{-it}$$

defines a continuous family of elements in M so that it consists of unitaries when $a = 1$ and $\sigma_t^\omega(a) = \omega^{it} a \omega^{-it}$ gives an automorphic action on M.

Remark 2. Here weights are introduced in a formal and restricted way but more serious treatments enable us to find that continuous one-parameter groups of unitaries of the form $\{u(t) \in M(it)\}$ are one-to-one correspondence with weights of generalized form on M.

At this stage, we can introduce two more classes of modular algebraic stuffs:

$$M(i\mathbb{R} + 1/2) = \sum_{t \in \mathbb{R}} M(it + 1/2), \quad M(i\mathbb{R} + 1) = \sum_{t \in \mathbb{R}} M(it + 1),$$

with

$$M(it + 1/2) = \sum_{\varphi \in M_*^+} M \varphi^{it+1/2} = \sum_{\varphi \in M_*^+} \varphi^{it+1/2} M$$

and

$$M(it + 1) = \sum_{\varphi \in M_*^+} M \varphi^{it+1} = \sum_{\varphi \in M_*^+} \varphi^{it+1} M$$

so that $M(1/2) = L^2(M)$ and $M(1) = M_*$. These are $i\mathbb{R}$-graded *-bimodules of $M(i\mathbb{R})$ and we have a natural module map $M(i\mathbb{R} + 1/2) \otimes_{M(i\mathbb{R})} M(i\mathbb{R}+1/2) \to M(i\mathbb{R}+1)$ which respects the grading in the sense that $M(is+1/2)M(it+1/2) = M(i(s+t)+1)$ for $s, t \in \mathbb{R}$. In particular, given a weight ω on M, we have $M(it+s) = M(s)\omega^{it} = \omega^{it} M(s)$ for $s = 1/2, 1$.

The evaluation of $\varphi \in M_*$ at the unit $1 \in M$ is called the **expectation** of φ and denoted by $\langle \varphi \rangle$. Note that the expectation satisfies the trace property for various combinations of multiplications such as $\langle a\varphi \rangle = \langle \varphi a \rangle$ and $\langle \varphi^{it} \xi \psi^{-it} \eta \rangle = \langle \psi^{-it} \eta \varphi^{it} \xi \rangle$ for $a \in M$, $\varphi, \psi \in M_*^+$ and $\xi, \eta \in L^2(M)$.

The scaling $\varphi \mapsto e^{-s}\varphi$ on M_*^+ gives rise to a *-automorphic action θ_s of $s \in \mathbb{R}$ (called the **scaling automorphism**) on these modular stuffs: $\theta_s(x\varphi^{it+r}) = e^{-ist-sr} x \varphi^{it+r}$ for $x \in M$ and $r \in \{0, 1/2, 1\}$.

Remark 3. Since elements in $L^2(M)$ and M_* are always 'finitely supported', we can decribe $M(it + s)$ $(s = 1/2, 1)$ without referring to weights.

We here collect basic analytic properties on these modular stuffs.

Lemma 9.4 (Modular Extension)**.** For $\varphi, \psi \in M_*^+$ and $a \in M$, $\mathbb{R} \ni t \mapsto \varphi^{it} a \psi^{1-it} \in M_*$ is extended analytically to a norm-continuous function $\varphi^{iz} a \psi^{1-iz}$ on the strip $-1 \leq \operatorname{Im} z \leq 0$ with a bound

$$\|\varphi^{it+r} a \psi^{-it+1-r}\| \leq \|\varphi a\|^r \|a\psi\|^{1-r} \quad (0 \leq r \leq 1)$$

in such a way that

$$(\varphi^{iz} a \psi^{1-iz})|_{z=t-i} = \varphi^{1+it} a \psi^{-it}, \quad (\varphi^{iz} a \psi^{1-iz})|_{z=t-i/2} = \varphi^{it+1/2} a \psi^{-it+1/2}.$$

Proof. By embedding $[\varphi]M[\psi]$ into a corner of $e(\varphi,\psi)M_2(M)e(\varphi,\psi)$ and replacing φ and ψ with $\varphi \oplus \psi$ on $e(\varphi,\psi)M_2(M)e(\varphi,\psi)$, we may assume that $\varphi = \psi$ is faithful on M. For $b \in M$, we have

$$\langle \varphi^{it} a \varphi^{1-it} b \rangle = (b^* \varphi^{1/2} | \sigma_t(a) \varphi^{1/2}) = (b^* \varphi^{1/2} | \Delta^{it}(a\varphi^{1/2})).$$

Since $M\varphi^{1/2} \subset D(\Delta^{1/2})$, $\Delta^{it}(a\varphi^{1/2}) \in L^2(M)$ is extended analytically to a norm-continuous function $\Delta^{iz}(a\varphi^{1/2})$ on the half strip $-1/2 \leq \operatorname{Im} z \leq 0$ so that

$$\|\Delta^r(a\varphi^{1/2})\|^2 \leq \|a\varphi^{1/2}\|^2 + \|\Delta^{1/2}(a\varphi^{1/2})\|^2 = \varphi(a^*a) + \varphi(aa^*)$$

for $0 \leq r \leq 1/2$, $\varphi^{it} a \varphi^{1-it}$ is extended to a norm-continuous analytic function $f_a(z) \in M_*$ on the half strip by

$$f_a(z) = \Delta^{iz}(a\varphi^{1/2}))\varphi^{1/2} \quad \text{with} \quad \|f_a(z)\| \leq \sqrt{\varphi(1)\varphi(aa^* + a^*a)}.$$

In view of the relation $\varphi^{iz} a \varphi^{1-iz} = (\varphi^{1+i\overline{z}} a^* \varphi^{-i\overline{z}})^*$, $f_a(z)$ is further extended to $-1 \leq \operatorname{Im} z \leq -1/2$ by $f_a(z) = (f_{a^*}(\overline{z} - i))^*$ with the same bound. Moreover, we have for $b \in M$

$$\langle b, f_a(t - i/2) \rangle = (b^* \varphi^{1/2} | \Delta^{it+1/2}(a\varphi^{1/2})) = \langle \varphi^{it+1/2} a \varphi^{-it+1/2} b \rangle$$

and

$$\langle b, f_a(t-i) \rangle = \langle b, f_{a^*}(t)^* \rangle = \overline{\langle \varphi^{it} a^* \varphi^{1-it} b^* \rangle} = \langle b \varphi^{1+it} a \varphi^{-it} \rangle.$$

Finally the three line theorem gives the bound estimate on $\|f_a(z)\|$. □

Corollary 9.5. Let $\varphi, \psi \in M_+^*$ and $a \in [\varphi]M[\psi]$. Then the function $\sigma_t^{\varphi,\psi}(a)\psi^{1/2} = \varphi^{it} a \psi^{-it} \psi^{1/2}$ of $t \in \mathbb{R}$ is analytically extended to an $L^2(M)$-valued continuous function $\varphi^{iz} a \psi^{-iz+1/2}$ of $z \in [-1/2, 0] + i\mathbb{R}$ so that $(\varphi^{iz} a \psi^{-iz+1/2})_{z=t-i/2} = \varphi^{1/2} \varphi^{it} a \psi^{-it} = \varphi^{1/2} \sigma_t^{\varphi,\psi}(a)$.

Theorem 9.6 (Multiple KMS condition)**.** Let $\phi_j \in M_*^+$ and $a_j \in M$ for $j = 0, 1, \dots, n$. Then the multilinear functional $\langle \phi_0^{1+it_0} a_0 \phi_1^{it_1} a_1 \dots \phi_n^{it_n} a_n \rangle$ of $(a_0, \dots, a_n) \in M^{n+1}$ which depends continuously on $(t_0, \cdots, t_n) \in$

$\mathbb{R}^{n+1}$ satisfying $t_0 + \cdots + t_n = 0$ is analytically extended to a multilinear functional $\langle \phi_0^{z_0} a_0 \phi_1^{z_1} a_1 \dots \phi_n^{z_n} a_n \rangle$ of $(a_0, \dots, a_n) \in M^{n+1}$ for $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$ satisfying $z_0 + \cdots + z_n = 1$ and $\Re z_j \geq 0$.

Moreover, we have

$$|\langle \phi_0^{z_0} a_0 \phi_1^{z_1} a_1 \dots \phi_n^{z_n} a_n \rangle| \leq \phi_0(1)^{\Re z_0} \dots \phi_n(1)^{\Re z_n} \|a_0\| \dots \|a_n\|.$$

Lemma 9.7. *Let $\omega \in M_*^+$ be faithful and let $a \in M$. Then the following conditions are equivalent.*

(i) *The inequality $a^* \omega a \leq \omega$ holds in M_*^+.*
(ii) *We can find a norm-bounded function $a(z) \in M$ of z in $-1/2 \leq \operatorname{Im} z \leq 0$ such that $a(t) = \omega^{it} a \omega^{-it}$ for $t \in \mathbb{R}$, $a(z)\xi \in L^2(M)$ is norm-continuous in z for any $\xi \in L^2(M)$ and $\|a(-i/2)\| \leq 1$.*
(iii) *We can find an element $b \in M$ satisfying $\|b\| \leq 1$ and $\omega^{1/2} a = b\omega^{1/2}$.*

Moreover, if this is the case, under the notation in (ii), $\xi a(z) \in L^2(M)$ is norm-continuous for any $\xi \in L^2(M)$.

Proof. (i) $\Longrightarrow$ (ii): By modular extension, the function $\langle x\omega^{1+it} a \omega^{-it} y \rangle$ with $x, y \in M$ gives rise to an analytic function $\langle x\omega^{iz+1/2} a \omega^{-iz+1/2} y \rangle$ of $z \in \mathbb{C}$ in the range $-1/2 \leq \operatorname{Im} z \leq 1/2$ so that it is estimated on the real line by

$$|\langle x\omega^{it+1/2} a \omega^{-it+1/2} y \rangle| = |(\omega^{1/2} x^* | a(t) \omega^{1/2} y)| \leq \|\omega^{1/2} x^*\| \|a\| \|\omega^{1/2} y\|$$

and on the line $\operatorname{Im} z = -1/2$ by

$$\begin{aligned}
|\langle x\omega^{it+1} a \omega^{-it} y \rangle| &= |(\omega^{1/2} x^* | \omega^{1/2} a(t) y)| \\
&\leq \|\omega^{1/2} x^*\| \|\omega^{1/2} a(t) y\| = \|\omega^{1/2} x^*\| \sqrt{\omega(a\omega^{-it} y y^* \omega^{it} a^*)} \\
&\leq \|\omega^{1/2} x^*\| \sqrt{\omega(\omega^{-it} y y^* \omega^{it})} = \|\omega^{1/2} x^*\| \|\omega^{1/2} y\|.
\end{aligned}$$

Thus the three line theorem for the region $-1/2 \leq \operatorname{Im} z \leq 0$ gives

$$|\langle x\omega^{iz+1/2} a \omega^{-iz+1/2} y \rangle| \leq \|a\|^{1+2\operatorname{Im} z} \|\omega^{1/2} x^*\| \|\omega^{1/2} y\|$$

and a bounded operator $a(z)$ on $L^2(M)$ is well-defined by the relation

$$(\omega^{1/2} x^* | a(z)(\omega^{1/2} y)) = \langle x\omega^{iz+1/2} a \omega^{-iz+1/2} y \rangle$$

with the bound $\|a(z)\| \leq \|a\|^{1+2\operatorname{Im} z}$. From the cyclic relation

$$\begin{aligned}
(\omega^{1/2} x^* | a(z)(\omega^{1/2} yb)) &= \langle x\omega^{iz+1/2} a \omega^{-iz+1/2} yb \rangle = \langle bx\omega^{iz+1/2} a \omega^{-iz+1/2} y \rangle \\
&= (\omega^{1/2} x^* b^* | a(z)(\omega^{1/2} y)) = (\omega^{1/2} x^* | \big(a(z)(\omega^{1/2} y)\big) b)
\end{aligned}$$

for $b \in M$, $a(z)$ in fact belongs to M. Since $a(z)$ is uniformly bounded in z and weakly analytic, we need to check the norm-continuity of $a(z)\omega^{1/2} \in L^2(M)$ as a function of z. In view of $a\omega^{1/2} \in D(\Delta^{1/2})$,

$\Delta^{iz}(a\omega^{1/2})$ is norm-continuous analytic function of z in $-1/2 \leq \operatorname{Im} z \leq 0$, which satisfies $\Delta^{it}(a\omega^{1/2}) = a(t)\omega^{1/2}$ for $t \in \mathbb{R}$ and hence $a(z)\omega^{1/2} = \Delta^{iz}(a\omega^{1/2})$ is norm-continuous in z.

(ii) $\Longrightarrow$ (iii): The relation $\Delta^{it}(a\omega^{1/2}) = a(t)\omega^{1/2}$ is analytically continued to the relation $\omega^{1/2}a = \Delta^{1/2}(a\omega^{1/2}) = \sigma(-i/2)\omega^{1/2}$ and we can put $b = a(-i/2)$.

(iii) $\Longrightarrow$ (i): For a positive element $x \in M$,

$$\begin{aligned}\omega(axa^*) &= \langle \omega^{1/2}axa^*\omega^{1/2}\rangle = \langle b\omega^{1/2}x\omega^{1/2}b^*\rangle \\ &= (\omega^{1/2}x\omega^{1/2})(b^*b) \leq (\omega^{1/2}x\omega^{1/2})(1) = \omega(x).\end{aligned}$$

Finally, under the notation and conditions in (ii) and (iii), $\omega^{1/2}a(t) = \Delta^{it}(b\omega^{1/2})$ is analytically extended to a norm-continuous function $\Delta^{iz}(b\omega^{1/2})$ of $z \in i[-1/2, 0] + \mathbb{R}$, while $(\eta|\omega^{1/2}a(t)) = (\omega^{1/2}|a(t)\eta^*)$ is analytically extended to $(\omega^{1/2}|a(z)\eta^*) = (\eta|\omega^{1/2}a(z))$ for any $\eta \in L^2(M)$. Thus $\omega^{1/2}a(z) = \Delta^{iz}(b\omega^{1/2})$ is norm-continuous in $z \in i[-1/2, 0]+\mathbb{R}$. Now the norm-boundedness of $a(z)$ is used to see that $x\omega^{1/2}a(z)$ norm-converges to $\xi a(z)$ uniformly in z when $\|x\omega^{1/2} - \xi\| \to 0$. □

Corollary 9.8. For $\varphi, \psi \in M_*^+$, the following conditions are equivalent.

(i) The inequality $\varphi \leq \psi$ holds in M_*^+.
(ii) $[\varphi] \leq [\psi]$ and the function $\varphi^{it}\psi^{-it}$ of $t \in \mathbb{R}$ is analytically extended to an M-valued function $\varphi^{iz}\psi^{-iz}$ of z in the range $-1/2 \leq \operatorname{Im} z \leq 0$ so that $\varphi^{iz}\psi^{-iz}\xi \in L^2(M)$ is norm-continuous in z for any $\xi \in L^2(M)$ and $\|\varphi^{1/2}\psi^{-1/2}\| \leq 1$.
(iii) We can find an element $c \in M$ satisfying $\|c\| \leq 1$ and $\varphi^{1/2} = c\psi^{1/2}$.

Moreover, if this is the case, $\xi\varphi^{iz}\psi^{-iz} \in L^2(M)$ is norm-continuous in $z \in i[-1/2, 0] + \mathbb{R}$ for any $\xi \in L^2(M)$.

Proof. Consider a faithful $\omega = \varphi \oplus \psi$ on $eM_2(M)e$ with $e = [\varphi] \oplus [\psi]$. For the choice

$$a = \begin{pmatrix} 0 & [\varphi] \\ 0 & 0 \end{pmatrix},$$

the condition $\varphi \leq \psi$ is equivalent to $a^*\omega a \leq \omega$ and

$$\omega^{it}a\omega^{-it} = \begin{pmatrix} 0 & \varphi^{it}\psi^{-it} \\ 0 & 0 \end{pmatrix}$$

is used to get (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) by the lemma, whereas (iii) implies $\varphi = \psi^{1/2}c^*c\psi^{1/2} \leq \psi$ in M_*^+. □

We now investigate continuity properties of families $\{M(it+s)\}_{t\in\mathbb{R}}$ for $s = 0, 1/2, 1$. Let us begin with simple observations on continuity of modular actions: Let $\varphi = \sum \varphi_j$ and $\psi = \sum \psi_k$ be weights on M. For $\xi \in L^2(M)$,

$$\varphi^{it}\xi\psi^{-it} = \sum_{j,k} \varphi_j^{it}\xi\psi_k^{-it}$$

is norm-continuous in $t \in \mathbb{R}$ as an absolute sum of $L^2(M)$-valued norm-continuous functions $\varphi_j^{it}\xi\psi_k^{-it}$. Any $\phi \in M_*^+$ has an expression $\xi\eta$ with $\xi, \eta \in L^2(M)$ and we see that

$$\varphi^{it}\phi\psi^{-it} = (\varphi^{it}\xi\omega^{-it})(\omega^{it}\eta\psi^{-it})$$

(ω being an auxiliary faithful weight) is an M_*-valued norm-continuous function of $t \in \mathbb{R}$ as a product of $L^2(M)$-valued norm-continuous functions.

Lemma 9.9. For a section $x = \{x(t)\}$ of $\{M(it)\}$, the following conditions are equivalent.

(i) There exists a faithful weight ω on M such that $\omega^{-it}x(t) \in M$ is weak*-continuous in $t \in \mathbb{R}$.
(ii) There exists a faithful weight ω on M such that $x(t)\omega^{-it} \in M$ is weak*-continuous in $t \in \mathbb{R}$.
(iii) For any faithful weight ω on M, $\omega^{-it}x(t) \in M$ is weak*-continuous in $t \in \mathbb{R}$.
(iv) For any faithful weight ω on M, $x(t)\omega^{-it} \in M$ is weak*-continuous in $t \in \mathbb{R}$.
(v) For any $\phi \in M_*^+$, $\phi^{-it}x(t) \in M$ is weak*-continuous in $t \in \mathbb{R}$.
(vi) For any $\phi \in M_*^+$, $x(t)\phi^{-it} \in M$ is weak*-continuous in $t \in \mathbb{R}$.

Moreover, if $\{x(t)\}$ satisfies these equivalent conditions, $\|x(t)\|$ is locally bounded in $t \in \mathbb{R}$.

We say that a section $\{x(t)\}$ is **weak*-continuous** if it satisfies any of these equivalent conditions.

Proof. Assume (i) and let ϕ be another weight. Then, for $\varphi \in M_*$,

$$\varphi(x(t)\phi^{-it}) = \langle \phi^{-it}\varphi\omega^{it}, \omega^{-it}x(t)\rangle$$

is continuous in $t \in \mathbb{R}$ as an evaluation of an M_*-valued norm-continuous function by an M-valued weak*-continuous function. Since ϕ is arbitrary, this menas that the four conditions (i) to (iv) are equivalent, which in turn imply (v) and (vi).

Now we show (vi) $\Longrightarrow$ (iv). By Banach-Steinhaus theorem, $\|x(t)[\phi]\|$ is locally bounded in t for any $\phi \in M_*^+$. If $\|x(t)\|$ is not locally bounded, we can find abounded sequence $\{t_n\}$ and a sequence $\{\xi_n\}$ of unit vectors in $L^2(M)$ such that $\|x(t_n)\xi_n\| \geq n$. Then, for $\phi = \sum \frac{1}{2^n}\phi_n$ with

$\phi_n = \xi_n\xi_n^*$, $\|x(t_n)[\phi]\| \geq n$, which contradicts with local boundedness of $\|x(t)[\phi]\|$. Thus $\|x(t)\|$ is locally bounded. Now in the expression

$$(\xi|x(t)\omega^{-it}\eta) = \sum_{j\in F}(\xi|x(t)\omega_j^{-it}\eta) + ((x(t)\omega^{-it})^*\xi|\sum_{j\notin F}\omega^{it}\omega_j^{-it}\eta)$$

with $\xi, \eta \in L^2(M)$ and F a finite subset of indices, the second term is estimated by

$$\begin{aligned}\left|((x(t)\omega^{-it})^*\xi|\sum_{j\notin F}\omega^{it}\omega_j^{-it}\eta)\right| &\leq \|x(t)\|\|\xi\|\|\sum_{j\notin F}\omega^{it}\omega_j^{-it}\eta\| \\ &= \|x(t)\|\|\xi\|\sqrt{\sum_{j\notin F}\|\omega^{it}\omega_j^{-it}\eta\|^2} \\ &\leq \|x(t)\|\|\xi\|\sqrt{\sum_{j\notin F}\|[\omega_j]\eta\|^2}.\end{aligned}$$

Since $\|x(t)\|$ is locally bounded, this reveals that the convergence

$$(\xi|x(t)\omega^{-it}\eta) = \lim_{F\nearrow I}\sum_{j\in F}(\xi|x(t)\omega_j^{-it}\eta)$$

is locally uniformly in $t \in \mathbb{R}$ and gives a continuous function of t, proving the weak* continuity of $x(t)\omega^{-it} \in M$ on $t \in \mathbb{R}$. $\square$

We here introduce the *-operation on sections by

$$x^*(t) = x(-t)^* \in M(it+s) \quad \text{for a section } \{x(t) \in M(it+s)\}.$$

As a consequence of the above lemma, for a section $x(t) \in M(it)$, $x^*(t)$ as well as $ax(t)b$ with $a, b \in M$ are weak*-continuous if so is $x(t)$.

Lemma 9.10. Let $p = 1$ or 2 with notation $L^1(M) = M_*$ for $p = 1$. Then the following conditions on a section $\xi = \{\xi(t)\}$ of $\{M(it+1/p)\}$ are equivalent.

(i) There exists a faithful weight ω on M such that $\omega^{-it}\xi(t) \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.
(ii) There exists a faithful weight ω on M such that $\xi(t)\omega^{-it} \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.
(iii) For any faithful weight ω on M, $\omega^{-it}\xi(t) \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.
(iv) For any faithful weight ω on M, $\xi(t)\omega^{-it} \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.
(v) For any $\phi \in M_*^+$, $\phi^{-it}\xi(t) \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.

(vi) For any $\phi \in M_*^+$, $\xi(t)\phi^{-it} \in L^p(M)$ is norm-continuous in $t \in \mathbb{R}$.

We say that a section $\{\xi(t)\}$ is **norm-continuous** if it satisfies any of these equivalent conditions. Notice here that $\xi^*(t) = \xi(-t)^*$ is norm-cotinuous if and inly if so is $\xi(t)$

Proof. Since $\omega^{it}(\cdot)\phi^{-it}$ gives a strongly continuous one-parameter group of isometries on $L^p(M)$, the condition (i) ensures norm-continuity of $\xi(t)\phi^{-it} = \omega^{it}(\omega^{-it}\xi)\phi^{-it}$, which implies the equivalence of conditions (i) to (iv).

Since

$$\begin{aligned}
\|\phi^{-it}\xi(t) - \phi^{-is}\xi(s)\| &= \|\phi^{-it}\omega^{it}\omega^{-it}\xi(t) - \phi^{-is}\omega^{is}\omega^{-is}\xi(s)\| \\
&\leq \|\phi^{-it}\omega^{it}(\omega^{-it}\xi(t) - \omega^{is}\xi(s))\| + \|(\phi^{-it}\omega^{it} - \phi^{-is}\omega^{is})\omega^{-is}\xi(s)\| \\
&\leq \|(\omega^{-it}\xi(t) - \omega^{is}\xi(s))\| + \|(\phi^{-it}\omega^{it} - \phi^{-is}\omega^{is})\omega^{-is}\xi(s)\|
\end{aligned}$$

and $\phi^{-it}\omega^{it}$ is continuous on t in strong operator topology, (v) follows from (i).

Assume (v). If $\omega^{-it}\xi(t)$ is not norm-continuous at $t = t_0$, we can find a $\delta > 0$ and a sequence $\{t_n\}$ which converges to s but $\|\omega^{-it_n}\xi(t_n) - \omega^{-it_0}\xi(t_0)\| \geq \delta$ for $n \geq 1$. Choose states $\phi_n \in M_*^+$ so that $[\phi_n]\xi(t_n) = \xi(t_n)$ for $n \geq 0$ and set $\phi = \sum_{n\geq 0} \phi_n/2^n \in M_*^+$. Then, by (v),

$$\begin{aligned}
\|\omega^{-it_n}\xi(t_n) - \omega^{-it_0}\xi(t_0)\| &= \|\omega^{-it_n}\phi^{it_n}\phi^{-it_n}\xi(t_n) - \omega^{-it_0}\phi^{it_0}\phi^{-it_0}\xi(t_0)\| \\
&\leq \|(\phi^{-it_n}\xi(t_n) - \phi^{it_0}\xi(t_0))\| + \|(\omega^{-it_n}\phi^{it_n} - \omega^{-it_0}\phi^{it_0})\phi^{-it_0}\xi(t_0)\|
\end{aligned}$$

converges to 0, a contradiction. □

Definition 9.11. A section $\{x(t)\}$ of $\{M(it)\}_{t\in\mathbb{R}}$ is said to be ***L^2*-continuous** if $x(t)\xi$ and $\xi x(t)$ are norm-continuous for any $\xi \in L^2(M)$. Notice that $x^*(t)$ is L^2-continuous if and only if so is $x(t)$ in view of $x^*(t)\xi = (\xi^* x(-t))^*$ and $\xi x^*(t) = (x(-t)\xi^*)^*$. Clearly L^2-continuous sections are weak*-continuous.

To control the norm of a weak*-continuous section $x = \{x(t) \in M(it)\}$, two norms are introduced by

$$\|x\|_\infty = \sup\{\|x(t)\|; t \in \mathbb{R}\}, \quad \|x\|_1 = \int_{\mathbb{R}} \|x(t)\|\, dt$$

and $x(t)$ is said to be **bounded** if $\|x\|_\infty < \infty$ and **integrable** if $\|x\|_1 < \infty$. Note here that $\|x(t)\|$ is locally bounded and lower-semicontinuous in $t \in \mathbb{R}$.

Lemma 9.12. The following conditions on a section $\{x(t)\}$ of $\{M(it)\}$ are equivalent.

(i) For any $\xi \in L^2(M)$, $\{x(t)\xi\}$ is a norm-continuous section of $\{M(it+1/2)\}$.
(ii) For any $\varphi \in M_*^+$ and any $\xi \in L^2(M)$, $x(t)\varphi^{-it}\xi \in L^2(M)$ is norm-continuous in $t \in \mathbb{R}$.
(iii) The norm function $\|x(t)\|$ is locally bounded and, for a sufficiently large $\phi \in M_*^+$, $x(t)\phi^{-it+1/2} \in L^2(M)$ is norm-continuous in $t \in \mathbb{R}$, i.e., given any $\varphi \in M_*^+$, we can find $\phi \in M_*^+$ such that $[\varphi] \leq [\phi]$ and $x(t)\phi^{-it+1/2} \in L^2(M)$ is norm-continuous in $t \in \mathbb{R}$.

Proof. (i) $\iff$ (ii): Let $\varphi \in M_*^+$. If (i) holds, $\varphi^{-it}x(t) \in M$ is continuous in $t \in \mathbb{R}$ with respect to the strong operator topology on M and hence

$$x(t)\varphi^{-it}\xi = \Delta_t^\varphi(\varphi^{-it}x(t))\Delta_{-t}^\varphi\xi$$

is an $L^2(M)$-valued norm-continuous function of $t \in \mathbb{R}$. Likewise, under the condition (ii), $\varphi^{-it}x(t)\xi = \Delta_{-t}^\varphi(x(t)\varphi^{-it})\Delta_t^\varphi\xi \in L^2(M)$ is norm-continuous in $t \in \mathbb{R}$.

Assume (i) and (ii). The section $x(t)$ is weak*-continuous by (i) and the local boundedness of $\|x(t)\|$ follows from Lemma 9.9, whereas the remaining condition in (iii) is included in (ii).

(iii) $\Longrightarrow$ (ii): Given $\varphi \in M_*^+$ and $\xi \in L^2(M)$, choose $\phi \in M_*^+$ so that $[\varphi] \leq [\phi]$ and ϕ supports ξ on the left. Then

$$\|x(t)\phi^{-it}(\phi^{1/2}a - \phi^{1/2}b)\| \leq \|x(t)\|\|\phi^{1/2}a - \phi^{1/2}b\|,$$

together with local boundedness of $\|x(t)\|$, shows that $x(t)\phi^{-it}\eta$ is norm-continuous for any $\eta \in \overline{\phi^{1/2}M}$ as a locally uniform limit of $L^2(M)$-valued norm-continuous functions of $t \in \mathbb{R}$. Now $x(t)\varphi^{-it}\xi = (x(t)\phi^{-it})(\phi^{it}\varphi^{-it}\xi)$ is norm-continuous in $t \in \mathbb{R}$ as a product of strongly continuous operator-valued function $x(t)\phi^{-it}$ on $\overline{\phi^{1/2}M}$ and a norm-continuous function $\phi^{it}\varphi^{-it}\xi$ in $\overline{\phi^{1/2}M}$. □

Corollary 9.13. A section $x(t) \in M(it)$ is L^2-continuous if and only if $\|x(t)\|$ is locally bounded and $L^2(M)$-valued functions $x(t)\phi^{-it+1/2}$, $\phi^{-it+1/2}x(t)$ are norm-continuous for a sufficiently large $\phi \in M_*^+$.

A family $\{x(t) \in M(it)\}_{t\in\mathbb{R}}$ is said to be **finitely supported** if we can find $\phi \in M_*^+$ so that $x(t) = [\phi]x(t)[\phi]$ for every $t \in \mathbb{R}$. We say that $\{x(t)\}$ is locally bounded (bounded) if so is the function $\|x(t)\|$ of t.

9.1. **Convolution Algebra.** Consider a bounded, L^2-continuous and integrable section $\{f(t) \in M(it)\}$ and identify it with a formal expression like $\int_{\mathbb{R}} f(t)\,dt$, which is compatible with the *-operation by

$$\left(\int_{\mathbb{R}} f(t)\,dt\right)^* = \int_{\mathbb{R}} f(t)^*\,dt = \int_{\mathbb{R}} f(-t)^*\,dt = \int_{\mathbb{R}} f^*(t)\,dt.$$

Moreover, a formal rewriting

$$\int_{\mathbb{R}} f(s)\,ds \int_{\mathbb{R}} g(t)\,dt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(s)g(t-s)\,ds\right)\,dt$$

suggests defining a product of f and g by

$$(fg)(t) = \int_{\mathbb{R}} f(s)g(t-s)\,ds = \int_{\mathbb{R}} f(t-s)g(s)\,ds.$$

The real definition, however, needs some procedures. Let φ be a weight on M which supports both f and g. Then

$$f(s)g(t-s)\varphi^{-it} = f(s)\varphi^{-is}\sigma_s^\varphi(g(t-s)\varphi^{-i(t-s)})$$

is a strongly continuous M-valued function of $(s,t) \in \mathbb{R}^2$ and

$$\int_{\mathbb{R}} f(s)g(t-s)\varphi^{-it}\,ds \in M$$

is well-defined as a Bochner integral with respect to the strong operator topology with a norm estimation

$$\left\| \int_{\mathbb{R}} f(s)g(t-s)\varphi^{-it}\,ds \in M \right\| \leq \int_{\mathbb{R}} \|f(s)\|\,\|g(t-s)\|\,ds.$$

In view of $\|f\|_1 < \infty$ and $\|g\|_\infty < \infty$, for each $\xi \in L^2(M)$, the Bochner integral

$$\int_{\mathbb{R}} f(s)g(t-s)\varphi^{-it}\xi\,ds \in L^2(M)$$

converges uniformly when t varies in a finite interval and gives a norm-continuous function of $t \in \mathbb{R}$.

Let ψ be another weight satisfying $[\varphi] \leq [\psi]$ (consequently it supports x and y as well). The equality

$$\left(\int_{\mathbb{R}} f(s)g(t-s)\psi^{-it}\,ds\right)\psi^{it}\varphi^{-it}\xi = \int_{\mathbb{R}} f(s)g(t-s)\varphi^{-it}\xi\,ds$$

then reveals that

$$\left(\int_{\mathbb{R}} f(s)g(t-s)\psi^{-it}\,ds\right)\psi^{it}[\varphi] = \left(\int_{\mathbb{R}} f(s)g(t-s)\varphi^{-it}\,ds\right)\varphi^{it}$$

and an element $\int_{\mathbb{R}} f(s)g(t-s)\,ds \in M$ is well-defined by the relation

$$\left(\int_{\mathbb{R}} f(s)g(t-s)\,ds\right)\phi^{-it} = \int_{\mathbb{R}} f(s)g(t-s)\phi^{-it}\,ds,$$

which is satisfied by an arbitrary weight ϕ on M.

When ϕ supports f and g, we have

$$\phi^{-it}\int f(s)g(t-s)\phi^{-it}\,ds\phi^{it} = \int \phi^{-it}f(s)g(t-s)\,ds,$$

which means that the convolution product also satisfies the left indentity

$$\phi^{it}\int \phi^{-it}f(s)g(t-s)\,ds = \int f(s)g(t-s)\,ds.$$

Moreover, for $\xi, \eta \in L^2(M)$, the identity

$$(\eta|\int f(s)g(t-s)\phi^{-it}\xi\,ds) = \int (\phi^{it}g(t-s)^*f(s)^*\eta|\xi)\,ds$$

is rephrased into

$$\phi^{it}(fg)(t)^* = \int \phi^{it}g^*(-t-s)f^*(s)\,ds \iff (fg)^*(t) = \int g^*(t-s)f^*(s)\,ds$$

and

$$\eta^*\int_{\mathbb{R}} f(s)g(t-s)\,ds = \int_{\mathbb{R}} \eta^*f(s)g(t-s)\,ds = \int_{\mathbb{R}} \eta^*f(t-s)g(s)\,ds.$$

Note that $\eta^*f(s)g(t-s) \in M(it+1/2)$ is norm-continuous in $s \in \mathbb{R}$ and Bochner-integrable.

We have so far checked $(fg)^* = g^*f^*$ as well as the left-and-right compatibility in the definition of convolution product. The aasociativity is now available by

$$\begin{aligned}
\langle \eta((fg)h)(t)\phi^{-it}\xi\rangle &= \int \langle \eta(fg)(s)\phi^{-is}\phi^{is}h(t-s)\phi^{-it}\xi\rangle\,ds\\
&= \int\int \langle \eta f(r)g(s-r)\phi^{-is}\phi^{is}h(t-s)\phi^{-it}\xi\rangle\,drds\\
&= \int\int \langle \eta f(r)g(s-r)h(t-s)\phi^{-it}\xi\rangle\,drds\\
&= \int\int \langle \eta f(r)g(s)h(t-r-s)\phi^{-it+ir}\phi^{-ir}\xi\rangle\,dsdr\\
&= \int \langle \eta f(r)(gh)(t-r)\phi^{-it}\xi\rangle\,dr\\
&= \langle \eta(f(gh))(t)\phi^{-it}\xi\rangle.
\end{aligned}$$

In this way, we have obtained a *-algebra of sections of $\{M(it)\}_{t\in\mathbb{R}}$. The scaling automorphism θ_s on $\{M(it)\}$ gives rise to a *-automorphic action the *-algebra of sections by $(\theta_s f)(t) = e^{-ist}f(t)$.

Here we shall apply formal arguments to illustrate how tracial functionals are associated to this kind of *-algebras.

A formal manipulation is an easy business: Imagine that a section $f(t)$ has an analytic extension to the region $-1 \leq \operatorname{Im} z \leq 0$ somehow so that $f^*(z) = f(-\overline{z})^*$ and define a linear functional by

$$\left\langle \int_{\mathbb{R}} f(t)\,dt \right\rangle = \langle f(-i)\rangle.$$

Note that $f(-i)$ in the right hand side belongs to $M(1) = M_*$. We then have

$$\begin{aligned}\langle f^*f\rangle &= \int_{\mathbb{R}} \langle f^*(s)f(-i-s)\rangle\,ds = \int_{\mathbb{R}} \langle f^*(s-i/2)f(-s-i/2)\rangle\,ds\\ &= \int_{\mathbb{R}} \langle f(-s-i/2)^*f(-s-i/2)\rangle\,ds \geq 0,\end{aligned}$$

where Cauchy's integral theorem is used in the first line. The trace property is seen by

$$\begin{aligned}\langle fg\rangle &= \int \langle f(s)g(-i-s)\rangle\,ds = \int \langle f(t-i)g(-t)\rangle\,dt\\ &= \int \langle g(-t)f(t-i)\rangle\,dt = \int \langle g(t)f(-t-i)\rangle\,dt = \langle gf\rangle,\end{aligned}$$

where Cauchy's integral theorem is used again in the first line.

Now we return to the sane track. It then turns out to be some chores to make the whole story rigorous along the above easy-minded lines. Instead we shall construct a Hilbert algebra as a halfway business in what follows, which is enough to take out a tracial functionsl (Appendix P).

A section $\{f(t) \in M(it)\}$ is said to be ***L^2*-analytic** if, for a sufficiently large $\phi \in M_*^+$, functions $f(t)\phi^{-it}$ and $\phi^{-it}f(t)$ of $t \in \mathbb{R}$ (ϕ-functions of x) are analytically extended to bounded M-valued functions $f(z)\phi^{-iz}$ and $\phi^{-iz}f(z)$ of $z \in i[-1/2,0]+\mathbb{R}$ so that $L^2(M)$-valued functions $f(z)\phi^{-iz}\xi$ and $\xi\phi^{-iz}f(z)$ of $z \in i[-1/2,0]+\mathbb{R}$ are norm-continuous for each $\xi \in L^2(M)$.

Note here that sufficient largeness in the condition has a meaning: For $\varphi \in M_*^+$ majorized by ϕ, $\phi^{it}\varphi^{-it}$ is analytically extended to a *strongly continuous function $\phi^{iz}\varphi^{-iz}$ of $z \in i[-1/2,0]+\mathbb{R}$ (Corollary 9.8) and therefore $f(t)\varphi^{-it}$ has an analytic extension of the form

$(f(z)\phi^{-iz})(\phi^{iz}\varphi^{-iz})$, which is *strongly continuous as a product of *strongly continuous locally bounded operator-valued functions.

Note also that the norm-continuity of $f(z)\phi^{-iz}\xi$ and $\xi\phi^{-iz}f(z)$ is equivalent to that of L^2-valued functions $(f(z)\phi^{-iz})\phi^{1/2}$ and $\phi^{1/2}(\phi^{-iz}f(z))$ respectively (Lemma 9.12). These are analytic extensions of $f(t)\phi^{1/2-it}$ and $\phi^{1/2-it}f(t)$ are then denoted by $f(z)\phi^{1/2-iz}$ and $\phi^{1/2-iz}f(z)$ respectively.

Warning: No separate meaning of $f(z)$ is assigned here.

Let $f(t) \in M(it)$ be an L^2-analytic section. Then $f^*(t) = f(-t)^* \in M(it)$ is also L^2-analytic: $f^*(t)\phi^{-it}\xi = (\xi^*\phi^{it}f(-t))^*$ and $\xi\phi^{-it}f^*(t) = (f(-t)\phi^{it}\xi^*)^*$ admit $\big(\xi^*(\phi^{i\overline{z}}f(-\overline{z}))\big)^*$ and $\big((f(-\overline{z})\phi^{i\overline{z}})\xi^*\big)^*$ as their analytic extensions respectively, i.e.,

$$f^*(z)\phi^{-iz} = \big(\phi^{i\overline{z}}f(-\overline{z})\big)^*, \quad \phi^{-iz}f^*(z) = \big(f(-\overline{z})\phi^{i\overline{z}}\big)^*.$$

To get the convolution product in a way applicable of Cauchy's integral theorem, we impose the following condition. An L^2-analytic section $f(t) \in M(it)$ is said to be **uniformly integrable** if it is integrable and the function $|f|_\phi$ defined by

$$|f|_\phi(t) = \sup\{\|f(t-ir)\phi^{-it-r}\| \vee \|\phi^{-it-r}f(t-ir)\|; 0 \leq r \leq 1/2\}$$

is integrable for a sufficiently large $\phi \in M_*^+$. Note that $|f|_\phi(t)$ is bounded by L^2-analyticity, lower-semicontinuous as a supremum of lower-semicontinuous functions and $|f^*|_\phi(t) = |f|_\phi(-t)$. In particular, f is uniformly integrable if and only if so is f^*.

We also point out here that, for a uniformly integrable section $f(t)$, we have the following property: Given any $\epsilon > 0$ and any $T > 0$, we can find $t_+ \geq T$ and $t_- \leq -T$ so that $|f|_\phi(t_\pm) < \epsilon$.

Example 9.14. Let $\varphi \in M_*^+$ and $a \in [\varphi]M[\varphi]$ be entirely analytic for σ_t^φ. Then $f(t) = a\varphi^{it}$ is L^2-analytic.

In fact, if $\phi \in M_*^+$ majorizes φ, $f(z)\phi^{-iz} = a(\varphi^{iz}\phi^{-iz})$ and $\phi^{-iz}f(z) = (\phi^{-iz}\varphi^{iz})\sigma_{-z}^\varphi(a)$ for $z \in i[-1/2,0] + \mathbb{R}$ are norm-bounded analytic extensions of $f(t)\phi^{-it}$ and $\phi^{-it}f(t)$ respectively. Note that both $\phi^{-iz}\varphi^{iz}$ and $\sigma_{-z}^\varphi(a)$ are norm-bounded strongly continuous functions of $z \in i[-1/2,0] + \mathbb{R}$, see Corollary 9.8 and Appendix A.

To get a uniformly integrable section, just put $f_{\alpha,\beta}(t) = e^{-\alpha t^2+\beta t}f(t)$ with $\alpha > 0$ and $\beta \in \mathbb{C}$, for example.

Lemma 9.15. Let $\{f(t)\}$ be a section of $\{M(it)\}$. If $f(t)$ is L^2-analytic, $f(t)$ is bounded.

Proof. L^2-analyticity includes boundedness of $\|f(t)\phi^{-it}\| = \|f(t)[\phi]\|$ for any $\phi \in M_*^+$. If $\|f(t)\|$ is not bounded, we can find a sequence t_n

so that $\|f(t_n)\| > n$ and then a sequence of unit vectors $\xi_n \in L^2(M)$ satisfying $\|f(t_n)\xi_n\| \geq n$. For the choice $\phi = \sum_n \xi_n\xi_n^*/2^n \in M_*^+$, $\|f(t_n)[\phi]\| \geq \|f(t_n)\xi_n\| \geq n$ is therefore not bounded, a contradiction. $\square$

Now let $\mathcal{N}$ be the vector space of L^2-analytic and uniformly integrable sections of $\{M(it)\}$, which is closed under taking the *-operation as already noticed. It is immediate to see that the scaling automorphisms leave $\mathcal{N}$ invariant globally in such a way that $(\theta_s f)(z)\phi^{-iz} = e^{-isz}f(z)\phi^{-iz}$.

Let $f, g \in \mathcal{N}$. Since L^2-analyticity connotes boundedness as well as L^2-continuity, the convolution product fg has a meaning and $(fg)(t)$ is a bounded L^2-continuous section. To have $fg \in \mathcal{N}$, we therefore need to check L^2-analyticity and uniform integrability of fg.

Lemma 9.16. Let ω be a weight which supports an L^2-analytic section $g(t) \in M(it)$. Then, for $\phi \in M_*^+$ and $\xi \in L^2(M)$,

$$\sigma_s^\omega(g(z-s)\phi^{-i(z-s)})\xi = \Delta_\omega^{is}\big(g(z-s)\phi^{-i(z-s)}\big)\Delta_\omega^{-is}\xi \in L^2(M)$$

is norm-continuous in $s \in \mathbb{R}$ and $z \in i[-1/2, 0] + \mathbb{R}$.

Proof. Operator product is continuous on norm-bounded sets with respect to strong operator topology. $\square$

We first describe the analytic extension of $(fg)(t)\phi^{-it}$ (similarly for $\phi^{-it}(fg)(t)$ by left-and-right symmetry). Choose an auxiliary weight ω supporting both f and g. Let $\xi \in L^2(M)$.

$$\int f(s)\omega^{-is}\sigma_s^\omega(g(z-s)\phi^{-i(z-s)})\omega^{is}\phi^{-is}\xi\, ds$$

is well-defined as a Bochner integral and

$$\left\|\int f(s)\omega^{-is}\sigma_s^\omega(g(z-s)\phi^{-i(z-s)})\omega^{is}\phi^{-is}\xi\, ds\right\| \leq \|\xi\| \int \|f(s)\|\, |g|_\phi(\Re z - s)\, ds$$

shows that $(fg)(z)\phi^{-iz} \in M$ is well-defined by

$$((fg)(z)\phi^{-iz})\xi = \int_{\mathbb{R}} f(s)\omega^{-is}\sigma_s^\omega(g(z-s)\phi^{-i(z-s)})\omega^{is}\phi^{-is}\xi\, ds$$

with its norm estimated by

$$\|(fg)(z)\phi^{-iz}\| \leq \int_{\mathbb{R}} \|f(s)\|\, |g|_\phi(\Re z - s)\, ds.$$

To see the continuity of $(fg)(z)\phi^{-iz}\xi$ on $z \in i[-1/2, 0] + \mathbb{R}$, consider

$$(fg)(z)\phi^{-iz+1/2} = \int f(s)\omega^{-is}\omega^{is}\big(g(z-s)\phi^{-i(z-s)}\phi^{1/2}\big)\phi^{-is}\, ds.$$

In view of the norm estimate

$$\|f(s)\omega^{-is}\omega^{is}\big(g(z-s)\phi^{-i(z-s)}\phi^{1/2}\big)\phi^{-is}\| \leq \phi(1)^{1/2}\||g|_\phi\|_\infty \|f(s)\|$$

of the integrand and the integrability of $\|f(s)\|$, the above Bochner integral is norm-continuous in z as a uniform limit of $L^2(M)$-valued norm-continuous functions of $z \in i[-1/2,0]+\mathbb{R}$.

From the inequality

$$\|(fg)(z)\phi^{-iz+1/2}a-(fg)(z)\phi^{-iz+1/2}b\| \leq \||g|_\phi\|_\infty \|\phi^{1/2}a-\phi^{1/2}b\| \int_{\mathbb{R}} \|f(s)\|\,ds,$$

one sees that $(fg)(z)\phi^{-iz}\phi^{1/2}a$ norm-converges to $(fg)(z)\phi^{-iz}\xi$ uniformly in z when $\|\phi^{1/2}a-\xi\| \to 0$ and the norm-continuity of $(fg)(z)\phi^{-iz}\xi$ follows.

Finally the uniform integrability of fg follows from

$$|fg|_\phi(t) \leq \int_{\mathbb{R}} \|f(s)\|\,|g|_\phi(t-s)\,ds \quad \text{or} \quad |fg|_\phi(t) \leq \int_{\mathbb{R}} |f|_\phi(t-s)\|g(s)\|\,ds.$$

So far $\mathcal{N}$ is checked to be a *-algebra and the scaling automorphisms gives a *-automorphic action of $\mathbb{R}$ on $\mathcal{N}$. We next furnish $\mathcal{N}$ with an inner product so that it makes $\mathcal{N}$ into a Hilbert algebra.

Lemma 9.17. The following identity holds for $\phi, \varphi \in M_*^+$ and $f \in \mathcal{N}$.

$$[\varphi]\big(f(t-i/2)\phi^{-it-1/2}\big)\phi^{it+1/2} = \varphi^{it+1/2}\big(\varphi^{-it-1/2}f(t-i/2)\big)[\phi]$$

(the left hand side is therefore independent of a choice of φ while the right hand side independent of a choice of ϕ and the common element in $M(it+1/2)$ is reasonably denoted by $[\varphi]f(t-i/2)[\phi]$).

Proof. For $a \in M$, the identity

$$\langle (f(t)\phi^{-it})\phi^{1/2}\sigma_t^{\phi,\varphi}(a)\varphi^{1/2}\rangle = \langle \varphi^{1/2}(\varphi^{-it}f(t))\phi^{1/2}a\rangle$$

is analytically continued from t to $t-i/2$ to get

$$\langle (f(t-i/2)\phi^{-it-1/2})\phi^{1/2}\phi^{1/2}\sigma_t^{\phi,\varphi}(a)\rangle = \langle \varphi^{1/2}(\varphi^{-it-1/2}f(t-i/2))\phi^{1/2}a\rangle$$

(use KMS-condition at $\sigma_t^{\phi,\varphi}(a)\varphi^{1/2}$) and, after a simple rewriting,

$$\langle (f(t-i/2)\phi^{-it-1/2})\phi^{it+1/2}\phi^{1/2}a\varphi^{-it}\rangle = \langle \varphi^{it+1/2}(\varphi^{-it-1/2}f(t-i/2))\phi^{1/2}a\varphi^{-it}\rangle.$$

□

In the identity $[\varphi]f(t-i/2)[\phi] = [\varphi]\big(f(t-i/2)\phi^{-it-1/2}\big)\phi^{it+1/2}$, take a limit $[\varphi] \to 1$ to get

$$f(t-i/2)[\phi] = \big(f(t-i/2)\phi^{-it-1/2}\big)\phi^{it+1/2},$$

which belongs to $M(it+1/2)[\phi]$. Likewise $[\varphi]f(t-i/2) \in [\varphi]M(it+1/2)$ is well-defined by

$$[\varphi]f(t-i/2) = \varphi^{it+1/2}\big(\varphi^{-it-1/2}f(t-i/2)\big).$$

Choose a faithful weight $\omega = \sum_{j\in I}\omega_j$ with $\omega_j \in M_*^+$ pairwise orthogonal and consider a family $\{\xi_j = f(t-i/2)[\omega_j]\}$ in $M(it+1/2)$. We claim that $\{j; \|\xi_j\| \geq 1/m\}$ is a finite set for any $m \geq 1$. If not, we can choose an injective sequence j_n so that $\|\xi_{j_n}\| \geq 1/m$ and the choice $\phi = \sum_{n\geq 1}\omega_{j_n}/2^n\omega_{j_n}(1) \in M_*^+$ shows that $f(t-i/2)[\phi] = \sum_{n\geq 1} f(t-i/2)[\omega_{j_n}]$ has a norm $\|f(t-i/2)[\phi]\|^2 = \sum_n \|\xi_{j_n}\|^2 = \infty$, a contradiction. Consequently, $J = \{j; \|\xi_j\| > 0\}$ is a countable set and there exists $\varphi \in M_*^+$ such that $[\varphi] = \sum_{j\in J}[\omega_j]$ and

$$f(t-i/2) = \lim_{[\phi]\nearrow 1} f(t-i/2)[\phi] = f(t-i/2)[\varphi].$$

In this way, we have obtained a section $\{f(t-i/2) \in M(it+1/2)\}$ so that, for each $t \in \mathbb{R}$, $f(t-i/2) = f(t-i/2)[\phi]$ for some $\phi \in M_*^+$. Now choose ϕ_r to each $r \in \mathbb{Q}$ and set $\phi = \sum_{r\in\mathbb{Q}}\epsilon_r\phi_r$ with $\epsilon_r > 0$ for $r \in \mathbb{Q}$ satisfying $\sum \epsilon_r < \infty$. Then, if $t \in \mathbb{Q}$, $f(t-i/2) = f(t-i/2)[\phi]$ and therefore, $[\varphi]f(t-i/2) = [\varphi]f(t-i/2)[\phi]$ for any $\varphi \in M_*^+$. Since $[\varphi]f(t-i/2)$ is continuous in $t \in \mathbb{R}$, this implies $[\varphi]f(t-i/2) = [\varphi]f(t-i/2)[\phi]$ for any $t \in \mathbb{R}$ and hence, by taking a limit $[\varphi] \to 1$, we finally get $f(t-i/2) = f(t-i/2)[\phi]$ for all $t \in \mathbb{R}$; the whole section $\{f(t-i/2)\}$ is right-supported by a single $\phi \in M_*^+$. In particular, $\{f(t-i/2)\}$ is a norm-continuous bounded integrable section of $\{M(it+1/2)\}$.

Remark 4. By an analytic continuation, one sees that any L^2-analytic section $\{f(t)\}$ of $\{M(it)\}$ is finitely supported in the sense that there exists $\phi \in M_*^+$ such that $f(t) = [\phi]f(t)[\phi]$ for every $t \in \mathbb{R}$.

Example 9.18. Let $\varphi \in M_*^+$ and $a, b \in [\varphi]M[\varphi]$ be entirely analytic for σ_t^φ. Then $f(t) = e^{-\alpha t^2+\beta t}a\varphi^{it}b$ belongs to $\mathcal{N}$ and its boundary section is $f(t-i/2) = e^{-\alpha(t-i/2)^2+\beta(t-i/2)}a\varphi^{it+1/2}b$.

The inner product is now introduced by

$$(f|g) = \int_{\mathbb{R}} (f(t-i/2)|g(t-i/2))\,dt = \int_{\mathbb{R}} \langle f(t-i/2)^*g(t-i/2)\rangle\,dt.$$

Clearly this is a positive sesquilinear form. To check the non-degeneracy, let $(f|f) = 0$. We then have $f(t-i/2) \equiv 0$ by the continuity of $(f(t-i/2)|f(t-i/2))$ for $t \in \mathbb{R}$ and consequently $f(t)\phi^{-it} \equiv 0$ for any $\phi \in M_*^+$ by an analytic continuation; $f(t) \equiv 0$.

Since it contains a dense set, the completed Hilbert space $\mathcal{H}$ is identified with the direct integral

$$\mathcal{H} = \oint_{\mathbb{R}} M(it + 1/2)\, dt.$$

Since the family $\{M(it + 1/2)\}$ is trivialized by obvious isomorphisms $L^2(M)\omega^{it} \cong L^2(M) \cong \omega^{it} L^2(M)$ in terms of a faithful weight ω on M, we have identifications $\mathcal{H} \cong L^2(M) \otimes L^2(\mathbb{R})$ in two ways. The Hilbert space $\mathcal{H}$ is made into a *-bimodule of $M(i\mathbb{R})$ by

$$a\omega^{is} \oint_{\mathbb{R}} \xi(t)\, dt = \oint_{\mathbb{R}} a\omega^{is} \xi(t - s)\, dt$$

and

$$\left(\oint \xi(t)\, dt \right)^* = \oint_{\mathbb{R}} \xi(-t)^*\, dt$$

in such a way that actions of $M(it)$ on $\mathcal{H}$ are normal.

Note that the scaling automorpshism θ_s satisfies $(\theta_s f)(t - i/2) = e^{-ist - s/2} f(t - i/2)$ and scales the inner product: $(\theta_s f|\theta_s g) = e^{-s}(f|g)$ for $f, g \in \mathcal{N}$.

Lemma 9.19. For a faithful weight ω, the multiplication of ω^{is} on $\mathcal{H}$ gives a continuous one-parameter group of unitaries. Moreover, if $\{\xi(t)\} \in \mathcal{H}$ is supported by $\phi \in M_*^+$ in the sense that $[\phi]\xi(t) = \xi(t)$ for almost all $t \in \mathbb{R}$, then $\phi^{is} \oint \xi(t)\, dt \in \mathcal{H}$ is norm-continuous in $s \in \mathbb{R}$.

Let $f \in \mathcal{N}$ be supported by $\phi \in M_*^+$. Then, for $h \in \mathcal{N}$, hf is right-supported by ϕ and we have

$$\begin{aligned}
(hf)(t - i/2) &= \big((hf)(t - i/2)\phi^{-it-1/2}\big)\phi^{it+1/2} \\
&= \int ds\, (h(s)\omega^{-is})\sigma_s^\omega\big(f(t - s - i/2)\phi^{-i(t-s-i/2)}\big)\omega^{is}\phi^{-is}\phi^{it+1/2} \\
&= \int ds\, h(s)\big(f(t - s - i/2)\phi^{-i(t-s)-1/2}\big)\phi^{i(t-s)+1/2} \\
&= \int ds\, h(s) f(t - s - i/2).
\end{aligned}$$

In other words,

$$\oint (hf)(t - i/2)\, dt = \int h(s)\, ds \oint f(t - i/2)\, dt$$

with

$$\|hf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \int \|h(s)\|\, ds.$$

Thus the left multiplication of h is bounded with respect to the inner product in such a way that $(hf|g) = (f|h^* g)$ for $f, g \in \mathcal{N}$.

Moreover, with the choice $h(t) = \sqrt{\alpha/\pi}e^{-\alpha t^2}\phi^{it}$, $\lim_{\alpha\to\infty} \|hf - f\|_{\mathcal{H}} \to 0$ shows the density of $\mathcal{N}\mathcal{N}$ in $\mathcal{H}$.

Finally $(f^*|f^*) = (f|f)$ follows from $f^*(t - i/2) = \big(f(-t - i/2)\big)^*$.

Exercise 47. Check $(hf|g) = (f|h^*g)$ and $f^*(t-i/2) = \big(f(-t-i/2)\big)^*$.

Theorem 9.20. Given a W*-algebra M, the associated $\mathcal{N}$ is a Hilbert algebra.

The associated von Neumann algebra is denoted by $N = M \rtimes \mathbb{R}$ and referred to as the **Takesaki dual** of M. The scale automorphisms θ_s of $\mathcal{N}$ induce a *-automorphic action (also denoted by θ_s) of $\mathbb{R}$ on N by $\theta_s(l(f)) = l(\theta_s f)$, which is referred to as the **dual action**.

Let τ be the associated trace on N, which is scaled by the dual action so that $\tau(\theta_s(x)) = e^{-s}\tau(x)$ for $x \in N_+$. In fact,

$$\tau(\theta_s(l(f)^*l(g)) = \tau(l(\theta_s f)^*l(\theta_s g)) = (\theta_s f|\theta_s g) = e^{-s}(f|g) = e^{-s}\tau(l(f)^*l(g)).$$

Here we slightly change the notation in Hilbert algebras: $\mathcal{N}$ is regarded as a *-subalgebra of N and we write $\mathcal{N}\tau^{1/2} = \tau^{1/2}\mathcal{N}$ to indicate the corresponding subspace in $\mathcal{H} = \oint_{\mathbb{R}} M(it + 1/2)\, dt$. Thus $h \in \mathcal{N}$ is identified with an operator on $\mathcal{H}$ satisfying $h(f\tau^{1/2}) = (hf)\tau^{1/2}$, for $f \in \mathcal{N}$, with $f\tau^{1/2} = \tau^{1/2}f = \oint_{\mathbb{R}} f(t - i/2)\, dt$. The scaling property $(\theta_s f)(t-i/2) = e^{-ist-s/2}f(t-i/2)$ is then compatible with this notation: $\theta_s \in \mathrm{Aut}(N)$ induces a unitary on $L^2(N)$ as seen from

$$\theta_s(f\tau^{1/2}) = (\theta_s f)(\tau \circ \theta_{-s})^{1/2} = e^{s/2}(\theta_s f)\tau^{1/2} = \oint_{\mathbb{R}} e^{-ist} f(t - i/2)\, dt.$$

Let $\mathcal{B} \supset \mathcal{N}$ be a dense *-ideal of N so that $\mathcal{B}\tau^{1/2} = \tau^{1/2}\mathcal{B}$ is the set of bounded vectors in $\mathcal{H}$.

To $\xi, \eta \in \mathcal{H}$, a sesquilinear element $\xi^*\eta \in N_*$ is associated by $\langle \xi^*\eta, x\rangle = (\xi|x\eta)$ and $a^*b\tau = \tau a^*b \in N_*$ is defined to be $(a\tau^{1/2})^*(b\tau^{1/2}$ for $a, b \in \mathcal{B}$.

As a square root of this correspondence, we have a unitary map $\mathcal{H} \to L^2(N)$ in such a way that $|a|\tau^{1/2} \mapsto (a^*a\tau)^{1/2}$ for $a \in \mathcal{B}$. Therefore, if we set $\mathcal{B}_+ = \mathcal{B} \cap N_+$, the closure of $\mathcal{B}_+\tau^{1/2} = \tau^{1/2}\mathcal{B}_+$ in $\mathcal{H}$ corresponds to the positive cone $L^2(N)_+$.

With this notation, the trace of $f^*g \in \mathcal{N}^2$ $(f, g \in \mathcal{N})$ is expressed by

$$\tau(f^*g) = (f\tau^{1/2}|g\tau^{1/2}) = \int_{\mathbb{R}} (f(t - i/2)|g(t - i/2))\, dt.$$

Question: Is there any complex analytic characterization of $\mathcal{B}$?

Let ω be a faithful weight on M. From the convolution form realization of $\mathcal{N}$ on $\mathcal{H}$, one sees that N contains M as well as ω^{is} as operators by left multiplication and these in turn generates N. Likewise right multiplications of M and ω^{it} generates the right action of N on $\mathcal{H}$.

Theorem 9.21 (Takesaki)**.** The fixed-point algebra N^θ of N under the dual action θ is identified with M; $M = N^\theta$.

Proof. Through $\mathcal{H} \cong L^2(M) \otimes L^2(\mathbb{R})$ adapted to the trivialization $M(it+1/2)\omega^{-it} = L^2(M)$ of $M(it+1/2)$, the right action of ω^{is} is realized on $L^2(\mathbb{R})$ by translations whereas θ_s by multiplication of e^{-ist} on $L^2(\mathbb{R})$. Since these generate $\mathcal{B}(L^2(\mathbb{R}))$ (Stone-von Neumann), N^θ is identified with $(\mathcal{B}(L^2(M)) \otimes 1) \cap \mathrm{End}(\mathcal{H}_M)$. Let $a \in M$ and $f, g \in L^2(\mathbb{R})$. For $\xi, \eta \in L^2(M)$,

$$(\eta \otimes g|(\xi \otimes f)a) = (\eta|\xi\sigma^\omega_{\overline{g}f}) \quad \text{with} \quad \sigma^\omega_{\overline{g}f} = \int_{\mathbb{R}} \overline{g(t)} f(t)\omega^{it} a \omega^{-it} \in M$$

shows that $T \in \mathcal{B}(L^2(M))$ belongs to N^θ if and only if it is in the commutant of the right action of $\{\sigma^\omega_h(a); h \in L^1(\mathbb{R})\}$ on $L^2(M)$. Since $\{\sigma^\omega_h(a); h \in L^1(\mathbb{R})\}$ generates M, this shows $N^\theta \subset M$. $\square$

Corollary 9.22. For $t \in \mathbb{R}$, $M(it) = \{y \in \widetilde{N}; \theta_s(y) = e^{-ist}y, \forall s \in \mathbb{R}\}$.

9.2. **Haagerup's trace formula.** Haagerup's trace formula in noncommutative integration theory is analysed in the framework of modular algebras. To avoid tautological faults, prior to this, we describe modular algebras as well as standard Hilbert spaces in terms of basic ingredients of Tomita-Takesaki theory. The semifiniteness of Takesaki's duals is then established by constructing relevant Hilbert algebras as a collaboration of modular algebras and complex analysis. Note that the known proof of the existence of trace is indirect in the sense that it is deduced from the innerness of modular automorphism groups together with Pedersen-Takesaki's Radon-Nykodym theorem.

Let $\delta > 0$, $\mu \in (0, \infty) + i\mathbb{R}$ and $s \in \mathbb{R}$. Consider a Fourier integral

$$f(\lambda) = \int_{-\infty}^{\infty} \frac{1}{\mu + it} \exp\left(-\frac{1}{2}\delta t^2 - ist + i\lambda t\right) dt.$$

From

$$\begin{aligned}\left(e^{\lambda\mu} f(\lambda)\right)' &= e^{\lambda\mu} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\delta t^2 - ist + i\lambda t\right) dt \\ &= \sqrt{\frac{2\pi}{\delta}} e^{\lambda\mu} \exp\left(-\frac{(\lambda - s)^2}{2\delta}\right)\end{aligned}$$

and $\lim_{\lambda\to-\infty} e^{\lambda\mu} f(\lambda) = 0$, we obtain an expression

$$f(\lambda) = \sqrt{\frac{2\pi}{\delta}} e^{-\lambda\mu} \int_{-\infty}^{\lambda} e^{\mu t} \exp\left(-\frac{(t-s)^2}{2\delta}\right) dt.$$

Since

$$\lim_{\delta\to+0} \sqrt{\frac{2\pi}{\delta}} e^{\mu t} \exp\left(-\frac{(t-s)^2}{2\delta}\right) = 2\pi e^{\mu s}\delta(t-s),$$

$f(\lambda)$ approaches $1_{[s,\infty)}(\lambda) 2\pi e^{-\lambda(t-s)}$ as $\delta \to +0$ and we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ist}}{\mu + it} e^{i\lambda t}\, dt = \begin{cases} e^{-\mu(\lambda - s)} & \text{if } \lambda \geq s \\ 0 & \text{otherwise.} \end{cases}$$

Consequently the relation $\theta(\lambda - a) e^{-\lambda\mu} \theta(\lambda - b) e^{-\lambda\nu} = \theta(\lambda - a \vee b) e^{-\lambda(\mu+\nu)}$ is converted into

$$\int_{-\infty}^{\infty} \frac{e^{(\mu - is)a}}{\mu + is} \frac{e^{(\nu - i(t-s))b}}{\nu + i(t-s)}\, ds = 2\pi \frac{e^{(\mu+\nu-it)(a\vee b)}}{\mu + \nu + it}.$$

Let ω be a weight on M. The left multiplication of ω^{it} on $\mathcal{H}$ has a kernel $(1 - [\omega])\mathcal{H}$ and its restriction to the supporting subspace $[\omega]\mathcal{H}$ is identified with the one-parameter group of translations on $L^2(\mathbb{R}) \otimes [\omega] L^2(M)$ under the unitary map

$$L^2(\mathbb{R}) \otimes [\omega] L^2(M) \ni f \otimes \xi \mapsto \int_{\mathbb{R}} f(t) \omega^{it} \xi \in [\omega]\mathcal{H} = \int_{\mathbb{R}} [\omega] M(it + 1/2)\, dt.$$

In terms of a spectral decomposition $\omega^{it} = \int_{\mathbb{R}} e^{it\tau} E(d\tau)$ in N,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\mu + it} \omega^{it}\, dt = \int_{[0,\infty)} e^{-\lambda\mu}\, E(d\lambda)$$

for $\mu \in (0,\infty) + i\mathbb{R}$ gives a bounded multiplicative family of elements in N, which is also denoted by $(1 \vee \omega)^{-\mu}$. Note that there is no need for worrying about integral boundaries because $E(\cdot)$ has no point spectrum.

Clearly $\big((1 \vee \omega)^{-\mu}\big)^* = (1 \vee \omega)^{-\overline{\mu}}$ and, when $\mu > 0$, positive $(1 \vee \omega)^{-\mu}$ is decreasing in μ and converges to the support projection $[1 \vee \omega]$ of $(1 \vee \omega)^{-\mu}$ in strong operator topology.

Theorem 9.23 (Haagerup's trace formula)**.** The trace of a positive operator $(1 \vee \omega)^{-\mu}$ with $\mu \geq -1$, which belongs to N_+ for $\mu \geq 0$ and is affiliated to N for $-1 < \mu < 0$, is given by

$$\left\langle \int_{\mathbb{R}} \frac{1}{\mu + it} \omega^{it}\, dt \right\rangle = \frac{\omega(1)}{\mu + 1}.$$

Notice that the right hand side is obtained by first taking a formal analytic continuation of the integrand to $t = -i$ and then by evaluating the result (which belongs to M_*) at $1 \in M$.

Moreover, when $\omega \in M_*^+$, the closed normal operator $(1 \vee \omega)^{-\mu}$ is in the trace class of N and the above trace formula remains valid for any $\mu \in (-1, \infty) + i\mathbb{R}$.

Proof. For $f \in \mathcal{N} \subset \mathcal{H}$, we observe that

$$\begin{aligned}\big((1 \vee \omega)^{-\mu} f\big)(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\mu + is} \omega^{is} f(t - s - i/2)\, ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\mu + is + 1/2} \omega^{is+1/2} f(t - s)\, ds = (\xi_\mu f)(t),\end{aligned}$$

where

$$\xi_\mu = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\mu + it + 1/2} \omega^{it+1/2}\, dt, \quad \Re\mu > -1/2$$

belongs to $\mathcal{H}$ if and only if $\omega \in M_*^+$. Note that, when $\omega \in M_*^+$, $\xi_\mu \to \xi_0$ in norm topology. The trace of $(1 \vee \omega)^{-r-is} = (1 \vee \omega)^{-r/2}(1 \vee \omega)^{-is-r/2}$ is now calculated by

$$\begin{aligned}(\xi_{r/2} | \xi_{is+r/2}) &= \frac{\omega(1)}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{1}{(r+1)/2 - it} \frac{1}{(r+1)/2 + i(t+s)}\, dt \\ &= \frac{\omega(1)}{(2\pi)^2} \oint_{t=-s+i(r+1)/2} \frac{1}{(r+1)/2 - it} \frac{1}{(r+1)/2 + i(t+s)}\, dt \\ &= \frac{\omega(1)}{2\pi(r + is + 1)}.\end{aligned}$$

□

Example 9.24. Let $M = \mathcal{B}(\mathcal{H})$ with the standard trace denoted by tr. For a trace class operator ρ (resp. a Hilbert-Schmidt class operator x) on $\mathcal{H}$, the associated element in M_* (resp. in $L^2(M)$) is then denoted by $\rho\mathrm{tr} = \mathrm{tr}\rho$ (resp. $x\mathrm{tr}^{1/2} = \mathrm{tr}^{1/2}x$). The trace tr is regarded as a weight with respect to any resolution of the identity $1 = \sum_{j \in J} e_j$ ($e_j\mathcal{H}$ being separable for any $j \in J$) and the associated one-parameter group of unitaries $(\mathrm{tr})^{it}$ is independent of the choice of resolutions.

A section $\{f(t)\}$ of $\{M(it)\}$ is then L^2-analytic if it is of the form $f(t) = x(t)\rho^{it}(\mathrm{tr})^{it}$, where $\rho \geq 0$ is a trace class operator and both $x(t)$ and $\rho^{-it}x(t)\rho^{it}$ are analytically extended to bistrongly continuous bounded M-valued functions $x(z)$ and $\rho^{-iz}x(z)\rho^{iz}$ of $z \in i[-1/2, 0] + \mathbb{R}$ respectively. It is then uniformly integrable if

$$\sup\{\|x(t - ir)\| \vee \|\rho^{-r}x(t - ir)\rho^r\|; 0 \leq r \leq 1/2\}$$

is an integrable function of $t \in \mathbb{R}$, which is assumed in the following.

The vector $f\tau^{1/2} \in L^2(N)$ then takes the form $\oint_{\mathbb{R}} \xi(t)\, dt$ with

$$\xi(t) = f(t - i/2) = x(t - i/2)\rho^{it+1/2}\mathrm{tr}^{it+1/2} \in M(it + 1/2)$$

and the trace of $f^*f \in \mathcal{N}^2 \subset N$ is calculated by

$$\int_{\mathbb{R}} (\xi(t)|\xi(t))\, dt = \int_{-\infty}^{\infty} \mathrm{tr}(x(t - i/2)^* x(t - i/2)\rho)\, dt.$$

Now set $\omega = \rho\mathrm{tr} \in M_*^+$ and consider $\omega^{is} = \rho^{is}(\mathrm{tr})^{is} \in M(is) \subset N$, which acts on $L^2(N)$ by $(\omega^{is}\xi)(t) = \rho^{is}\xi(t - s)\mathrm{tr}^{is}$. To get the spectral decomposition, we introduce the relevant Fourier transform $L^2(N) \ni \xi \mapsto \widehat{\xi} \in L^2(\mathbb{R}, L^2(M)) = L^2(\mathbb{R}) \otimes L^2(M)$ by

$$\widehat{\xi}(\tau) = \int_{\mathbb{R}} e^{it\tau}\xi(t)(\mathrm{tr})^{-it}\, dt.$$

Then

$$\widehat{\omega^{is}\xi}(\tau) = e^{is\tau}\rho^{is}\widehat{\xi}(\tau), \quad \widehat{\theta_s\xi}(\tau) = \widehat{\xi}(\tau - s)$$

and the spectral decomposition of ω^{is} is reduced to that of ρ: Let $\rho = \sum_j \rho_j e_j$ be a spectral expression of ρ with $\rho_j > 0$ and $\{e_j\}$ mutually orthogonal finite-rank projections in $M = \mathcal{B}(\mathcal{H})$.

Thus ω^{is} is realized as a diagonalizable operator on $L^2(\mathbb{R}, L^2(M))$ by

$$\mathbb{R} \ni \tau \mapsto \sum_j e^{is\tau}\rho_j^{is} e_j \in M$$

and one sees that

$$(1 \vee \omega)^{-\mu} = \oint_{\mathbb{R}} d\tau \sum_j 1_{[-\log\rho_j, \infty)}(\tau) e^{-\mu\tau}\rho_j^{-\mu} e_j.$$

Since ω^{is} for various ρ and $s \in \mathbb{R}$ generates N, $N = L^\infty(\mathbb{R}) \otimes M$ on $L^2(\mathbb{R}) \otimes L^2(M)$. The trace on N is now identified with $e^{-\tau}d\tau \otimes \mathrm{tr}$:

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\, e^{-\tau} 1_{[-\log\rho_j,\infty)}(\tau) e^{-\mu\tau}\rho_j^{-\mu}\mathrm{tr}(e_j) &= \int_{-\log\rho_j}^{\infty} d\tau\, e^{-\tau}e^{-\mu\tau}\rho_j^{-\mu}\mathrm{tr}(e_j) \\ &= \frac{1}{\mu + 1}\rho_j \mathrm{tr}(e_j),\end{aligned}$$

which is summed up for j to coincide with $\mathrm{tr}(\rho)/(\mu + 1) = \langle(1 \vee \omega)^{-\mu}\rangle$. Since $(1 \vee \omega)^{-\mu}$ for various ρ and $\mu > -1$ also genetes N, the identification is justified.

The effect of θ_s on $N = L^\infty(\mathbb{R}) \otimes M$ is just the translational shift $(\theta_s x)(\tau) = x(\tau - s)$ and the identity $\theta_s((1 \vee \omega)^{-\mu}) = (1 \vee e^{-s}\omega)^{-\mu}$ is checked explicitly by

$$\oint_{\mathbb{R}} d\tau \sum_j 1_{[-\log\rho_j,\infty)}(\tau - s)e^{-\mu(\tau-s)}\rho_j^{-\mu}e_j$$
$$= e^{\mu s}\oint_{\mathbb{R}} d\tau \sum_j 1_{[s-\log\rho_j,\infty)}(\tau)e^{-\mu\tau}\rho_j^{-\mu}e_j.$$

Haagerup's ingeneous observation is that the whole $L^p(M)$'s are realized as measurable operators on $\mathcal{H}$.

Let $h \geq 0$ be an N-measurable operator satisfying $\theta_s(h) = e^{-s}h$ for $s \in \mathbb{R}$. and $e = [1 \vee h]$ be the support projection of $1 \vee h$. By the relative invariance of h, $\theta_s(e)$ is then the support projection of $e^s \vee h$ and we have a Stieltjes integral representation of h:

$$h = -\int_{-\infty}^{\infty} e^s\, d\theta_s(e) = \int_{-\infty}^{\infty} e^{-s} d\theta_{-s}(e).$$

Thus we have a one-to-one correspondence between densely defined positive self-adjoint operator h and a projection $e \in N$ satisfying $\theta_s(e) \leq e$ for $s \geq 0$.

The multiplicative family $\{(1\vee h)^{-\mu}\}_{\Re\mu>-1}$ of τ_M-measurable normal operators is in the trace class of N and, for $\Re\mu > -1/2$, we have

$$(1 \vee h)^{-\mu}\tau^{1/2} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\mu + it + 1/2}h^{it+1/2}\, dt = (1 \vee h)^{-\mu-1/2}h^{1/2}.$$

To see this, we need some KMS-property for h^{it}.

Let $x \in M$ and compute

$$\begin{aligned}\langle hx(1 \vee h)^{-\mu}\rangle &= \langle x(1 \vee h)^{-\mu}h\rangle = \langle x(1 \vee h)^{1-\mu}\rangle \\ &= -\int_0^{\infty} e^{(1-\mu)s}d\langle x\theta_s(e)\rangle = -\int_0^{\infty} e^{(1-\mu)s}d(e^{-s})\langle xe\rangle \\ &= \langle xe\rangle \int_0^{\infty} e^{-\mu s}ds = \frac{1}{\mu}\langle xe\rangle.\end{aligned}$$

This result for $\Re\mu > 0$ is compatible with

$$\begin{aligned}\int_{\mathbb{R}} \theta_s(x(1 \vee h)^{-\mu})\, ds &= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\mu + it}x\theta_s(h^{it})\, dsdt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\mu + it}e^{-ist}xh^{it}\, dsdt \\ &= \int_{\mathbb{R}} \frac{1}{\mu + it}xh^{it}\delta(t)\, dt = \frac{1}{\mu}x[h]\end{aligned}$$

We claim that $\sigma_t(x) = h^{it}xh^{-it}$ satisfies the KMS-condition for $\varphi(x) = \langle xe\rangle$. First notice that $\sigma_t(x) \in M$ by the θ-invariance and $[h] = [\varphi]$.

In fact, from the definition of φ and the faithfulness of the standard trace, $(1-[\varphi])e = 0$, which means that $e \leq [\varphi]$ and then $[h] = \lim_{s\to-\infty}\theta_s(e) \leq \theta_s([\varphi]) = [\varphi]$. Conversely, from $(1-[h])e = 0$, $1-[h] \leq 1-[\varphi]$ gives the reverse inequality.

Let $x \in M$ and consider

$$\varphi(x^*\sigma_t(x)) = \langle x^*h^{it}xh^{-it}e\rangle.$$

The Stieltjes integral expression for h is used in $xh^{-it}e = -\int_0^\infty e^{-ist}d\theta_s(xe)$ to get

$$\begin{aligned}
-\varphi(x^*\sigma_t(x)) &= \int_0^\infty e^{-ist}d\langle x^*h^{it}\theta_s(xe)\rangle \\
&= \int_0^\infty e^{-ist}d\langle\theta_s(x^*\theta_{-s}(h^{it})xe)\rangle = \int_0^\infty e^{-ist}d(e^{-s}e^{ist})\langle x^*h^{it}xe\rangle \\
&= (it-1)\langle x^*h^{it}xe\rangle
\end{aligned}$$

and then

$$-\langle x^*h^{it}xe\rangle = \int_{-\infty}^0 e^{ist}d\langle x^*\theta_s(e)xe\rangle + \int_0^\infty e^{ist}d\langle x^*\theta_s(e)xe\rangle$$

with

$$\begin{aligned}
\int_0^\infty e^{ist}d\langle x^*\theta_s(e)xe\rangle &= \int_0^\infty e^{ist}d(e^{-s}\langle x^*ex\theta_{-s}(e)\rangle) \\
&= \int_0^\infty e^{ist}e^{-s}d\langle x^*ex\theta_{-s}(e)\rangle + \int_0^\infty e^{ist}e^{-s}\langle x^*ex\theta_{-s}(e)\rangle\,ds
\end{aligned}$$

reveals that $-\langle x^*h^{it}xe\rangle$ is anlaytically extended to a bounded continuous function

$$\begin{aligned}
-\langle x^*h^{iz}xe\rangle &= \int_{-\infty}^\infty e^{isz}d\langle x^*\theta_s(e)xe\rangle = \int_{-\infty}^\infty e^{isz}d(e^{-s}\langle x^*ex\theta_{-s}(e)\rangle) \\
&= \int_{-\infty}^\infty e^{isz}e^{-s}d\langle x^*ex\theta_{-s}(e)\rangle - \int_{-\infty}^\infty e^{isz}e^{-s}\langle x^*ex\theta_{-s}(e)\rangle\,ds.
\end{aligned}$$

of $z = t - ir \in \mathbb{R} + i[-1,0]$. Note that $\langle x^*ex\theta_{-s}(e)\rangle$ ($\langle x^*\theta_s(e)xe\rangle$) is positive, increasing (decreasing) and continuous in $s \in \mathbb{R}$, whence $d\langle x^*ex\theta_{-s}(e)\rangle$ and $-d\langle x^*\theta_s(e)xe\rangle$ give positive finite measures on $\mathbb{R}$.

With helpf of integration-by-parts, we get

$$\begin{aligned}\varphi(x^*\sigma_{t-ir}(x)) &= (it+r-1)\int_{-\infty}^{\infty} e^{(it+r-1)s} d\langle x^*ex\theta_{-s}(e)\rangle \\ &\qquad - (it+r-1)\int_{-\infty}^{\infty} e^{(it+r-1)s}\langle x^*ex\theta_{-s}(e)\rangle\, ds \\ &= (it+r)\int_{-\infty}^{\infty} e^{(it+r-1)s} d\langle x^*ex\theta_{-s}(e)\rangle \\ &\qquad - \Big[e^{(it+r-1)s}\langle x^*ex\theta_{-s}(e)\rangle\Big]_{-\infty}^{\infty}.\end{aligned}$$

For $0<r<1$, we see

$$\lim_{s\to\infty} e^{(it+r-1)s}\langle x^*ex\theta_{-s}(e)\rangle = 0,$$
$$\lim_{s\to-\infty} e^{(it+r-1)s}\langle x^*ex\theta_{-s}(e)\rangle = \lim_{s\to-\infty} e^{(it+r)s}\langle x^*\theta_s(e)x(e)\rangle = 0$$

at the boundary values and therefore

$$\varphi(x^*\sigma_{t-ir}(x)) = (it+r)\int_{-\infty}^{\infty} e^{(it+r-1)s} d\langle x^*ex\theta_{-s}(e)\rangle.$$

Since both sides are continuous in $r\in[0,1]$, the equality holds at the boundaries as well. We compare this expression with

$$\begin{aligned}\varphi(\sigma_t(x)x^*) = \langle eh^{it}xh^{-it}x^*\rangle &= -\int_0^{\infty} e^{ist} d\langle\theta_s(e)xh^{-it}x^*\rangle \\ &= -\int_0^{\infty} e^{ist} d\langle\theta_s\big(ex\theta_{-s}(h^{-it})x^*\big)\rangle \\ &= -\int_0^{\infty} e^{ist} d(e^{-s-ist})\langle exh^{-it}x^*\rangle \\ &= \langle exh^{-it}x^*\rangle(it+1) \\ &= -(it+1)\int_{-\infty}^{\infty} e^{-ist} d\langle ex\theta_s(e)x^*\rangle \\ &= (it+1)\int_{-\infty}^{\infty} e^{ist} d\langle ex\theta_{-s}(e)x^*\rangle\end{aligned}$$

to conclude that $\varphi(x^*\sigma_{t-i}(x)) = \varphi(\sigma_t(x)x^*)$ for $t\in\mathbb{R}$.

So far we have checked that $h^{it}xh^{-it} = \varphi^{it}x\varphi^{-it}$ for $x\in[\varphi]M[\varphi]$. Then $u(t) = h^{it}\varphi^{-it}$ is a central unitary in $[\varphi]M[\varphi]$. Since φ^{it} commute with the reduced center, $u(t)$ is a one-parameter group of unitaries in the reduced algebra. Let $u(t) = \int_{\mathbb{R}} e^{it\tau}\, E(d\tau)$ be the spectral decomposition in $[\varphi]M[\varphi]$. Then $a_n = \int_{[-n,n]} e^{\tau/2}\, E(d\tau)$ is an increasing sequence of positive elemnts in the reduced center and $\varphi_n = a_n\varphi a_n \in M_*^+$

satisfies $\varphi_n^{it} = h^{it}[a_n] = [a_n]h^{it}$ for $t \in \mathbb{R}$. Set $h_n = h[a_n] = [a_n]h$, which is also N-measurable and satisfies $\theta_s(h_n) = e^{-s}h_n$. From the equalities

$$\frac{\varphi_n(x)}{\mu} = \langle x(1\vee\varphi_n)^{1-\mu}\rangle = \langle x(1\vee h_n)^{1-\mu}\rangle = \langle x[a_n](1\vee h)^{1-\mu}\rangle = \frac{\varphi(x[a_n])}{\mu}$$

for $x \in M$, one sees that $\varphi_n = \varphi[a_n] = [a_n]\varphi$ and then $\varphi_n^{it} = \varphi^{it}[a_n]$ for $t \in \mathbb{R}$. Finally we get

$$h^{it} = \lim_{n\to\infty} h_n^{it} = \lim_{n\to\infty} \varphi^{it}[a_n].$$

We next check the additivity of the correspondence $h_\varphi \leftrightarrow \varphi$. For this, we first establish the averaging relation: Let $f(t) = 1/(\mu + it)$ with $\mu > 0$. Usefulness of these functions is in their simple behavior under taking convolution products. Recall that $\int_{\mathbb{R}} f(t)\omega^{it}\,dt$ belongs to B_+ with

$$\int_{\mathbb{R}} dt\,\frac{1}{\mu+it}\omega^{it}\tau^{1/2} = \frac{1}{2\pi}\oint_{\mathbb{R}} \frac{1}{\mu+it+1/2}\omega^{it+1/2}\,dt.$$

For $y = \int_{\mathbb{R}} f(t)x^*\omega^{it}x\,dt \in B_+^2$ with $x \in M$ and $\omega \in M_*^+$, we have

$$\langle h^{1/2}yh^{1/2}\rangle = 2\pi f(0)\varphi(x^*x), \qquad \int \theta_s(y)\,ds = 2\pi f(0)x^*x.$$

The first equality is checked as follows:

$$\begin{aligned}
\langle hy\rangle &= -\lim_{n\to\infty}\int_{-n}^{n} e^s d\langle\theta_s(e)y\rangle \\
&= \lim_{n\to\infty}\left(\int_{-n}^{n} e^s\langle\theta_s(e)y\rangle\,ds - e^n\langle\theta_n(e)y\rangle + e^{-n}\langle\theta_{-n}(e)y\rangle\right) \\
&= \int_{-n}^{n} ds\,e^s\int_{\mathbb{R}} dt\,f(t)\langle\theta_s(e)x^*\omega^{it}x\rangle - \langle e\theta_{-n}(y)\rangle \\
&= \int_{-n}^{n} ds\int_{\mathbb{R}} dt\,e^{ist}f(t)\langle ex^*\omega^{it}x\rangle - \int_{\mathbb{R}} dt\,e^{int}f(t)\langle ex^*\omega^{it}x\rangle.
\end{aligned}$$

Note here that $\langle ex^*\omega^{it}x\rangle = (x\xi_0|\omega^{it}x\xi_0)$ with

$$\xi_0 = e\tau^{1/2} = \frac{1}{2\pi}\oint_{\mathbb{R}} dt\,\frac{1}{it+1/2}\varphi^{it+1/2}$$

and

$$\omega^{is}x\xi_0 = \frac{1}{2\pi}\oint_{\mathbb{R}} dt\,\frac{1}{i(t-s)+1/2}\omega^{is}x\varphi^{-is}\varphi^{it+1/2}.$$

We have the following expression for

$$\begin{aligned}\langle ex^*\omega^{is}x\rangle &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dt\,\frac{1}{-it+1/2}\frac{1}{i(t-s)+1/2}(x\varphi^{it+1/2}|\omega^{is}x\varphi^{-is}\varphi^{it+1/2})\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dt\,\frac{1}{-it+1/2}\frac{1}{i(t-s)+1/2}\varphi(x^*\omega^{is}x\varphi^{-is})\\ &= \frac{1}{2\pi}\frac{1}{1-is}\varphi(x^*\omega^{is}x\varphi^{-is}),\end{aligned}$$

which shows that the function $f(t)\langle ex^*\omega^{it}x\rangle$ is integrable and we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} dt\,e^{int}f(t)\langle ex^*\omega^{it}x\rangle = 0.$$

To deal with the first term, we use the form $f = g^* * g$ ($g(t) = 1/(it+\mu/2)$) to see

$$\int_{\mathbb{R}} e^{ist}f(t)\omega^{it}\,dt = \int_{\mathbb{R}} dt'\,e^{-ist'}\overline{g(t')}\omega^{-it'}\int_{\mathbb{R}} dt\,e^{ist}g(t)\omega^{it}$$

and then

$$\begin{aligned}&\int_{\mathbb{R}} dt\,e^{ist}f(t)(x\xi_0|\omega^{it}x\xi_0)\\ &\qquad= (\int_{\mathbb{R}} dt'\,e^{ist'}g(t')\omega^{it'}x\xi_0|\int_{\mathbb{R}} dt\,e^{ist}g(t)\omega^{it}x\xi_0)\\ &\qquad= \sum_j(\int_{\mathbb{R}} dt'\,e^{ist'}g(t')\omega^{it'}x\xi_0|\delta_j)(\delta_j|\int_{\mathbb{R}} dt\,e^{ist}g(t)\omega^{it}x\xi_0)\\ &\qquad= \sum_j\int_{\mathbb{R}} dt\,e^{ist}(F_j^* * F_j)(t) = \sum_j|\widehat{F_j}(s)|^2,\end{aligned}$$

where $F_j(t) = g(t)(\delta_j|\omega^{it}x\xi_0)$ and $\widehat{F_j}(s) = \int_{\mathbb{R}} e^{ist}F_j(t)\,dt$ belong to $L^2(\mathbb{R})$. Thus the Fourier transform of $f(t)\langle ex^*\omega^{it}x\rangle$ is integrable and we get the first equality

$$\langle hy\rangle = \int_{-\infty}^{\infty} ds\int_{\mathbb{R}} dt\,e^{ist}f(t)\langle ex^*\omega^{it}x\rangle = 2\pi f(0)(x\xi_0|x\xi_0) = 2\pi f(0)\langle ex^*x\rangle.$$

Similarly and more easily, the second equality follows from

$$\begin{aligned}\int_{\mathbb{R}}(\xi|\theta_s(y)\xi) &= \int ds\int dt\,f(t)(x\xi|\theta_s(\omega^{it})x\xi)\\ &= \int ds\int dt\,e^{-ist}f(t)(x\xi|\omega^{it}x\xi)\\ &= 2\pi f(0)(\xi|x^*x\xi)\end{aligned}$$

for each $\xi\in L^2(N)$.

Remark 5.

$$
\begin{aligned}
\langle (1 \vee h)^{1-\mu} y \rangle &= - \lim_{n\to\infty} \int_0^n e^{(1-\mu)s} d\langle \theta_s(e) y\rangle \\
&= \lim_{n\to\infty} (1-\mu) \int_0^n e^{(1-\mu)s} \langle \theta_s(e) y \rangle \, ds - \lim_{n\to\infty} e^{-\mu n} \langle e\theta_{-n}(y)\rangle + \langle ey\rangle \\
&= (1-\mu) \int_0^\infty ds\, e^{-\mu s} \int_{\mathbb{R}} dt\, f(t) e^{ist} \langle ex\omega^{it} x^* \rangle + \langle ey \rangle \\
&= \frac{1-\mu}{2\pi} \int_0^\infty ds\, e^{-\mu s} \int_{\mathbb{R}} dt\, e^{ist} \frac{f(t)}{1-it} \varphi(x^* \omega^{it} x \varphi^{-it}) + \langle ey \rangle \\
&= \frac{1-\mu}{2\pi} \int_{\mathbb{R}} dt\, \frac{f(t)}{1-it} \varphi(x^* \omega^{it} x \varphi^{-it}) \int_0^\infty ds\, e^{(it-\mu)s} + \langle ey \rangle \\
&= \frac{1-\mu}{2\pi} \int_{\mathbb{R}} dt\, \frac{f(t)}{1-it} \varphi(x^* \omega^{it} x \varphi^{-it}) \frac{1}{\mu - it} \\
&\quad + \frac{1}{2\pi} \int_{\mathbb{R}} dt\, \frac{f(t)}{1-it} \varphi(x^* \omega^{it} x \varphi^{-it}) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(t)}{\mu - it} \varphi(x^* \omega^{it} x \varphi^{-it})\, dt.
\end{aligned}
$$

More generally, if $a \in N$ satisfies $\theta_t(a) \leq a$ for $t \geq 0$, then $\theta_t(a)$ is decreasing in $t \in \mathbb{R}$ and the sum

$$
-\sum_{j=1}^n e^{t_{j-1}} \big(\theta_{t_j}(a) - \theta_{t_{j-1}}(a) \big)
$$

for a division $s_0 \leq s_1 \leq \cdots \leq s_n$ is increasing in refinement, whence it converges to a not necessarily densely defined positive operator

$$
-\int_{-\infty}^\infty e^t \, d\theta_t(a) = - \lim_{r\to\infty} \int_{-\infty}^r e^t \, d\theta_t(a).
$$

Note that the convergence is in the strong operator topology in N for the bounded operator part $-\displaystyle\int_{-\infty}^r e^t \, d\theta_t(a) \in N_+$ and the relation

$$
\theta_s \left(\sum_{j=1}^n e^{t_{j-1}} \big(\theta_{t_j}(a) - \theta_{t_{j-1}}(a) \big) \right) = \sum_{j=1}^n e^{t_{j-1}} \big(\theta_{t_j+s}(a) - \theta_{t_{j-1}+s}(a) \big)
$$

passes into

$$
\theta_s \left(\int_{-\infty}^\infty e^t \, d\theta_t(a) \right) = e^{-s} \int_{-\infty}^\infty e^t \, d\theta_t(a),
$$

In view of the associated positive quadratic form on $L^2(N)$, we see that it is densely defined if and only if the form domain specified by

$$-(\xi|\left(\int_{-\infty}^{\infty} e^t\, d\theta_t(a)\right)\xi) = -\int_{-\infty}^{\infty} e^t\, d(\xi|\theta_t(a)\xi) < \infty$$

is dense in $L^2(N)$. A densely-defined positive operator of this kind is then N-measurable if $\tau([a]) < \infty$.

Theorem 9.25 (Haagerup correspondence)**.** There is a linear isomorphism between M_* and the linear space of N-measurable operators h on $L^2(N)$ satisfying $\theta_s(h) = e^{-s}h$ and so that $\varphi \in M_*^+$ corresponds to the analytic generator h_φ of the one-parameter group $\{\varphi^{it}\}$ of partial isometries in N.

Moreover the correspondence preserves N^*-bimodule structures as well as positivity.

Proof. The correspondence is already established for positive parts, which is semilinear by

$$\frac{2\pi}{\mu}\phi(x^*x) = \langle (h_\varphi + h_\psi)x^*(1\vee\omega)^{-\mu}x\rangle = \frac{2\pi}{\mu}(\varphi(x^*x) + \psi(x^*x)).$$

Here $\varphi, \psi \in M_*^+$ and $\phi \in M_*^+$ is specified by $h_\phi = h_\varphi + h_\psi$. □

Here we filled up the key ingredients in non-commutative L^p-theory without referreing to weight theory. Note that the additivity is comparably easy once the surjectivity of the positive part together with the averaging relation

$$\tau(hy) = \varphi\left(\int_{\mathbb{R}} \theta_s(y)\, ds\right) \quad \text{for } y \in N_+$$

is established. All these are handled with the theory of operator-valued weights and we refer to the Haagerup's papers for the original method.

9.3. **Modular Algebras.** The Haagerup's correspondence is immediately extended to an embedding of $M(1+it)$ into $\widetilde{N}$ by $h_{x\phi^{1+it}} = xh_\phi^{1+it}$, which is well-defined and compatible with the *-bimodule structures of $M(i\mathbb{R})$ on $M(1+i\mathbb{R})$ and $\widetilde{N}$, thanks to the relation $h_\varphi^{it}h_\psi^{-it} = \varphi^{it}\psi^{-it}$.

We now consider the Hilbert spaces $M(it+1/2)$. We set $\widetilde{N}(z) = \{y \in \widetilde{N}; \theta_s(y) = e^{-sz}y, \forall s \in \mathbb{R}\}$ for $z \in [0,\infty) + i\mathbb{R}$. Note that, for $\varphi \in M_*^+$, $h_\varphi^z \in \widetilde{N}(z)$.

Lemma 9.26. If $\varphi \leq \psi$ and $z \in [0, 1/2] + i\mathbb{R}$, $h_\varphi^z = (\varphi^z\psi^{-z})h_\psi^z = h_\psi^z(\psi^{-z}\varphi^z)$.

9.4. **Complex Interpolation.** Given a positive Haagerup operator h on $L^2(N)$ and $a, b \in M$, $\phi^{-it} a h^{it} b \in M$ and hence $\phi^{1-it} a h^{it} b \in M_*$ Claim that $\langle \phi^{1-it} a h^{it} b \rangle$ is analytically continued to $z \in i[-1, 0] + \mathbb{R}$ so that

$$|\langle \phi^{1-iz} a h^{iz} b \rangle| \leq \phi(1)^{1-r} \tau(e)^r \|a\| \|b\|$$

for $z = t - ir$.

$$\|\phi^{1/2-r-it} a_0 \phi_1^{z_1} a_1 \cdots \phi_n^{z_n} a_n\| / \phi(1)^r$$

where $z_1 + \cdots + z_n = r + it$ with $\Re z_j \geq 0$ and $r \leq 1/2$.

Consider the vector space of finitely supported one-analytic sections of $M(i\mathbb{R})$. Introduce a seminorm by

$$\sup_{\phi \in M_*^+} \|\phi^{1-r-it} x(t - ir)\| / \phi(1)^r.$$

For a partial isometry $u \in M$ satisfying $u[\phi] = u$, the identity

$$\langle u \phi^{1-it} x(t) \rangle = \langle x(t) u \phi^{1-it} \rangle = \langle x(t) (u \phi u^*)^{1-it} u \rangle$$

is analytically continued to

$$\langle u \phi^{1-r-it} x(t - ir) \rangle = \langle x(t - ir) (u \phi u^*)^{1-r-it} u \rangle.$$

Since $u\phi u^*(1) = \phi(u^* u) \leq \phi(1)$, this implies

$$\begin{aligned} |\langle u \phi^{1-r-it} x(t - ir) \rangle| / \phi(1)^r &\leq |\langle x(t - ir) (u \phi u^*)^{1-r-it} u \rangle| / (u \phi u^*)(1)^r \\ &\leq \|x(t - ir) (u \phi u^*)^{1-r-it}\| / (u \phi u^*)(1)^r \end{aligned}$$

and, by taking supremum first on u and then on ϕ,

$$\sup_{\phi \in M_*^+} \|\phi^{1-r-it} x(t - ir)\| / \phi(1)^r \leq \sup_{\phi \in M_*^+} \|x(t - ir) \phi^{1-r-it}\| / \phi(1)^r.$$

By symmetry, we have also the reverse inequality and they coincide.

For $\phi_j \in M_*^+$ and $a_j \in M$ $(j = 0, 1, \ldots, n)$, let

$$\langle \phi_0^{z_0} a_0 \phi_1^{z_1} a_1 \phi_2^{z_2} \ldots a_{n-1} \phi_n^{z_n} a_n \rangle \quad \text{with } z_0 = 1 - z_1 - \cdots - z_n$$

be the analytic continuation of

$$\langle \phi_0^{1-i(t_1+\cdots+it_n)} a_0 \phi_1^{it_1} a_1 \phi_2^{it_2} \ldots a_{n-1} \phi_n^{it_n} a_n \rangle$$

to the tube domain $I^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n; \Re z_j \geq 0, \sum_j \Re z_j \leq 1\}$.

Let $f(it_1, \cdots, it_n) \in M(i(t_1 + \cdots + t_n))$ be a norm-bounded function of $(it_1, \cdots, it_n) \in i\mathbb{R}^n$ and assume that

(i) $\phi^{-it} f(t_1, \cdots, t_n) \in M$ and $f(t_1, \cdots, t_n) \phi^{-it} \in M$ are weak*-continuous and

(ii) $\phi^{1-i(t_1+\cdots+t_n)} f(it_1, \cdots, it_n) \in M_*$ and $\phi^{1-i(t_1+\cdots+t_n)} f(it_1, \cdots, it_n) \in M_*$ are analytically extended to bounded norm-continuous functions of $z = (z_1, \cdots, z_n) \in I^n$.

Given $0 \neq (r_1, \cdots, r_n) \in [0, \infty)^n$,

$$F(z) = f\left(\frac{r_1}{r_1 + \cdots + r_n} z, \cdots, \frac{r_n}{r_1 + \cdots + r_n} z\right)$$

We now choose and fix a faithful φ (with a suffix φ being abbreviated for modular stuffs like $\sigma_t^\varphi = \sigma_t$) for the time being to do operator algebraic analysis.

9.5. **Miscellanea.** Consider the following condition on an M-valued weak* continuous function $x(t)$ of $t \in \mathbb{R}$:

(i) The function $\mathbb{R}^2 \ni (s,t) \mapsto \sigma_s(x(t)) \in M$ is extended to a weak* continuous analytic function on $\mathbb{C} \times (\mathbb{R} - i[0,1])$.

(ii) Given any $\rho > 0$, we can find $\epsilon > 0$ such that

$$\sup_{|s| \leq \rho, 0 \leq r \leq 1} \|\sigma_{is}(x(t - ir))\| = O(e^{-\epsilon t^2}).$$

Recall that weak* continuity implies local boundedness in norm.

For example, $x(t) = f(t)a$ with a an entirely analytic element and $f(t) = \alpha e^{-\beta t^2 + \gamma t}$ ($\alpha, \gamma \in \mathbb{C}$, $\beta > 0$) satisfies the condition.

Let $\mathcal{N}$ be the totality of functions fulfilling the condition. Under the interpretation of $x(t)$ as representing an expression

$$\int_{\mathbb{R}} x(t) \varphi^{it} \, dt,$$

$\mathcal{N}$ is made into a *-algebra by

$$x^*(t) = \sigma_t(x(-t))^*, \quad (xy)(t) = \int_{\mathbb{R}} x(s) \sigma_s(y(t-s)) \, ds.$$

The analytic extension of $\sigma_s(x^*(t))$ is given by $\sigma_z(x^*(w)) = \sigma_{z+w}(x(-\overline{w})^*)$ and it satisfies the Gaussian decay condition. The analytic extension of $\sigma_s((xy)(t))$ is given by

$$\sigma_z((xy)(w)) = \int_{\mathbb{R}} \sigma_z(x(s)) \sigma_{z+s}(y(w-s)) \, ds,$$

where the exponentially decaying property of relevant functions ensures that the right hand side is a well-defined Bochner integral and it varies norm-continuously in $z \in \mathbb{C}$ and $w \in \mathbb{R} - i[0,1]$. Note that the exponential decay is uniform for local changes of z and w. Moreover,

$$\begin{aligned} \|\sigma_z((xy)(t - ir))\| &\leq \int_{\mathbb{R}} \|\sigma_z(x(s))\| \|\sigma_z(y(t - s - ir))\| \, ds \\ &\leq C \int_{\mathbb{R}} e^{-\alpha s^2} e^{-\beta(t-s)^2} \, ds = C \sqrt{\frac{\pi}{\alpha + \beta}} e^{-\frac{\alpha\beta}{\alpha+\beta} t^2} \end{aligned}$$

shows that xy inherits the Gaussian decay property.

Note that the *-algebra $\mathcal{N}$ allows elements in $\mathcal{M}\varphi^{is} = \varphi^{is}\mathcal{M}$ as multipliers: For $a \in \mathcal{M}$ and $s \in \mathbb{R}$,

$$a\varphi^{is}\Big(\int x(t)\varphi^{it}\,dt\Big) = \int a\sigma_s(x(t-s))\varphi^{it}\,dt,$$
$$\Big(\int x(t)\varphi^{it}\,dt\Big)\,a\varphi^{is} = \int x(t-s)\sigma_{t-s}(a)\varphi^{it}\,dt.$$

We now introduce a linear functional $\langle\cdot\rangle : \mathcal{N} \to \mathbb{C}$ by

$$\langle\int x(t)\varphi^{it}\,dt\rangle = \varphi(x(-i)).$$

$$\begin{aligned}
\langle x^*x\rangle &= \varphi\Big(\int_{\mathbb{R}} x^*(s)\sigma_s(x(-i-s))\,ds\Big) = \int_{\mathbb{R}} \varphi\Big(\sigma_s(x(-s)^*)\sigma_s(x(-i-s))\Big)\,ds\\
&= \int_{\mathbb{R}} \varphi\Big(x(-s)^*x(-i-s)\Big)\,ds = \int_{\mathrm{Im}\,\zeta=-1/2} \varphi\Big(x(-\overline{\zeta})^*x(-i-\zeta)\Big)\,d\zeta\\
&= \int_{\mathbb{R}} \varphi\Big(x(-s-i/2)^*x(-s-i/2)\Big)\,ds\\
&= \int_{\mathbb{R}} \varphi\Big(x(s-i/2)^*x(s-i/2)\Big)\,ds \geq 0.
\end{aligned}$$

Here Cauchy's integral theorem is used in the second line.

$$\begin{aligned}
\langle xy\rangle &= \int \varphi\Big(x(s)\sigma_s(y(-i-s))\Big)\,ds = \int \varphi\Big(x(t-i)\sigma_{t-i}(y(-t))\Big)\,dt\\
&= \int \varphi\Big(\sigma_t(y(-t))x(t-i)\Big)\,dt = \int \varphi\Big(y(-t)\sigma_{-t}(x(t-i))\Big)\,dt\\
&= \int \varphi\Big(y(t)\sigma_t(x(-t-i))\Big)\,dt = \langle yx\rangle.
\end{aligned}$$

Again Cauchy's integral theorem is used in the first line.

Now $\mathcal{N}$ is completed relative to the inner product $\langle x^*y\rangle$ to get a Hilbert space $L^2(\mathcal{N})$, which is obviously identified with $L^2(\mathbb{R}) \otimes L^2(M)$ by the correspondence

$$\int y(t)\varphi^{it}\,dt \mapsto \oint_{\mathbb{R}} y(t-i/2)\varphi^{1/2}\,dt = \int_{\mathbb{R}} y(t-i/2)\varphi^{1/2+it}\,dt.$$

Here we put the dummy syombol φ^{it} at the last expression to indicate that it is realized as a formal analytic continuation.

Moreover, the left multiplication of $\int x(t)\varphi^{it}\,dt$ transforms the $L^2(M)$-valued function $\eta(t) = y(t - i/2)\varphi^{1/2}$ into a function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}} ds\, x(s)\Delta^{is}\eta(t-s) \in L^2(M),$$

which is consequently bounded with a bound $\displaystyle\int_{\mathbb{R}} \|x(s)\|\,ds$.

10. Reduction Theory

The von Neumann's disintegration theory on representations. Dixmier, Pedersen, Reed-Simon §IV.5, Bratteli-Robinson §4.4.1.

10.1. **Commutative Ampliations.** We shall start with describing $L^2(\Omega,\mu)\otimes\mathcal{H}$ in terms of $\mathcal{H}$-valued measurable functions.

Each $\xi \in L^2(\Omega,\mu)\otimes\mathcal{H}$ has an expression

$$\xi = \lim_{n\to\infty}\sum_{j=1}^{N_n} f_j^{(n)}\otimes\delta_j^{(n)}$$

with $f_j^{(n)} \in L^2(\Omega,\mu)$ and $\delta_j^{(n)} \in \mathcal{H}$, which shows that the range of partial evaluation $\{\langle\xi\rangle_{f\in L^2(\Omega,\mu)}\}$ is included in a separable closed subspace $\mathcal{H}_\xi$ of $\mathcal{H}$, i.e., $\xi \in L^2(\Omega,\mu)\otimes\mathcal{H}_\xi$. Let $\{\delta_j\}$ be an orthonormal basis in $\mathcal{H}_\xi$ and write $\xi = \sum_j f_j\otimes\delta_j$ with $f_j \in L^2(\Omega,\mu)$. Since

$$\|\xi\|^2 = \sum_j \|f_j\|^2 = \sum_j \int_\Omega |f_j(\omega)|^2\,\mu(d\omega) < \infty,$$

we see that $\{f_j(\omega)\}_{j\geq 1} \in \ell^2$ for μ-a.e. $\omega\in\Omega$ and ξ is represented by an $\mathcal{H}_\xi$-valued function of $\omega\in\Omega$ defined by

$$\xi(\omega) = \sum_j f_j(\omega)\delta_j.$$

As an $\mathcal{H}$-valued function on Ω, $\xi(\xi)$ is weakly μ-measurable in the sense that the function $\Omega \ni \omega \mapsto (\alpha|\xi(\omega))$ is μ-measurable for every $\alpha\in\mathcal{H}$.

Conversely, given an $\mathcal{H}$-valued function $\xi(\cdot)$ on Ω which is weakly μ-measurable and satisfies the separability condition on the range of ξ,

$$\|\xi(\omega)\|^2 = \sum_{j\geq 1}(\xi(\omega)|\delta_j)(\delta_j|\xi(\omega))$$

is a μ-measurable function of $\omega\in\Omega$ and the square-integrability condition

$$\int_\Omega \|\xi(\omega)\|^2\,\mu(d\omega) < \infty$$

has a meaning.

The set of $\mathcal{H}$-valued functions on Ω satisfying the weak μ-measurability, the range separability and the square-integrability is an inner product space $L^2(\Omega, \mu; \mathcal{H})$ with the inner product given by

$$(\xi|\eta) = \int_\Omega (\xi(\omega)|\eta(\omega))\, \mu(d\omega).$$

Proposition 10.1. The inner product space $L^2(\Omega, \mu; \mathcal{H})$ is complete and the correspondence $f \otimes \alpha \mapsto \xi$ with $\xi(\omega) = f(\omega)\alpha$ is extended to a unitary map of $L^2(\Omega, \mu) \otimes \mathcal{H}$ onto $L^2(\Omega, \mu; \mathcal{H})$.

From various reasons, it is reasonable to impose the separability condition on relevant Hilbert spaces for doing measure theoretical analysis further. So we shall assume the σ-finiteness on measures and the separability on Hilbert spaces in what follows.

Choose an orthonormal basis $\{\delta_j\}_{j\geq 1}$ in $\mathcal{H}$ and set $\mathcal{D} = \sum_{j\geq 1}(\mathbb{Q} + i\mathbb{Q})\delta_j$.

For $\alpha, \beta \in \mathcal{H}$, a σ-weakly continuous linear map $\langle \alpha, \beta \rangle : \mathcal{B}(L^2(\Omega, \mu) \otimes \mathcal{H}) \to \mathcal{B}(L^2(\Omega, \mu))$ is defined by the relation

$$(f|\langle \alpha, a\beta \rangle g) = (f \otimes \alpha | a(g \otimes \beta)).$$

When $a \in (L^\infty(\Omega, \mu) \otimes 1)'$, the operator $\langle \alpha, a\beta \rangle$ on $L^2(\Omega, \mu)$ commutes with $L^\infty(\Omega, \mu)$ and hence it belongs to $L^\infty(\Omega, \mu)$ (Example 4.21). Let $a_{j,k}$ be a μ-measurable function which represents $\langle \delta_j, a\delta_k \rangle$. Then, for $\alpha, \beta \in \mathcal{D}$,

$$\sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta)$$

represents $\langle \alpha, a\beta \rangle \in L^\infty(\Omega, \mu)$ and the inequality $\|\langle \alpha, a\beta \rangle\| \leq \|a\|\,\|\alpha\|\,\|\beta\|$ implies that the set

$$N_{\alpha,\beta} = \left\{ \omega \in \Omega; \left| \sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta) \right| > \|a\|\,\|\alpha\|\,\|\beta\| \right\}$$

is μ-negligible and so is their countable union $N = \cup_{\alpha,\beta\in\mathcal{D}} N_{\alpha,\beta}$. Now the μ-measurable function

$$a_{\alpha,\beta}(\omega) = \begin{cases} \sum_{j,k} a_{j,k}(\omega)(\alpha|\delta_j)(\delta_k|\beta) & \text{if } \omega \notin N, \\ 0 & \text{otherwise,} \end{cases}$$

which is a representative of $\langle \alpha, a\beta \rangle$, depends on α, β in a sesquilinear fashion and satisfies

$$|a_{\alpha,\beta}(\omega)| \leq \|a\|\,\|\alpha\|\,\|\beta\| \quad \text{for any } \omega \in \Omega \text{ and } \alpha, \beta \in \mathcal{D}.$$

Now express each $\alpha, \beta \in \mathcal{H}$ in the form $\alpha = \lim_n \alpha_n$, $\beta = \lim_n \beta_n$ with $\alpha_n, \beta_n \in \mathcal{D}$. Then

$$\begin{aligned} |a_{\alpha_m,\beta_m}(\omega) - a_{\alpha_n,\beta_n}(\omega)| &\leq |a_{\alpha_m-\alpha_n,\beta_m}(\omega)| + |a_{\alpha_n,\beta_m-\beta_n}(\omega)| \\ &\leq \|a\|\|\beta_m\|\|\alpha_m - \alpha_n\| + \|a\|\|\alpha_n\|\|\beta_m - \beta_n\| \end{aligned}$$

shows that the sequence of functions $\{a_{\alpha_n,\beta_n}(\cdot)\}$ converges uniformly on Ω to a function $a_{\alpha,\beta}$, which represents $\langle \alpha, a\beta \rangle$, satisfies the inequality $|a_{\alpha,\beta}(\omega)| \leq \|a\| \, \|\alpha\| \, \|\beta\|$ for $\alpha, \beta \in \mathcal{H}$ and $\omega \in \Omega$. Moreover, $\alpha \times \beta \mapsto a_{\alpha,\beta}(\omega)$ gives a sesquilinear form at every $\omega \in \Omega$. By Riesz' lemma, we obtain a family of bounded operators $\{a(\omega)\}$ by the relation $a_{\alpha,\beta}(\omega) = (\alpha|a(\omega)\beta)$ for $\alpha, \beta \in \mathcal{H}$ with the obvious bound $\|a(\omega)\| \leq \|a\|$.

Conversely, given a uniformly bounded $\mathcal{B}(\mathcal{H})$-valued function $a(\omega)$ such that $(\alpha|a(\omega)\beta)$ is μ-measurable for any $\alpha, \beta \in \mathcal{H}$, the $\mathcal{H}$-valued function $a(\omega)\xi(\omega) \in \mathcal{H}$ is weakly μ-measurable for any $\xi \in L^2(\Omega, \mu; \mathcal{H})$ in view of the expression

$$(\alpha|a(\omega)\xi(\omega)) = \sum_j (\alpha|a(\omega)\delta_j)(\delta_j|\xi(\omega))$$

for $\alpha \in \mathcal{H}$, and the inequality

$$\int_\Omega \|a(\omega)\xi(\omega)\|^2 \, \mu(d\omega) \leq \|a\|^2 \int_\Omega \|\xi(\omega)\|^2 \, \mu(d\omega)$$

shows that it is square-integrable. Here the function

$$\|a(\omega)\| = \sup\{|(\alpha|a(\omega)\beta)|; \alpha, \beta \in \mathcal{D}, \|\alpha\| \leq 1, \|\beta\| \leq 1\}$$

is μ-measurable and $\|a\|$ is equal to its essential supremum.

Thus the totality $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ of such functions $a(\omega)$ (two $\mathcal{B}(\mathcal{H})$-valued functions $a(\omega)$ and $b(\omega)$ satisfying $\|a - b\| = 0$ being identified) is identified with the commutant of $L^\infty(\Omega, \mu) \otimes 1_{\mathcal{H}}$ on $L^2(\Omega, \mu) \otimes \mathcal{H} = L^2(\Omega, \mu; \mathcal{H})$.

Remark 6. As already appeared in the above discussion, the following measure-theoretical fact will be repeatedly used without explicit qualification: Let $P_j(\omega)$ be a sequence of propositions on $\omega \in \Omega$. If, for each $j \geq 1$, $P_j(\omega)$ holds at almost every $\omega \in \Omega$, then $\wedge_{j \geq 1} P_j(\omega)$ holds at almost every $\omega \in \Omega$.

Definition 10.2. Given a W*-algebra M on a separable Hilbert space $\mathcal{H}$, let $L^\infty(\Omega, \mu; M)$ be a *-subalgebra of $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ consisting of M-valued μ-measurable functions.

Proposition 10.3. The commutant of $L^\infty(\Omega, \mu; M)$ on $L^2(\Omega, \mu; \mathcal{H})$ is equal to $L^\infty(\Omega, \mu; M')$.

Proof. Let M be generated by a sequence $\{u_n\}$ of unitaries (cf. Example 4.24 and Example 2.24). Then each $a' \in L^\infty(\Omega, \mu; M)'$ belongs to $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$ and satisfies $u_n a' u_n^* = a'$ for $n \geq 1$. Then we can find a representative $a'(\omega)$ so that $u_n a'(\omega) u_n^* = a'(\omega)$ for any ω and any $n \geq 1$. Thus a' is in $L^\infty(\Omega, \mu; M')$, showing

$$L^\infty(\Omega, \mu; M)' \subset L^\infty(\Omega, \mu; M').$$

Since the reverse inclusion is obvious, we have the equality. $\square$

Corollary 10.4.

$$(L^\infty(\Omega, \mu) \otimes M)' = L^\infty(\Omega, \mu; M') = L^\infty(\Omega, \mu) \otimes M'.$$

Proof. Since the W*-algebra $L^\infty(\Omega, \mu) \otimes M$ is generated by $L^\infty \otimes 1$ and $1 \otimes M$ and since M is generated by $\{u_n\}$, the above proof shows that $(L^\infty \otimes M)' \subset L^\infty(\Omega, \mu; M')$ and then the equality $(L^\infty \otimes M)' = L^\infty(\Omega, \mu; M')$ because the reverse inclusion is trivial. $\square$

Remark 7. The equality $(L^\infty(\Omega, \mu) \otimes M)' = L^\infty(\Omega, \mu) \otimes M'$ is also a special case of the Tomita's commutant theorem.

10.2. **Measurable Fields.** Assume that a commutative W*-algebra $L^\infty(\Omega, \mu)$ is faithfully represented in a separable Hilbert space $\mathcal{H}$. The representation is then unitarily equivalent to a subrepresentation of $L^\infty(\Omega, \mu)$ on $L^2(\Omega, \mu) \otimes \ell^2$ by Theorem 5.9. The projection e realizing this subrepresentation is in $L^\infty(\Omega, \mu; \mathcal{B}(\ell^2))$ and $\mathcal{H}$ is unitarily isomorphic to $e(L^2(\Omega) \otimes \ell^2)$. Thanks to a function realization $e(\omega)$ of e, we obtain a family of separable Hilbert spaces $\{\mathcal{H}_\omega = e(\omega)\ell^2\}$, which is μ-measurable in the sense that there is a sequence of sections $\{\xi_n\}_{n\geq 1}$ satisfying the following conditions.

(i) $\{\xi_n(\omega)\}_{n\geq 1}$ is total in $\mathcal{H}_\omega$ at almost every $\omega \in \Omega$.
(ii) functions $\omega \mapsto (\xi_m(\omega)|\xi_n(\omega))$ $(m, n \geq 1)$ are μ-measurable.

We call a section $\{\xi(\omega) \in \mathcal{H}_\omega\}_{\omega\in\Omega}$ measurable with respect to the family $\{\xi_n\}_{n\geq 1}$ if $\omega \mapsto (\xi_n(\omega)|\xi(\omega))$ is measurable for every $n \geq 1$. Each $\xi \in e(L^2(\Omega, \mu; \ell^2))$ is then characterised as a measurable section satisfying

$$\int_\Omega \|\xi(\omega)\|^2 \, \mu(d\omega) < \infty.$$

More generally, given a family $\{\mathcal{H}_\omega\}$ of separable Hilbert spaces, a sequence of sections $\{\xi_n\}$ is called a measurability sequence if it satisfies the conditions stated above. Given a measurability sequence, the measurability of a section is defined exactly in the same way.

Let $\{\eta_n\}_{n\geq 1}$ be another measurability sequence. We say that $\{\xi_n\}$ and $\{\eta_n\}$ are equivalent if they give rise to the same classes of measurable sections. It is immediate to see that $\{\xi_n\}$ and $\{\eta_n\}$ are equivalent if and only if functions $\omega \mapsto (\xi_m(\omega)|\eta_n(\omega))$ $(m, n \geq 1)$ are measurable.

A family $\{\mathcal{H}_\omega\}$ of separable Hilbert spaces is called a **measurable field** if it is equipped with an equivalence class of measurability sequences. The direct integral of $\{\mathcal{H}_\omega\}$ with respect to a measure μ is now an obvious analogue of the Hilbert space of square-intergrable functions: If square-integrable measurable sections are identified with respect to the positive sesquilinear form

$$(\xi|\eta) = \int_\Omega (\xi(\omega)|\eta(\omega))\, \mu(d\omega),$$

we obtain the **direct integral Hilbert space**

$$\int_\Omega^\oplus \mathcal{H}_\omega\, \mu(d\omega)$$

with a square-integrable measurable section ξ denoted by

$$\int_\Omega^\oplus \xi(\omega)\, \mu(d\omega)$$

when it is regarded as an element in the direct integral space.

Clearly $L^\infty(\Omega, \mu)$ is represented in $\int_\Omega^\oplus \mathcal{H}_\omega\, \mu(d\omega)$ by multiplication and an operator in this class is said to be **diagonal**.

Let $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ be measurable fields of Hilbert spaces over a common measure space (Ω, μ). A family of bounded linear maps $\{T_\omega : \mathcal{H}_\omega \to \mathcal{K}_\omega\}$ is called **measurable** if $(\eta(\omega)|T_\omega\, \xi(\omega))$ is measurable whenever ξ and η are measurable sections. A measurable family $\{T_\omega\}$ is defined to be essentially bounded if the essential supremum $\|T\|_\infty$ of the function $\|T_\omega\|$ is finite. If this is the case, a bounded linear map $T : \int_\Omega^\oplus \mathcal{H}_\omega\, \mu(d\omega) \to \int_\Omega^\oplus \mathcal{K}_\omega\, \mu(d\omega)$ is defined by

$$T\left(\int_\Omega^\oplus \xi(\omega)\, \mu(d\omega)\right) = \int_\Omega^\oplus T_\omega\, \xi(\omega)\, \mu(d\omega)$$

with the operator norm $\|T\|$ equal to $\|T\|_\infty$. We call a bounded linear map of this type **decomposable** and denoted by

$$T = \int_\Omega^\oplus T_\omega\, \mu(d\omega).$$

Measurable fields $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ are then said to be unitarily equivalent if we can find a decomposable unitary map between $\int_\Omega^\oplus \mathcal{H}_\omega \mu(d\omega)$ and $\int_\Omega^\oplus \mathcal{K}_\omega \mu(d\omega)$.

Here are obvious algebraic relations between measurable operator families and integrated decomposable operators.

Proposition 10.5. Let $\{S_\omega : \mathcal{K}_\omega \to \mathcal{L}_\omega\}$ be another measurable family of bounded linear maps. Then $\{S_\omega T_\omega\}$ and $\{T_\omega^*\}$ are measurable and, if $\|S\|_\infty < \infty$, the following holds.

(i)

$$\left(\int_\Omega^\oplus S_\omega\,\mu(d\omega)\right)\left(\int_\Omega^\oplus T_\omega\,\mu(d\omega)\right) = \int_\Omega^\oplus S_\omega T_\omega\,\mu(d\omega).$$

(ii)

$$\left(\int_\Omega^\oplus T_\omega\,\mu(d\omega)\right)^* = \int_\Omega^\oplus T_\omega^*\,\mu(d\omega).$$

Theorem 10.6 (multiplicity decomposition)**.** Two measurable fields $\{\mathcal{H}_\omega\}$ and $\{\mathcal{K}_\omega\}$ are unitarily equivalent if and only if their dimension functions $\dim \mathcal{H}_\omega$ and $\dim \mathcal{K}_\omega$ coincide for μ-a.e. ω.

Proof. If one applies the Gram-Schmidt orthogonalization to a measurability sequence $\{\xi_n\}$, then we obtain a sequence $\{\delta_n\}$ of measurable sections satisfying

(i) $\sum_{j=1}^n \mathbb{C}\xi_j = \sum_{j=1}^n \mathbb{C}\delta_j$ for $n \geq 1$,
(ii) (semi-orthonormality) $(\delta_j(\omega)|\delta_k(\omega)) = 0$ for $j \neq k$ and $\|\delta_n(\omega)\| \in \{0, 1\}$ for $n \geq 1$.

Let $\Omega_n = \{\omega \in \Omega; \delta_n(\omega) \neq 0\}$ and rearrange δ_n by cutting and pasting in the following way: Set

$$\delta_1'(\omega) = \begin{cases} \delta_1(\omega) & \text{if } \omega \in \Omega_1, \\ \delta_2(\omega) & \text{if } \omega \in \Omega_2 \setminus \Omega_1, \\ \dots & \\ \delta_n(\omega) & \text{if } \omega \in \Omega_n \setminus (\Omega_1 \cup \dots \Omega_{n-1}) \\ \dots & \end{cases}$$

and

$$\delta_n'(\omega) = \begin{cases} \delta_n(\omega) & \text{if } \omega \in \Omega_n \cap (\Omega_1 \cup \dots \Omega_{n-1}) \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 2$. Then we have the equality of algebraic sums $\sum_{n\geq 1} \mathbb{C}\delta_n(\omega) = \sum_{n\geq 1} \mathbb{C}\delta_n'(\omega)$ for any $\omega \in \Omega$ and $\{\delta_n'\}_{n\geq 1}$ is semi-orthonormal.

Repeat the rearrangement to the semi-orthonormal system $\{\delta_n'|_{\Omega'}\}_{n\geq 2}$ ($\Omega' = \cup_{n\geq 1}\Omega_n$) to get $\{\delta_n''\}_{n\geq 2}$, $\Omega'' = \bigcup_{n\geq 2} \Omega_n' \subset \Omega'$ and so on. Now the diagonal choice

$$\epsilon_n(\omega) = \begin{cases} \delta_n^{(n)}(\omega) & \text{if } \omega \in \Omega^{(n)}, \\ 0 & \text{if } \omega \in \Omega \setminus \Omega^{(n)} \end{cases}$$

satisfies $\sum_{n\geq 1} \mathbb{C}\delta_n(\omega) = \sum_{n\geq 1} \mathbb{C}\epsilon_n(\omega)$ at each $\omega \in \Omega$ and we observe that, if $\omega \in \Omega^{(n)} \setminus \Omega^{(n+1)}$ for $n \geq 0$ or $\omega \in \Omega^{(\infty)}$ for $n = \infty$, $\{\epsilon_j(\omega)\}_{1\leq j\leq n}$ is an orthonormal basis for $\mathcal{H}(\omega)$ with $\epsilon_j(\omega) = 0$ for $j > n$. Note here that

$$\Omega^{(n)} = \{\omega \in \Omega; \dim \mathcal{H}_\omega \geq n\}$$

with $\Omega^{(0)} = \Omega$ and $\Omega^{(\infty)} = \bigcap_{n\geq 1} \Omega^{(n)}$. □

Corollary 10.7. Let $\Omega_n = \{\omega \in \Omega; \dim \mathcal{H}_\omega = n\}$ and $\ell^2(n)$ be the standard Hilbert space of dimension n for $n = 1, 2, \cdots, \infty$. Then $\{\mathcal{H}_\omega\}_{\omega\in\Omega_n}$ is unitarily equivalent to the constant field $\{\ell^2(n)\}_{\omega\in\Omega_n}$.

Theorem 10.8. A bounded linear map T between direct integral Hilbert spaces $\int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ and $\int_\Omega^\oplus \mathcal{K}_\omega \, \mu(d\omega)$ is decomposable if and only if T intertwines the diagonal representations of $L^\infty(\Omega, \mu)$.

Proof. This follows from the multiplicity decomposition of measurable fields of Hilbert spaces and results on tensor products. □

Lemma 10.9. Assume that a uniformly bounded sequence $\{a_n\}_{n\geq 1}$ of decomposable operators converges to a in the strong operator topology. Then we can find a subsequence $\{n_k\}_{k\geq 1}$ so that

$$\lim_{k\to\infty} a_{n_k}(\omega) = a(\omega)$$

in the strong operator topology of $\mathcal{B}(\mathcal{H}_\omega)$ at almost every $\omega \in \Omega$.

Proof. By replacing a_n with $a_n - a$, we may assume that $a = 0$. For each $\xi = \int^\oplus \xi(\omega) \, \mu(d\omega)$, choose a subsequence $\{n'\}_{n\geq 1}$ so that $\|a_{(n+1)'}\xi - a_{n'}\xi\| \leq 1/2^n$ for $n \geq 1$. By the subadditivity of L^2-norm, we have

$$\left(\int_\Omega \left(\sum_{k=1}^n \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\| \right)^2 \mu(dx) \right)^{1/2} \leq \sum_{k=1}^n \|a_{(k+1)'}\xi - a_{k'}\xi\| \leq 1$$

and then, by taking the limit $n \to \infty$,

$$\int_\Omega \left(\sum_{k=1}^\infty \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\| \right)^2 \mu(dx) \leq 1.$$

Thus

$$\sum_{k=1}^{\infty} \|a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega)\| < \infty \quad \text{for } \mu\text{-a.e. } \omega.$$

and hence

$$\lim_{n\to\infty} a_{n'}(\omega)\xi(\omega) = a_{1'}(\omega)\xi(\omega) + \sum_{k=1}^{\infty}(a_{(k+1)'}(\omega)\xi(\omega) - a_{k'}(\omega)\xi(\omega))$$

is norm-convergent at almost every $\omega \in \Omega$.

In view of $\|a_n(\omega)\xi(\omega)\| \leq \sup\{\|a_n\|\}\|\xi(\omega)\|$ at almost every ω and $\int \|\xi(\omega)\|^2 \, \mu(d\omega) = \|\xi\|^2 < \infty$, the dominated convergence theorem is here applied to get

$$\int_\Omega \lim_{n\to\infty} \|a_{n'}(\omega)\xi(\omega)\|^2 \, \mu(d\omega) = \lim_{n\to\infty} \|a_{n'}\xi\|^2 = 0,$$

i.e., $\lim_{n\to\infty} \|a_{n'}(\omega)\xi(\omega)\| = 0$ at almost every $\omega \in \Omega$.

Let $\{\xi_j\}_{j\geq 1}$ be a measurability sequence. If one applies a Cantor's diagonal argument to choosing subsequences for the convergence $\lim_{n\to\infty} \|a_n(\omega)\xi_j(\omega)\| = 0$, we can find a subsequence $\{n_k\}_{k\geq 1}$ so that

$$\lim_{k\to\infty} \|a_{n_k}(\omega)\xi_j(\omega)\| = 0$$

at almost every $\omega \in \Omega$ for $j \geq 1$. By the totality of $\{\xi_j(\omega)\}_{j\geq 1}$ in $\mathcal{H}_\omega$ and the uniform boundedness $\sup\{\|a_n\|; n \geq 1\} < \infty$, this implies

$$\lim_{k\to\infty} a_{n_k}(\omega) = 0$$

in the strong operator topology at almost every $\omega \in \Omega$. □

Let $\{\mathcal{H}_\omega\}$ be a measurable field of separable Hilbert spaces and $\{M_\omega \subset \mathcal{B}(\mathcal{H}_\omega)\}_{\omega\in\Omega}$ be a family of W*-algebras on $\{\mathcal{H}_\omega\}$. A measurable operator family $\{a(\omega)\}$ is said to be **adapted** to $\{M_\omega\}$ if $a(\omega) \in M_\omega$ at almost every $\omega \in \Omega$. Thanks to the previous lemma, we see that the set of decomposable operators associated to adapted operator families is a W*-algebra on $\int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ and is denoted by

$$\int_\Omega^\oplus M_\omega \, \mu(d\omega).$$

Definition 10.10. A family of W*-algebras $\{M_\omega\}_{\omega\in\Omega}$ on $\{\mathcal{H}_\omega\}$ is called **measurable** if we can find a sequence of adapted families $\{a_n(\omega)\}$ $(n = 1, 2, \cdots)$ such that M_ω is generated by $\{a_n(\omega)\}_{n\geq 1}$ at μ-a.e. $\omega \in \Omega$. Such a sequence $\{a_n(\omega)\}$ is referred to as a generating sequence of measurability.

Example 10.11. Let $C \subset M$ be a central W*-subalgebra of a W*-algebra M in a seprable Hilbert space $\mathcal{H}$ and realize C as $C = L^\infty(\Omega, \mu)$ with $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$ the associated direct integral decomposition. The W*-algebra $M \subset C'$ then consists of decomposable operators.

Since M_* is separable, we can find a sequence $\{a_n(\omega)\}_{n\geq 1}$ of measurable families of operators such that the integrated decomposable operators $\{a_n\}_{n\geq 1}$ are dense in M with repsect to the σ-strong* topology and, if M_ω denotes the W*-algebra on $\mathcal{H}_\omega$ generated by $\{a_n(\omega); n \geq 1\}$, the family $\{M_\omega\}$ of W*-algebras is measurable.

Proposition 10.12. Let $\{M_\omega\}$ be a measurable family of W*-algebras and e (resp. e') be a decomposable projection satisfying $e(\omega) \in M_\omega$ (resp. $e'(\omega) \in M_\omega$) at almost every $\omega \in \Omega$. Then the reduced family $\{e(\omega)M_\omega e(\omega)\}$ (resp. the induced family $\{e'(\omega)M_\omega\}$) is measurable.

Theorem 10.13. Let $\{M_\omega\}$ be a measurable family of W*-algebras. Then $\int_\Omega^\oplus M_\omega \, \mu(d\omega)$ is equal to the W*-algebra M generated by any generating sequence $\{a_n\}_{n\geq 1}$ of measurability together with the diagonal algebra $L^\infty(\Omega, \mu)$.

Proof. Since $M' = \{a_n, a_n^*; n \geq 1\}' \cap L^\infty(\Omega, \mu)'$ is realized on a separable Hilbert space, it is generated by a sequence $\{a'_n\}_{n\geq 1}$ of decomposable operators. From $a_j a'_k = a'_k a_j$ for $j, k \geq 1$, we see that $a_j(\omega)a'_k(\omega) = a'_k(\omega)a_j(\omega)$ at almost every $\omega \in \Omega$. Thus, $a'_k(\omega) \in M'_\omega$ for all $k \geq 1$ at almost every $\omega \in \Omega$. □

Example 10.14. The W*-algebra M in Example 10.11 is recovered from the measurable family $\{M_\omega\}$ as the integrated W*-algebra.

Moreover, the family $\{M_\omega\}$ does not depend on the choice of countable generators up to μ-negligible sets of Ω.

In fact, let $\{b_n\}$ be another sequence of generators of M and set $N_\omega = \{b_n(\omega)\}''$. From $b_n \in \int_\Omega^\oplus M_\omega \, \mu(d\omega)$, $b_n(\omega) \in M_\omega$ at almost all ω, which implies $N_\omega \subset M_\omega$ at almost all ω. By symmetry, we also have $M_\omega \subset N_\omega$ at almost all ω. Thus $M_\omega = N_\omega$ at almost all $\omega \in \Omega$.

Theorem 10.15. Let $\{M_\omega\}$ be a measurable family of W*-algebras on a measurable field $\{\mathcal{H}_\omega\}$ of Hilbert spaces. Then the family $\{M'_\omega\}$ of commutants is measurable and the integrated W*-algebra $\int_\Omega^\oplus M'_\omega \, \mu(d\omega)$ is the commutant of $\int_\Omega^\oplus M'_\omega \, \mu(d\omega)$ on $\int_\Omega^\oplus \mathcal{H}_\omega \, \mu(d\omega)$.

Proof. Although the measurability of $\{M'_\omega\}$ seems to be very reasonable in appearance, its proof is not so obvious as can be witnessed in Dixmier §II.3.3, Pedersen §4.11.7 or Takesaki §IV.8. We shall see this with the help of standard spaces below.

Once the measurability of commutants is established, the commutant relation between integrated W*-algebras is immediate: Let $\{a'_n(\omega\}$ be a generating sequence for $\{M'_\omega\}$ and assume that a decomposable operator $a = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ is in the commutant of $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$. Since $\int_\Omega^\oplus a'_n(\omega)\,\mu(d\omega)$ belongs to $\int_\Omega^\oplus M'_\omega\,\mu(d\omega)$, we have $a(\omega)a'_n(\omega) = a'_n(\omega)a(\omega)$ for $n \geq 1$ at almost every $\omega \in \Omega$, whence $a(\omega) \in M'_\omega$ at almost every $\omega \in \Omega$. □

Given a measurable family of W*-algebras $\{M_\omega\}$ on $\{\mathcal{H}_\omega\}$, denote by $\mathcal{M}$ the *-algebra of adapted operator families. A family $\{\phi_\omega : M_\omega \to \mathbb{C}\}$ of normal functionals is said to be **measurable** if $\phi_\omega(a(\omega))$ is a measurable function of ω for every $\{a(\omega)\} \in \mathcal{M}$.

Example 10.16. There are plenty of measurable families of normal functionals: Let $\{\xi(\omega)\}$ and $\{\eta(\omega)\}$ be measurable sections of $\{\mathcal{H}_\omega\}$. Then $\{\phi_\omega(\cdot) = (\xi(\omega)|(\cdot)\eta(\omega))\}$ is a mesurable families of normal functionals of $\{M_\omega\}$.

Moreover, if $\{\xi_n\}_{n\geq 1}$ is a sequence of measurability of $\{\mathcal{H}_\omega\}$, then

$$\varphi_\omega(x) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{(\xi_n(\omega)|x\xi_n(\omega))}{(\xi_n(\omega)|\xi_n(\omega))}, \quad x \in M_\omega$$

defines a measurable family of faithful normal states on $\{M_\omega\}$.

Proof. Given measurable families $\{\varphi_\omega\}$, $\{\psi_\omega\}$ of positive normal functionals, we have two measurable families of positive sesquilinear forms $\{(\varphi_\omega)_L\}$ and $\{(\varphi_\omega)_R\}$ on $\{M_\omega\}$ by

$$(\varphi_\omega)_L(a,b) = \varphi_\omega(a^*b), \quad (\psi_\omega)_R(a,b) = \psi_\omega(ba^*)$$

for $a, b \in M_\omega$ and then a measurable family of sesquilinear forms as their geometric means. Thus functions of $\omega \in \Omega$

$$(a(\omega)\varphi_\omega^{1/2}|\psi_\omega^{1/2}b(\omega)) = \sqrt{(\varphi_\omega)_L(\psi_\omega)_R}(a(\omega), b(\omega)),$$

for $\{a(\omega)\}$, $\{b(\omega)\} \in \mathcal{M}$ are measurable, which makes $\{L^2(M_\omega)\}$ into a measurable field of separable Hilbert spaces.

Let $J_\omega : L^2(M_\omega) \to L^2(M_\omega)$ be the canonical conjugation. Then the family $\{J_\omega\}$ is measurable[14] as the phase part of the polar decomposition of the measurable family $\{S_\omega : a(\omega)\varphi_\omega^{1/2} \mapsto a(\omega)^*\varphi_\omega^{1/2}\}$ of closable operators, where $\{\varphi_\omega\}$ is a measurable family of faithful normal states on $\{M_\omega\}$.

Thus $\{M'_\omega = J_\omega M_\omega J_\omega\}$ is a measurable family on $\{L^2(M_\omega)\}$ because it is generated by the sequence $\{J_\omega a_n(\omega) J_\omega\}$ of measurable operator

[14]See Appendix I.1 for the measurable version of polar decomposition.

families if $\{a_n(\omega)\}$ is a generating sequence of $\{M_\omega\}$. Now the measurable version of Dixmier's theorem on normal homomorphisms in Appendix N.3 shows the measurabilty of commutants in the general situation. $\square$

We shall now identify the standard space $L^2(M)$ for $M = \int_\Omega^\oplus M_\omega\,\mu(d\omega)$ with $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$. Given a measurable family $\{\varphi_\omega\}$ of normal positive functionals on $\{M_\omega\}$ satisfying

$$\int_\Omega \|\varphi_\omega^{1/2}\|^2\,\mu(d\omega) = \int_\Omega \varphi_\omega(1)\,\mu(d\omega) < \infty,$$

we define a positive normal functional φ on M by

$$\begin{aligned}\varphi(a) &= \Big(\int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)\Big|\Big(\int_\Omega^\oplus a(\omega)\,\mu(d\omega)\Big)\int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)\Big)\\ &= \int_\Omega \varphi_\omega(a(\omega))\,\mu(d\omega).\end{aligned}$$

Since the sesquilinear forms φ_L is realized in the form of integration as

$$\varphi(a^*b) = \int_\Omega \varphi_\omega(a(\omega)^*b(\omega))\,\mu(d\omega)$$

and similarly for ψ_R with $\{\psi_\omega\}$ another measurable family satisfying $\psi(1) < \infty$, we have

$$\begin{aligned}(a\varphi^{1/2}|\psi^{1/2}b) = \sqrt{\varphi_L\psi_R}(a,b) &= \int_\Omega \sqrt{(\varphi_\omega)_L(\psi_\omega)_R}(a(\omega),b(\omega))\,\mu(d\omega)\\ &= \int_\Omega (a(\omega)\varphi_\omega^{1/2}|\psi_\omega^{1/2}b(\omega))\,\mu(d\omega)\end{aligned}$$

for $a = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ and $b = \int_\Omega^\oplus a(\omega)\,\mu(d\omega)$ in M. Thus the correspondence $\varphi^{1/2} \mapsto \int_\Omega^\oplus \varphi_\omega^{1/2}\mu(d\omega)$ is extended to a unitary map from $L^2(M)$ onto $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$ so that it intertwines the bimodule actions of M.

Moreover, as the antiunitary part in the polar decomposition of

$$S = \int_\Omega^\oplus S_\omega\mu(d\omega)$$

with respect to a faithful normal state $\varphi = \int_\Omega^\oplus \varphi_\omega\mu(d\omega)$, we see that the canonical conjugation in $L^2(M)$ is identified with $\int_\Omega^\oplus J_\omega\,\mu(d\omega)$.

Conversely, given $\varphi \in M_*^+$, let

$$\varphi^{1/2} = \int_\Omega^\oplus \varphi^{1/2}(\omega)\,\mu(d\omega)$$

be the decomposition in $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$. Then vector functionals $\varphi_\omega(\cdot) = (\varphi^{1/2}(\omega)|(\cdot)\varphi^{1/2}(\omega))$ constitute a measurable family of positive normal functionals on $\{M_\omega\}$ and $\varphi^{1/2}$ is realized by the vector $\int_\Omega^\oplus \varphi_\omega^{1/2}\,\mu(d\omega)$. Thus $\varphi^{1/2}(\omega) = \varphi_\omega^{1/2}$ at almost every $\omega \in \Omega$.

Theorem 10.17. Given a measurable family $\{M_\omega\}$ of W*-algebras, the standard space $L^2(M)$ of the integrated W*-algebra $M = \int_\Omega^\oplus M_\omega \mu(d\omega)$ is naturally identified with $\int_\Omega^\oplus L^2(M_\omega)\,\mu(d\omega)$ as *-bimodules of M in such a way that $\xi = \int_\Omega^\oplus \xi(\omega)\,\mu(d\omega) \in L^2(M)$ belongs to $L^2(M)_+$ if and only if $\xi(\omega) \in L^2(M_\omega)_+$ at almost every $\omega \in \Omega$.

Theorem 10.18. Let (Ω, μ) be a σ-finite measure space and suppose that, for each $n \geq 1$, we are given a measurable family $\{M_n, \omega\}$ of W*-algebras on a measurable field $\{\mathcal{H}_\omega\}$ of Hilbert spaces. Then $\{\vee_{n\geq 1} M_{n,\omega}\}$, $\{\cap_{n\geq 1} M_{n,\omega}\}$ are measurable families and

$$\bigvee_{n\geq 1} \int_\Omega^\oplus M_{n,\omega}\,\mu(d\omega) = \int_\Omega^\oplus \bigvee_{n\geq 1} M_{n,\omega}\,\mu(d\omega),$$
$$\bigcap_{n\geq 1} \int_\Omega^\oplus M_{n,\omega}\,\mu(d\omega) = \int_\Omega^\oplus \bigcap_{n\geq 1} M_{n,\omega}\,\mu(d\omega).$$

Proof. Let $\{a_{n,k}(\omega)\}_{k\geq 1}$ be a generating sequence for $\{M_{n,\omega}\}$. Then $\{\vee_{n\geq 1} M_{n,\omega}\}$ is generated by $\{a_{n,k}(\omega)\}_{n,k\geq 1}$. Now Theorem 9.13 is applied to get

$$\bigvee_{n\geq 1} M_n = \{a_{n,k}, f; f \in L^\infty(\Omega), n, k \geq 1\}'' = \int_\Omega^\oplus \bigvee_{n\geq 1} M_{n,\omega}\,\mu(d\omega).$$

The relation for intersections is then obtained by taking commutants. □

11. Group Symmetry

Bratteli-Robinson, §2.5.3 and chapter 2.7.
Takesaki 2,
Perdersen, §7.4.

Symmetries on operator algebras are most straightforwardly studied through their automorphisms. Recall here that the relevant topology of an operator algebra (i.e., a C*-algebra or a W*-algebra) is determnined by its order structure and therefore *-automorphisms are automatically continuous. Let $\mathrm{Aut}(A)$ be the group of *-automorphisms of an operator algebra A. An **automorphic action** of a group G on A is, by definition, a group homomorphism $\theta : G \to \mathrm{Aut}(A)$.

Here groups are typically of the form of vectorial translations, linear transformations associated with space-time geometry, permutations of physical elements and compact groups which govern conserved quantities called charges. Among these most fundamental is the group of time development and has been investigated much. Although all of these are (not necessarily connected) Lie groups, general features of automorphic actions can be capturted under the existence of invariant measures; we shall work with locally compact second countable groups. Note that the existence of invariant measures is equivalent to requiring local compactness by Weil's converse theorem.

To take their topological nature into account, we impose continuity on the actions somehow.

For an automorphic action on a C*-algebra, we have several candidates as continuity:

(i) For each $a \in A$, $G \ni g \mapsto \theta_g(a) \in A$ is norm-continuous.
(ii) For each $a \in A$ and $\phi \in A^*$, $G \ni g \mapsto \phi(\theta_g(a))$ is continuous.
(iii) For each $\phi \in A^*$, $G \ni g \mapsto \phi \circ \theta_g \in A^*$ is norm-continuous.

As seen in Appendix, (i) and (ii) are equivalent and adopted as a definition of continuity, whereas (iii) is stronger than these.

Example 11.1. Let Ω be a locally compact space. Then an automorphic action of G on the commutative C*-algebra $A = C_0(\Omega)$ is continuous if and only if $G \times \Omega \to \Omega$ is continuous.

The dualized action of G on A^* is however not necessarily norm-continuous. In fact, if G moves the support S of a probability measure ϕ on Ω transversally, two states ϕ and $g\phi$ are disjoint, whence $\|g\phi - \phi\| = 2$ for $g \neq e$.

Related to the continuity (iii), the following notion is important.

Definition 11.2. Given an automorphic action $\theta : G \to \mathrm{Aut}(A)$, a **covariant representation** of θ is a *-representation π of A together with a continuous unitary representation u of G on a common Hilbert space $\mathcal{H}$ which satisfies the covariance relation $\pi(\theta_g(a)) = u(g)\pi(a)u(g)^*$.

Given a covariant representation, any vector state satisfies the norm continuity: Let $\phi(a) = (\xi|\pi(a)\xi) = \mathrm{trace}(\pi(a)\xi\xi^*)$. Then $\phi(\theta_g(a)) = \mathrm{trace}(\pi(a)u(g)^*\xi\xi^*u(g))$ and

$$\begin{aligned}\|\phi\theta_g - \phi\| &\leq \|a\|\|u(g)^*\xi\xi^*u(g) - \xi\xi^*\|_1 \\ &\leq \|a\|\|u(g)^*\xi\xi^*u(g) - \xi\xi^*u(g)\|_1 + \|a\|\|\xi\xi^*u(g) - \xi\xi^*\|_1 \\ &\leq 2\|a\|\|\xi\|\|u(g)^*\xi - \xi\|.\end{aligned}$$

Let $A^*_G = \{\phi \in A^*; G \ni g \mapsto \phi \circ \theta_g \in A^* \text{ is norm-continuous}\}$, which is a norm-closed G-invariant subspace of A^*. Let M be a universal W*-algebra which is generated by A and G with the relation $gx = \theta_g(x)g$ and the condition: Any covariant representation (π, u) of (A, G) on a Hilbert space $\mathcal{H}$ is extended to a normal representation of M on $\mathcal{H}$.

Theorem 11.3. The Banach space A^*_G should be identified with the predual of M.

Definition 11.4. Let a locally compact group G act on a Banach space X by isometries. We say that an element $x \in X$ is G-continuous if $G \ni g \mapsto gx$ is norm-continuous. Note that the condition is equivalent to the norm-continuity at $g = e$.

Theorem 11.5. Let θ be an automorphic action of a locally compact group G on a W*-algebra A. Then the following conditions are equivalent.

(i) $G \ni g \mapsto \phi(\theta_g(a))$ is continuous for any $a \in A$ and any $\phi \in A_*$.
(ii) $G \ni g \mapsto \phi \circ \theta_g \in A_*$ is norm-continuous for any $\phi \in A_*$.
(iii) The unitary representation of G on $L^2(A)$ is continuous.

Moreover, under these equivalent conditions, the set of G-continuous elements in A is a weak*-dense C*-subalgebra of A.

Proof. (ii) $\Longrightarrow$ (i) is obvious. The equivalence (ii) $\Longleftrightarrow$ (iii) is a consequence of Powers-Störmer-Araki inequality, whereas (i) $\Longrightarrow$ (ii) is a special case of Theorem K.3.

We can apply a similar (and easier) argument to the action $G \times A \to A$ to get the remaining assertion. □

Definition 11.6. We say that an automorphic action θ of a locally compact group G on a W*-algebra A is continuous if it satisfies the equivalent conditions in the above theorem.

Given a *-representation π of A which makes θ weakly measurable in the sense that $G \ni g \mapsto (\xi|\pi(\theta_g(a))\eta)$ is measurable for every $a \in A$ and $\xi, \eta \in \mathcal{H}$, we can introduce a covariant representaion as follows.

Let $L^2(G) \otimes \mathcal{H}$ be identified with $\mathcal{H}$-valued square-integrable functions on G. It is mnemonic to write

$$\oint_G \xi(g)\, dg = \int_G g\xi(g)\, dg,$$

which suggests

$$h \int_G g\xi(g)\, dg = \int_G g\xi(h^{-1}g)\, dg = \oint_G \xi(h^{-1}g)\, dg$$

and

$$a\int_G g\xi(g)\,dg = \int_G g\pi(g^{-1}ag)\xi(g)\,dg = \oint_G \pi(\theta_{g^{-1}}a)\xi(g)\,dg.$$

Clearly $a \in A$ is represented on $L^2(G)\otimes\mathcal{H}$ as a decomposable operator $\{\pi(\theta_{g^{-1}}(a))\}_{g\in G}$ and it, together with a unitary representation $\lambda_h \otimes 1$, constitutes a covariant representation, which is referred to as an **induced covariant representation**.

When A is a W*-algebra (with continuity of $G \times A_* \to A_*$) and π is normal, so is the induced representation. In fact, for the choice $\xi(g) = f(g)\eta$ with $f \in C_c(G)$ and $\eta \in \mathcal{H}$,

$$(\xi|a\xi) = \int_G \overline{f(g)}f(g)(\xi|\pi(\theta_{g^{-1}}a)\xi)\,dg,$$

which is an evaluation of $\int_G |f(g)|^2\phi\circ\theta_{g^{-1}}\,dg \in A_*$ at $a \in A$. Here $\phi \in A_*$ is set to be $\phi(a) = (\eta|\pi(a)\eta)$.

Returning to the C*-case, assume that θ_g is implemented on $\pi(A)$ by a unitary u_g in $\mathcal{B}(\mathcal{H})$, i.e., $u_g\pi(a)u_g^* = \pi(\theta_g(a))$, so that it depends on g measurably. Then another mnemonic of commutation relations $g\eta = u_g\eta$ for $g \in G$ and $\eta \in \mathcal{H}$ gives rise to a unitary map

$$L^2(G)\otimes\mathcal{H} \ni \int_G g\xi(g)\,dg \mapsto \int_G u_g\xi(g)g\,dg \in \mathcal{H}\otimes L^2(G)$$

or a unitary operator U on $L^2(G,\mathcal{H})$ by $(U\xi)(g) = u_g\xi(g)$, which provides us another expression for the induced covariant representation:

$$(Ua\xi)(g) = u_g\pi(\theta_{g^{-1}}a)\xi(g) = \pi(a)(U\xi)(g), \quad (Uh\xi)(g) = u_g\xi(h^{-1}g).$$

Example 11.7. Let G act on a W*-algebra M continuously and represent M on $L^2(M)$ by left multiplication. Then θ_g on M has a canonical implementation on $L^2(M)$, which depends on g continuously. The induced covarianct representation is realized on both of $L^2(G)\otimes L^2(M)$ and $L^2(M)\otimes L^2(G)$, which is referred to as a **regular covariant representation** of $M \curvearrowleft G$.

Definition 11.8 (Crossed products)**.** Let $f \in C_c(G,A)$ be identified with a formal expression

$$\int f(g)g\,dg$$

and make $C_c(G, A)$ into a *-algebra by

$$\begin{aligned}\int f_1(g)g\,dg \int f_2(h)h\,dh &= \iint f_1(g)\theta_g(f_2(h))gh\,dhdg \\ &= \iint f_1(g)\theta_g(f_2(g^{-1}h))h\,dhdg \\ &= \int \left(\int f_1(g)\theta_g(f_2(g^{-1}h))\,dg\right) h\,dh\end{aligned}$$

and

$$\begin{aligned}\left(\int f(g)g\,dg\right)^* &= \int g^{-1}f(g)^*\,dg = \int gf(g^{-1})^*\frac{dg^{-1}}{dg}\,dg \\ &= \int \theta_g(f(g^{-1}))^*\frac{dg^{-1}}{dg}g\,dg.\end{aligned}$$

The universal C*-algebra generated by $C_c(G, A)$ is then denoted by $A \rtimes_\theta G$ or simply $A \rtimes G$ and called the **C*-crossed product** of A by G with respect to θ.

For an action on a W*-algebra M, the **W*-crossed product** $M \rtimes G$ is defined to be the von Neumann algebra on $L^2(M) \otimes L^2(G)$ generated by the regular covariant representation of $M \curvearrowleft G$.

Given a covariant representation (π, u) of $A \curvearrowleft G$ on $\mathcal{H}$,

$$\int f(g)g\,dg \mapsto \int_G \pi(f(g))u(g)\,dg \in \mathcal{B}(\mathcal{H})$$

defines a *-repsentation of $A \rtimes G$. Conversely, given a *-representation $\tilde{\pi}$ of $A \curvearrowleft G$ on $\mathcal{H}$, a covariant representation (π, u) is recovered by

$$\pi(a)(\tilde{\pi}(f)\xi) = \tilde{\pi}(af)\xi, \quad u(g)(\tilde{\pi}(f)\xi) = \tilde{\pi}(gf)\xi,$$

where $(af)(g') = af(g')$ and $(gf)(g') = f(g^{-1}g')$ for $g' \in G$.

Thus there is one-to-one correspondence between covariant representations and representations of the crossed algebra so that both generate the same von Neumann algebra.

$$(\pi(A) \cup u(G))'' = \tilde{\pi}(A \rtimes G)''.$$

Exercise 48. Show the generating property.

The following identification is fundamental but its proof is not easy.

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Theorem 11.9. $L^2(M \rtimes G) = L^2(M) \otimes L^2(G)$.

Proof. Use the theory of left Hilbert algebra. □

Proposition 11.10. Let G continuously act on a W*-algebra M. Given a normal representation of M on $\mathcal{H}$, the induced covariant representation gives rise to a normal representation of $M \rtimes G$. Moreover, the normal representation of $M \rtimes G$ on $L^2(G) \otimes \mathcal{H}$ is faithful if so is M on $\mathcal{H}$.

Proof. By Theorem 5.9, we may assume that ${}_M\mathcal{H} = {}_M L^2(M)^{\oplus I} e$ with e a projection in $M_I(M)$. Then the induced representation is realized on

$$(L^2(G) \otimes L^2(M))^{\oplus I} e = L^2(M \rtimes G)^{\oplus I} e$$

by left muntiplication. Since $e \in M_I(M) \subset M_I(M \rtimes G)$, this is a normal representation of $M \rtimes G$.

The representation of M on $\mathcal{H}$ is then faithful if and only if the central support of e in $M_I(M)$ is the identity. (The center of $M_I(M)$ is naturally indentified with that of M.) In view of $1 = \vee_{u \in \mathcal{U}(M)} ueu^*$, the central support of e in $M_I(M \rtimes G)$ is also identity and the induced representation is faithful. □

Consider an action G on a W*-algebra M which admits a unitary implementation in M: we can find a continuous unitary homomorphism $v : G \to M$ such that $\theta_g(x) = v(g)xv(g)^*$ for $x \in M$ and $g \in G$. Let (π, u) be a covariant representation on a Hilbert space $\mathcal{H}$. Then the unitary $\rho(g) = u(g)\pi(v(g)^*)$ is in the commutant of $\pi(M)$. Moreover,

$$\begin{aligned} \rho(gh) &= u(gh)\pi(v(gh)^*) = u(g)\rho(h)\pi(v(g)^*) \\ &= u(g)\pi(v(g)^*)u(h)\pi(v(h)^*) = \rho(g)\rho(h) \end{aligned}$$

shows that $G \ni g \mapsto \rho(g) \in \pi(M)'$ is a unitary representation of G and the von Neumann algebra $(\pi(M) \cup u(G))''$ on $\mathcal{H}$ is generated by $\pi(M)$ and $\rho(G) \subset \pi(M)'$.

For the regular covariant representation

$$(\pi(a)\xi)(g) = a\xi(g), \quad (u(h)\xi)(g) = v(h)\xi(h^{-1}g)v(h)^*$$

on $L^2(M) \otimes L^2(G)$, we observe $(\rho(h)\xi)(g) = \xi(h^{-1}g)v(h)^*$ and ρ is further transformed into $1 \otimes \lambda$ by a unitary operator $\xi(g) \mapsto \xi(g)v(g)^{-1}$ on $L^2(M) \otimes L^2(G)$, where λ denotes the left regular representation of G. The crossed product $M \rtimes G$ is therefore isomorphic to $M \otimes \lambda(G)''$.

Appendix A. Analytic Elements

Let X be a dual Banach space with a predual X_*. An X-valued function f on a topological space Ω is said to be weak*-continuous if $\phi \circ f$ is continuous for each $\phi \in X_*$.

Exercise 49. An X-valued function f is weak*-continuous if and only if $f : \Omega \to X$ is continuous when X is furnished with the weak* topology.

Let Ω be locally compact and μ be a complex Radon measure on Ω. If the function f is norm-bounded, i.e.,

$$\|f\|_\infty = \sup\{\|f(\omega)\|; \omega \in \Omega\} < \infty,$$

then we can define an element

$$\int_\Omega f(\omega)\, \mu(d\omega) \in X$$

by the relation

$$\left\langle \int_\Omega f(\omega)\, \mu(d\omega), \phi \right\rangle = \int_\Omega \langle f(\omega), \phi\rangle\, \mu(d\omega)$$

in view of the estimate

$$\left| \int_\Omega \langle f(\omega), \phi\rangle\, \mu(d\omega) \right| \leq |\mu|(\Omega) \|f\|_\infty \|\phi\|.$$

Note that, when Ω is compact, the weak*-continuity of f is enough to have the norm-boundedness $\|f\|_\infty < \infty$ thanks to the principle of uniform boundedness.

Proposition A.1. Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \to X$ be a weak*-continuous function. Then the following conditions are equivalent.

(i) The function $\|f(w)\|$ of $w \in \Omega$ is locally bounded and $\langle f(w), \phi\rangle$ is holomorphic function of $w \in \Omega$ for ϕ in a dense subset of X_*.

(ii) For each $\phi \in X_*$, $\langle f(w), \phi\rangle$ is a holomorphic function of $w \in \Omega$.

(iii) If $z \in \Omega$ and $r > 0$ satisfies $\{w \in \mathbb{C}; |w - z| < r\} \subset \Omega$, then we can find a sequence $\{f_n\}_{n\geq 0}$ in X such that

$$f(w) = \sum_{n=0}^\infty (w - z)^n f_n$$

holds in an absolutely norm-convergent manner for $|w - z| < r$.

If f satisfies these equivalent conditions, we say that f is holomorphic on Ω.

Proof. (iii) $\Longrightarrow$ (ii) is trivial, whereas (ii) $\Longrightarrow$ (i) is remarked already.

(i) $\Longrightarrow$ (iii): By Cauchy's integral formula,

$$\oint_{|\zeta - z| = r - \epsilon} \frac{\langle f(\zeta), \phi\rangle}{(\zeta - z)^{n+1}}\, d\zeta \in X,$$

with $\phi \in X_*$ is independent of the choice of $0 < \epsilon < r$ and we can define $f_n \in X$ by

$$f_n = \frac{1}{2\pi i} \oint_{|\zeta - z| = r - \epsilon} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \in X.$$

Then the Cauchy's estimate $\|f_n\| \leq \|f\|_\infty/(r-\epsilon)^n$ shows that the series in question is absolutely norm-convergent if $|w - z| < r$. $\square$

Definition A.2. An X-valued weak*-continuous function f defined on a subset D of C is said to be **analytic** if it is holomorphic when restricted to the inner part $D \setminus \partial D$.

Let $I_t : X \to X$ be a one-parameter group of isometries on X and suppose that

(i) $\mathbb{R} \ni t \mapsto I_t(x) \in X$ is weak*-continuous for each $x \in X$.
(ii) For $\phi \in X_*$ and $t \in \mathbb{R}$, the functional $\langle I_t(\cdot), \phi \rangle$ belongs to X_*.

Remark 8. The above assumption is equivalent to requiring the norm-continuity of $\mathbb{R} \ni t \mapsto \phi \circ I_t \in X_*$ for every $\phi \in X_*$, see Theorem K.3.

Definition A.3. An element $x \in X$ is said to be **analytic** for $\{I_t\}$ if we can find $r > 0$ and an analytic function on the strip domain $f : \mathbb{R} + i(-r, r) \to X$ such that $f(t) = \langle I_t(x), \phi \rangle$ for $\phi \in X_*$ and $t \in \mathbb{R}$.

An analytic element $x \in X$ is said to be **entirely analytic** if we can find an analytic function $f : \mathbb{C} \to X$ such that $f(t) = \langle I_t(x), \phi \rangle$ for $\phi \in X_*$ and $t \in \mathbb{R}$.

We have plenty of entirely analytic elements: Let

$$x_n = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-nt^2} I_t(x) \, dt.$$

Since $\sqrt{n/\pi} e^{-nt^2}$ gives an approsimate delta function, $x_n \to I_0(x) = x$ as $n \to \infty$ in the weak*-topology. Moreover x_n is entirely analytic because

$$I_t(x_n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} I_{s+t}(x) \, ds = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-t)^2} I_s(x) \, ds.$$

indicates to set

$$f(z) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} I_s(x) \, ds$$

for $z \in \mathbb{C}$, which is analytic.

Proposition A.4. Let Δ be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with a trivial kernel. Then, for $r \in \mathbb{R}$, the following conditions on $\xi \in \mathcal{H}$ are equivalent.

(i) $\xi \in D(\Delta^r)$.
(ii) The continuous function $\Delta^{it}\xi$ of $t \in \mathbb{R}$ is analytically continued to the strip region $\mathbb{R} - ir[0,1]$.

Proof. We may assume that $r > 0$. Let $\Delta = \int_0^\infty \lambda E(d\lambda)$ be the spectral decomposition.

(i) $\Longrightarrow$ (ii): For $0 \le s \le r$,

$$\begin{aligned}\|\Delta^s\xi\|^2 = \int_0^\infty \lambda^{2s}(\xi|E(d\lambda)\xi) &\le \int_{(0,1)}(\xi|E(d\lambda)\xi) + \int_{[1,\infty)}\Delta^{2r}(\xi|E(d\lambda)\xi)\\ &\le (\xi|\xi) + (\Delta^r\xi|\Delta^r\xi) < \infty\end{aligned}$$

shows that $\xi \in D(\Delta^r)$ and then the dominated convergence theorem ensures that the function

$$\Delta^{iz}\xi = \int_0^\infty \lambda^{iz}E(d\lambda)$$

is norm-continuous on $\mathbb{R} - ir[0,1]$ and analytic in $\mathbb{R} - ir(0,1)$.

(ii) $\Longrightarrow$ (i): Assume that the function $\Delta^{it}\xi$ is analytically continued to $f(z)$ $(z \in \mathbb{R} - ir[0,1])$. If $\eta \in D(\Delta^r)$, then $(\eta|\Delta^{it}\xi) = (\Delta^{-it}\eta|\xi)$ is analytically continued to the relation $(\eta|f(z)) = (\Delta^{-i\bar{z}}\eta|\xi)$. Thus $(\eta|f(-ir)) = (\Delta^r\eta|\xi)$ for $\eta \in D(\Delta^r)$, which means $\xi \in D((\Delta^r)^*) = D(\Delta^r)$ and $\Delta^r\xi = f(-ir)$. □

Example A.5. Let $\mathcal{A}$ be the set of entirely analytic elements for $\sigma_t = \mathrm{Ad}(\Delta^{it})$ on $\mathcal{B}(\mathcal{H})$. Then, for $\xi \in D(\Delta^r)$ $(r \in \mathbb{R})$ and $a \in \mathcal{A}$, the relation $\Delta^{it}(a\xi) = \sigma_t(a)\Delta^{it}\xi$ is analytically continued to $\Delta^r(a\xi) = \sigma_{-ir}(a)\Delta^r\xi$ with $a\xi \in D(\Delta^r)$.

Definition A.6. Let h be a densely defined hermitian operator on a Hilbert space $\mathcal{H}$. An element $\xi \in \mathcal{H}$ is called an **analytic vector** for h if $\xi \in D(h^n)$ for $n = 1, 2, \cdots$ and

$$\sum_{n=0}^\infty \frac{1}{n!}\|h^n\xi\|\, r^n < \infty \quad \text{for some } r > 0.$$

Example A.7. Let h be a self-adjoint operator and $\xi \in D(e^{rh}) \cap D(e^{-rh})$ for some $r > 0$. Then ξ is an analytic vector for h:

$$e^{zh}\xi = \sum_{n=0}^\infty \frac{z^n}{n!}h^n\xi \quad \text{for any } z \in \mathbb{C} \text{ satisfying } |z| < r.$$

In fact, by the assumption $(\xi|e^{\pm 2rh}\xi) < \infty$, we have

$$\sum_{n=0}^{\infty} \frac{(2r)^n}{n!} \| |h|^{n/2}\xi\|^2 = \sum_{n=0}^{\infty} \frac{(2r)^n}{n!} \int_{\mathbb{R}} |\lambda|^n (\xi|E(d\lambda)\xi)$$
$$\leq (\xi|e^{2rh}\xi) + (\xi|e^{-2rh}\xi) < \infty$$

and hence

$$\|h^n\xi\| \leq \| |h|^n\xi\| \leq \ell^2 \left(\frac{\sqrt{(2n)!}}{(2r)^n} \right) = \ell^2 \left(\frac{n!}{r^n} \right).$$

Thus $\sum_n |z|^n \|h^n\xi\|/n! < \infty$ for $|z| < r$. The Taylor expansion identity follows from the dominated convergence theorem in view of inequalities

$$\left\| e^{zh}\xi - \sum_{k=0}^{n} \frac{z^k}{k!} h^k \xi \right\|^2 = \int_{\mathbb{R}} \left| e^{z\lambda} - \sum_{k=0}^{n} \frac{(z\lambda)^k}{k!} \right|^2 (\xi|E(d\lambda)\xi)$$
$$\leq \int_{\mathbb{R}} \left(\sum_{k>n} \frac{|z\lambda|^k}{k!} \right)^2 (\xi|E(d\lambda)\xi)$$
$$\leq \int_{\mathbb{R}} e^{2r|\lambda|} (\xi|E(d\lambda)\xi) \leq \|e^{rh}\xi\|^2 + \|e^{-rh}\xi\|^2$$

for $|z| \leq r$.

A linear combination of analytic vectors is again analytic by taking the common domain of convergence and, if ξ is an analytic vector for h, $h^n\xi$ $(n = 1, 2, \cdots)$ is analytic with the same radius of convergence by the Cauchy-Hadamard formula.

Theorem A.8 (E. Nelson)**.** If a densely defined hermitian operator h has a total set of analytic vectors, then it is essentially self-adjoint, i.e., $h^* = \overline{h}$.

Proof. By the previous observation, we may assume that the domain D of h consists of anaytic vectors and satisfies $hD \subset D$. By the von Neumann's criterion of self-adjointness, it suffices to prove the density of $(h \pm i)D$ in $\mathcal{H}$. To see this, we shall show that $\eta \in \mathcal{H}$ orthogonal to $(h - i)D$ or $(h + i)D$ satisfies $(\eta|D) = 0$.

For $\xi \in D$ with a radius $r > 0$ of convergence, let p be the projection to $\overline{\mathbb{C}[h]\xi} \subset \mathcal{H}$ and let $h_\xi = h|_{\mathbb{C}[h]\xi}$ be a restriction of h. Then the conjugation Γ in $p\mathcal{H}$ defined by

$$\Gamma \left(\sum_{k=0}^{n} \lambda_k h^k \xi \right) = \sum_{k=0}^{n} \overline{\lambda_k} h^k \xi$$

commutes with h_ξ. Consequently, $\Gamma(\ker(h_\xi^* + i)) = \ker(h_\xi^* - i)$ and h_ξ has a self-adjoint extension H. Let $H = \int_{\mathbb{R}} \lambda E(d\lambda)$ be the spectral decomposition. Then, for $0 < s \le r/2$ and $m = 0, 1, 2, \dots,$

$$\begin{aligned}
\|e^{\pm sH}\xi\|^2 &= \int_{\mathbb{R}} e^{\pm 2s\lambda}(\xi|E(d\lambda)\xi) \le \sum_{n=0}^{\infty} \frac{(2s)^n}{n!} \int_{\mathbb{R}} |\lambda|^n (\xi|E(d\lambda)\xi) \\
&= \sum_{n=0}^{\infty} \frac{(2s)^n}{n!} (\xi||H|^n\xi) \le \sum_{n=0}^{\infty} \frac{(2s)^n}{n!} \|\xi\| \, \|H^n\xi\| \\
&= \sum_{n=0}^{\infty} \frac{(2s)^n}{n!} \|\xi\| \, \|h^n\xi\| \le \sum_{n=0}^{\infty} \frac{r^n}{n!} \|\xi\| \, \|h^n\xi\| < \infty
\end{aligned}$$

shows that $\xi \in D(e^{zH})$ for $z \in [-r/2, r/2] + i\mathbb{R}$ and

$$e^{zH}\xi = \int_{\mathbb{R}} e^{\lambda z} E(d\lambda)\xi$$

is an $\mathcal{H}$-valued analytic function of $z \in [-r/2, r/2] + i\mathbb{R}$ with the Taylor expansion at $z = 0$ given by

$$e^{zH}\xi = \sum_{n=0}^{\infty} \frac{z^n}{n!} h^n \xi, \quad |z| \le \frac{r}{2}$$

because of

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{|z|=r/2} dz \, \frac{1}{z^{n+1}} e^{zH}\xi &= \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\oint_{|z|=r/2} \frac{e^{\lambda z}}{z^{n+1}} \, dz \right) E(d\lambda)\xi \\
&= \frac{1}{n!} \int_{\mathbb{R}} \lambda^n E(d\lambda)\xi = \frac{1}{n!} H^n \xi = \frac{1}{n!} h^n \xi.
\end{aligned}$$

Now assume that $(\eta|(h - i)D) = 0$. Then η satisfies $(\eta|h^{n+1}\xi) = i(\eta|h^n\xi)$ for $n = 0, 1, \cdots$, whence $(\eta|h^n\xi) = i^n(\eta|\xi)$. Thus,

$$(\eta|e^{zH}\xi) = \sum_{n=0}^{\infty} \frac{z^n}{n!} (\eta|h^n\xi) = (\eta|\xi)e^{iz}$$

first for $|z| \le r/2$ and then for $z \in [-r/2, r/2] + i\mathbb{R}$ by analytic continuation. Since the left hand side is bounded for $z \in i\mathbb{R}$, we conclude that $(\eta|\xi) = 0$. □

Example A.9. Let $\{\Delta^{it}\}$ be a one-parameter group of unitaries on a Hilbert space $\mathcal{H}$, M be a W*-algebra on $\mathcal{H}$ such that $\Delta^{it} M \Delta^{-it} = M$ for $t \in \mathbb{R}$, and $\mathcal{M}$ be the set of entirely analytic elements of M with respect to $\{\sigma_t(\cdot) = \Delta^{it}(\cdot)\Delta^{-it}\}$. Let $\xi \in \mathcal{H}$ be cyclic for M and satisfy $\Delta^{it}\xi = \xi$ for $t \in \mathbb{R}$. Then $\mathcal{M}\xi$ is a core of the positive self-adjoint Δ^r for any $r \in \mathbb{R}$.

In fact, for $f \in C_c^2(\mathbb{R})$, its inverse Fourier transform $\widehat{f}$ is an entirely analytic integrable function and

$$\sigma_{\widehat{f}}(a) = \int_{\mathbb{R}} \widehat{f}(t)\sigma_t(a)\,dt$$

belongs to $\mathcal{M}$ for any $a \in M$. Since Δ^r operates on

$$\sigma_{\widehat{f}}(a)\xi = \int_{\mathbb{R}} \widehat{f}(t)\Delta^{it}(a\xi)\,dt = 2\pi \int_{\mathbb{R}} f(\lambda)E(d\lambda)(a\xi)$$

boundedly, this is an entirely analytic vector for Δ^r and, if we denote by $\mathcal{M}_0$ the set of all such vectors, $\overline{\Delta^r|_{\mathcal{M}_0\xi}}$ is self-adjoint by the Nelson's theorem. As a self-adjoint extension of $\overline{\Delta^r|_{\mathcal{M}_0\xi}}$, Δ^r coincides with this and $\overline{\Delta^r|_{\mathcal{M}\xi}}$ is equal to Δ^r as an intermediate extension.

For the use in the text, we record here facts on entire analyticity for automorphic actions on operator algebras. Let σ_t be a weak*-continuous one-parameter group of *-automorphisms of a W*-algebra M with $\mathcal{M}$ the set of entirely analytic elements in M. Then $\mathcal{M}$ is weak*-dense *-subalgebra of M. For $a \in \mathcal{M}$, $\sigma_z(a) \in \mathcal{M}$ and $\mathbb{C} \times \mathcal{M} \ni (z,a) \mapsto \sigma_z(a) \in \mathcal{M}$ gives an automorphic action of the additive group $\mathbb{C}$ on $\mathcal{M}$ so that $\sigma_z(a)^* = \sigma_{\overline{z}}(a^*)$.

For an automorphic action on a C*-algebra A, it is common to assume the norm-continuity of $\sigma_t(a)$. In that case, a norm-continuous function $f : \Omega \to A$ on an open subset Ω of $\mathbb{C}$ is said to be holomorphic if it satisfies the following equivalent conditions:

(i) For each $\phi \in A^*$, $\langle f(w), \phi\rangle$ is a holomorphic function of $w \in \Omega$.

(ii) If $z \in \Omega$ and $r > 0$ satisfies $\{w \in \mathbb{C}; |w - z| < r\} \subset \Omega$, then we can find a sequence $\{f_n\}_{n\geq 0}$ in X such that

$$f(w) = \sum_{n=0}^{\infty} (w-z)^n f_n$$

holds in an absolutely norm-convergent manner for $|w - z| < r$.

An element $a \in A$ is said to be **entirely analytic** if $\mathbb{R} \ni t \mapsto \sigma_t(a) \in A$ is extended to a holomorphic function on $\mathbb{C}$. The set $\mathcal{A}$ of entirely analytic elements is then a norm-dense *-subalgebra of A and, if we denote by $\sigma_z(a)$ the holomorphic extension of $\sigma_t(a)$, then σ_z gives an automorphic action of $\mathbb{C}$ on $\mathcal{A}$ so that $\sigma_z(a)^* = \sigma_{\overline{z}}(a^*)$ for $a \in \mathcal{A}$ and $z \in \mathbb{C}$. Note that the Gaussian regularization in the proof of density takes the form of Bochner integral and the associated holomorphic extension is norm-continuous.

Appendix B. Haar Measure

V.S. Varadarajan, Geometry of Quantum Theory.

Appendix C. Pontryagin Duality

For a locally compact abelian group G, the set

$$\widehat{G} = \{\chi : G \to \mathbb{T}; \chi \text{ is continuous and satisfies } \chi(ab) = \chi(a)\chi(b)\}$$

is a subgroup of the product group $\prod_{g \in G} \mathbb{T}$ and, if we furnish it with the topology of uniform convergence on compact subsets of G, $\widehat{G}$ is a locally compact group, called the Pontryagin dual of G.

Theorem C.1 (Pontryagin)**.** The second dual $\widehat{\widehat{G}}$ is naturally identified with G as a locally compact abelian group.

Given Haar measures dg on G and a function f in $L^1(G)$,

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)\, dg$$

defines a function in $C_0(\widehat{G})$, which belong to $L^2(\widehat{G})$ for $f \in L^1(G) \cap L^2(G)$. The correspondence $L^1(G) \cap L^2(G) \ni f \mapsto \widehat{f} \in L^2(\widehat{G})$ gives rise to a unitary map between $L^2(G)$ and $L^2(\widehat{G})$ when the Haar measure of $\widehat{G}$ is appropriately normalized relative to dg.

Appendix D. Group Representations

By a unitary representation of a locally compact group G on a Hilbert space, we shall mean a group homomorphism $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ such that $G \ni g \mapsto U_g\xi \in \mathcal{H}$ is continuous for any $\xi \in \mathcal{H}$. Here the weak continuity is enough to have the norm-continuity of $U_g\xi$ in view of $\|U_g\xi - \xi\|^2 = 2(\xi|\xi) - (\xi|U_g\xi) - (U_g\xi|\xi)$.

Let dg be a left Haar measure. The inequality

$$\int_G |f(g)(\xi|U_g\eta)|\, dg \leq \|\xi\|\, \|\eta\| \int_G |f(g)|\, dg$$

for $f \in L^1(G)$ implies the existence of a bounded operator U_f satisfying

$$(\xi|U_f\eta) = \int_G f(g)(\xi|U_g\eta)\, dg.$$

The Banach space $L^1(G)$ is then made into a Banach *-algebra so that $f \mapsto U_f$ is a *-homomorpshim:

$$(f_1 f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\, dh, \quad f^*(g) = \frac{dg^{-1}}{dg}\overline{f(g^{-1})}.$$

Conversely, given a *-representation π of $L^1(G)$ on a Hilbert space $\mathcal{H}$, a unitary representation U_g is recovered by

$$U_g(\pi(f)\xi) = \pi(gf)\xi,$$

where $(gf)(h) = f(g^{-1}h)$ $(g, h \in G)$. Thus there exits a one-to-one correspondence between unitary representations of G on $\mathcal{H}$ and *-represenations of $L^1(G)$ on $\mathcal{H}$. By the way of construction, the one-to-one correspondence is valid for a dense *-subalgebra $\mathcal{A}$ of $L^1(G)$ satisfying $ga \in \mathcal{A}$ for $g \in G$ and $a \in \mathcal{H}$. Example: $\mathcal{A} = C_c(G)$ or $\mathcal{A} = C_c^\infty(G)$ for a Lie group G.

The universal C*-algebra $C^*(G)$ of $L^1(G)$ (or any smaller $\mathcal{A}$ which is dense in $L^1(G)$) is referred to as the group C*-algebra. In this way, we have established a canonical correspondence between unitary representations of G and *-representations of $C^*(G)$.

Note here that the correspondence between group representations and algebra representations is also valid if we weaken the continuity condition on group representations to the measurability one with respect to the Haar measure class:

Theorem D.1 (von Neumann)**.** If a measurable family $\{T_g\}$ of unitary operators on a Hilbert space $\mathcal{H}$ satisfies $T_gT_h = T_{gh}$ for almost all $(g, h) \in G \times G$, then we can find a unitary representation U_g on G such that $T_g = U_g$ for almost all $g \in G$.

The correspondence of representations also enables us to embed G into $C^*(G)^{**}$ in such a way that

$$\int_G f(g)g\,dg \in C^*(G)^{**}$$

belongs to $C^*(G) \subset C^*(G)^{**}$ for $f \in L^1(G)$. In fact, it is equal to the image of $f \in L^1(G)$ in $C^*(G)$.

A continuous function $\varphi(g)$ of $g \in G$ is said to be **positive definite** if

$$\sum_{1 \leq j,k \leq n} \varphi(g_j^{-1}g_k)\overline{z_j}z_k \geq 0$$

for any $\{z_j\}_{j=1}^n \in \mathbb{C}^n$.

Proposition D.2. There is a one-to-one correspondence between positive definition functions and positive functionals on $C^*(G)$.

Now restrict ourselves to the case of abelian groups. Then a character $\chi : C^*(G) \to \mathbb{C}$ is nothing but a *-representation on a one-dimensional Hilbert space $\mathbb{C}$ and it is rephrased as a unitary representation of G on $\mathbb{C}$, i.e., a continuous group homomorphism $G \to \mathbb{T}$. In this way, the

Gelfand spectrum $\sigma_{C^*(G)}$ of $C^*(G)$ is identified with the dual group $\widehat{G}$ and the Gelfand transform of $f \in L^1(G)$ with the function

$$\widehat{G} \ni \chi \mapsto \int_G f(g)\chi(g)\, dg,$$

which is nothing but the Fourier transform of f.

Example D.3 (unitary spectral decomposition)**.** Any unitary representation of the additive group $\mathbb{Z}$ on $\mathcal{H}$ corresponds to a single unitary U on $\mathcal{H}$ and we can find a projection-valued measure E on $\widehat{\mathbb{Z}} = \mathbb{T}$ so that

$$U = \int_{\mathbb{T}} z\, E(dz).$$

For the vector group $\mathbb{R}^n$, its dual group is identified with $\mathbb{R}^n$ itself by

$$\langle s, t\rangle = e^{is\cdot t}.$$

Example D.4 (Stone)**.** Given a unitary representation U of the vector group $\mathbb{R}^n$, we can find a projection-valued measure E so that

$$U_t = \int_{\mathbb{R}^n} e^{is\cdot t}\, E(ds).$$

Example D.5 (Bochner)**.** Given a continuous positive definite function φ of a locally compact abelian group G, we can find a finite Radon measure μ on the dual group $\widehat{G}$ so that

$$\varphi(g) = \int_{\widehat{G}} \chi(g)\, \mu(d\chi).$$

Appendix E. Projective Representations

A.A. Kirillov, Elements of the Theory of Representations, Springer, 1976.

A. Kleppner, Multipliers on Abelian Groups, Math. Annalen 158(1965), 11–34.

Baggett-Kleppner, Multiplier representations of abelian groups, JFA, 14(1973), 299-324.

Let a locally compact group G be represented by a measurable family of unitaries U_g $(g \in G)$ in a projective way:

$$U_g U_{g'} = \gamma(g, g')U_{gg'}$$

with $\gamma(g, g') \in \mathbb{T}$ a measurable function. Here the function $\gamma(g, g')$ is referred to as a Schur multiplier or simply a cocyle of G. From associativity, we see that γ satisfies the cocycle condition

$$\gamma(g, g')\gamma(gg', g'') = \gamma(g', g'')\gamma(g, g'g'') \quad \text{for all } g, g', g'' \in G$$

and two cocycles γ and γ' belong to the same projective representation if and only if they are equivalent in the sense that we can find a measurable function $\beta : G \to \mathbb{T}$ so that $\gamma'(g, g') = \gamma(g, g')\beta(g)\beta(g')\beta(gg')^{-1}$.

In a reverse way, given a cocycle γ of G, a γ-representation of G is an assignment of unitaries U_g satisfying $U_g U_{g'} = \gamma(g, g')U_{gg'}$.

A cocycle γ is said to be normalized if $\gamma(g, e) = \gamma(e, g) = 1$ for $g \in G$. Any cocycle is equivalent to a normalized one: If we put $g' = e$ in the cocycle condition, the relation $\gamma(g, e) = \gamma(e, g'')$ for $g, g'' \in G$ implies $\gamma(g, e) = \gamma(e, e) = \gamma(e, g)$ for $g \in G$ and therefore γ is equivalent to $\gamma(g, g')/\gamma(e, e)$ ($\beta(g) = \gamma(e, e)^{-1}$). Problems related to Schur multipliers are consequently redeuced to normalized ones.

Given a normalized cocycle γ of G, the product set $G \times \mathbb{T}$ is made into a group (denoted by $G \times_\gamma \mathbb{T}$) so that $(g, z) \mapsto zU_g$ is a unitary representation for any γ-representation U_g of G:

$$(g, z)(g', z') = (gg', zz'\gamma(g, g')).$$

Note that $(e, 1)$ is a unit element in the group $G \times_\gamma \mathbb{T}$ and $(zU_g)^{-1} = z^{-1}U_g^{-1} = z^{-1}\gamma(g, g^{-1})^{-1}\gamma(e, e)^{-1}U_{g^{-1}}$ implies

$$(g, z)^{-1} = (g^{-1}, z^{-1}\gamma(g, g^{-1})^{-1}).$$

Note also that $\mathbb{T}$ is identified with the central subgroup of $G \times_\gamma \mathbb{T}$ by the embedding $\mathbb{T} \ni z \mapsto (e, z) \in G \times \mathbb{T}$.

Let $\xi \in L^2(G)$ be identified with a measurable function ξ on $G \times \mathbb{T}$ satisfying $\xi(g, z) = z\xi(g, 1)$ for $(g, z) \in G \times \mathbb{T}$. Then a γ-representation $\{R_g^\gamma\}$ of G on $L^2(G)$ is obtained via the right translation:

$$(R_g^\gamma \xi)(h, 1) = \xi((h, 1)(g, 1)) = \xi(hg, \gamma(h, g)) = \gamma(h, g)\xi(hg, 1).$$

Exercise 50. Check the relation $R_a^\gamma R_b^\gamma = \gamma(a, b)R_{ab}^\gamma$ for $a, b \in G$.

$$(R_a^\gamma R_b^\gamma \xi)(g) = \gamma(g, a)\gamma(ga, b)\xi(gba) = \gamma(a, b)(R_{ab}^\gamma \xi)(g).$$

In what follows we focus on locally compact second countable abelian groups G. Given a measurable cocycle γ of G, let $[\gamma]$ be another cocycle defined by $[\gamma](a, b) = \gamma(a, b)/\gamma(b, a)$. By the cocycle condition on γ, together with the commutativity in G, we see that $[\gamma]$ is a bicharacter of G, whence it is continuous. Note here that cocycles equivalent to γ give the same bicharacter $[\gamma]$.

Conversely, if $[\gamma] \equiv 1$, then γ is coboundary; we can find a measurable function $\beta : G \to \mathbb{T}$ such that $\gamma(g, g') = \beta(g)\beta(g')\beta(gg')^{-1}$ for all $(g, g') \in G \times G$. In fact, if $[\gamma](g, g') \equiv 1$, then $G \times_\gamma \mathbb{T}$ is commutative and, as an irreducible component of a measurable representation of $G \times_\gamma \mathbb{T}$, we can find a measurable homomorphism $\alpha : G \times_\gamma \mathbb{T} \to \mathbb{T}$

satisfying $\alpha(1, z) = z$.

$\alpha(g, 1)\alpha(g', 1) = \alpha((g, 1)(g', 1)) = \alpha((gg', 1)(1, c(g, g')) = \alpha(gg', 1)c(g, g')$.

Exercise 51. By using the averaging method, show that any separately continuous bicharacter $\langle g, g'\rangle$ is jointly continuous in $(g, g') \in G \times G$.

The point is the continuity at (e, e); $\langle a_n, b_n\rangle \to 1$ if $(a_n, b_n) \to (e, e)$. Since $\langle a, b\rangle$ is separately continuous, the integrals in

$$\int \langle a_n, g\rangle f(b_n^{-1}g)\, dg = \langle a_n, b_n\rangle \int \langle a_n, g\rangle f(g)\, dg$$

converge as $n \to \infty$, which implies $\langle a_n, b_n\rangle \to 1$.

Theorem E.1. The map $\gamma \mapsto [\gamma]$ induces a group-isomorphism from the second cohomology group of G into the group of alternating bicharacters of G.

Let $H = \{h \in G; \gamma(g, h) = \gamma(h, g) \forall g \in G\}$ be the kernel of $[\gamma]$, which is a closed subgroup of G. Since γ is symmetric when restricted to H, we can find a measurable function $\alpha : H \to \mathbb{T}$ so that $\gamma(h, h') = \alpha(h)\alpha(h')\alpha(hh')^{-1}$ for $h, h' \in H$. By choosing a measurable extension $\beta : G \to \mathbb{T}$ of α and replacing γ with $\gamma(g, g')d\beta(g, g')$, we may assume that $\gamma(h, h') = 1$ for $h, h' \in H$ from the outset.

Given a γ-representstion U_g of G on a separable Hilbert space $\mathcal{H}$, consider $U_g \otimes R_g$ ($R_g = R_g^\gamma$ for $\gamma \equiv 1$). Define a unitary operator S on $L^2(G) \otimes \mathcal{H}$ by $(S\xi)(g) = U_g\xi(g)$. Then

$$(S(R_g \otimes U_g)\xi)(h) = U_hU_g\xi(hg) = \gamma(h, g)U_{hg}\xi(hg) = \gamma(h, g)(S\xi)(hg)$$

shows that $S(R_g \otimes U_g)S^* = R_g^\gamma \otimes 1$ for $g \in G$.

If one applies the Fourier transform on the $L^2(G)$ part, the representation $\{R_g \otimes U_g\}$ takes the form $\{T_g\}$ with

$$(T_g\widehat{\xi})(\chi) = \chi(g)U_g\widehat{\xi}(\chi)$$

If the bicharacter $[\gamma]$ gives an isomorphism $G \to \widehat{G}$ of abelian groups, the unitary operator Φ defined by

$$(\Phi\xi)(g) = \int_G [\gamma](g, h)\xi(h)\, dh$$

satisfies

$$[\gamma](h, g)U_g(\Phi\xi)(h) = U_hU_gU_h^*(\Phi\xi)(h) = S(1 \otimes U_g)S^*(\Phi\xi)(h),$$

whence $\{R_g \otimes U_g\}$ is unitarily equivalent to $\{1 \otimes U_g\}$. Thus, $1_{L^2(G)} \otimes U$ is unitarily equivalent to $R^\gamma \otimes 1_{\mathcal{H}}$ for any γ-representation U on $\mathcal{H}$.

Theorem E.2 (Stone-von Neumann)**.** Under the condition that $G \cong \widehat{G}$ via $[\gamma]$, all the γ-representations are quasi-equivalent. In particular, G has a unique (up to unitary equivalence) irreducible γ-representation.

Example E.3. Let $G = H \times \widehat{H}$ with H a locally compact abelian group and $\gamma((a,\chi),(a',\chi')) = \chi(a')$. Then the bicharacter

$$[\gamma]((a,\chi),(a',\chi')) = \frac{\chi(a')}{\chi'(a)}$$

satisfies the condition $G \cong \widehat{G}$.

Let G be as above and define a γ-representation of G on $L^2(H)$ by

$$(U_{(a,\chi)}\xi)(b) = \chi(ba)^{-1}\xi(ba).$$

Then it is irreducible. In fact, $U_{(1,\chi)}$ generates $L^\infty(H)$ on $L^2(H)$, whence the commutant $\{U_g\}'$ is in the fixed point algebra of $L^\infty(H)' = L^\infty(H)$ under the adjoint action of $\{U_{a,1}\}$.

Appendix F. Tensor Products

Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, their tensor product is a Hilbert space $\mathcal{H} \otimes \mathcal{K}$ together with a bilinear map $\mathcal{H} \times \mathcal{K} \ni (\xi,\eta) \mapsto \xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ satisfying the following properties.

(i) For $\xi, \xi' \in \mathcal{H}$ and $\eta, \eta' \in \mathcal{K}$, $(\xi \otimes \eta | \xi' \otimes \eta') = (\xi|\xi')\,(\eta|\eta')$.

(ii) Linear combinations of elements of the form $\xi \otimes \eta$ are dense in $\mathcal{H} \otimes \mathcal{K}$.

If $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$ are orthonormal bases in $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{\xi_i \otimes \eta_j\}_{(i,j) \in I \times J}$ is an orthonormal basis in $\mathcal{H} \otimes \mathcal{K}$.

Proposition F.1. Tensor product exists and is unique.

Clearly, given $\xi \in \mathcal{H}$, the linear map $\mathcal{K} \ni \eta \mapsto \xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ is a scalar multiplication of an isometry and its adjoint, denoted by $\langle \cdot \rangle_{\xi \otimes 1}$, is specified by $\xi' \otimes \eta \mapsto (\xi|\xi')\eta \in \mathcal{K}$ and referred to as a partial evaluation by $\xi \in \mathcal{H}$.

*-representations $\mathcal{B}(\mathcal{H}) \ni a \mapsto a \otimes 1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and $\mathcal{B}(\mathcal{K}) \ni b \mapsto \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ are defined by

$$(a \otimes 1)(\xi \otimes \eta) = a\xi \otimes \eta, \quad (1 \otimes b)(\xi \otimes \eta) = \xi \otimes b\eta,$$

and set $a \otimes b = (a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1)$.

Given W*-algebras M on $\mathcal{H}$ and N on $\mathcal{K}$, their tensor product $M \otimes N$ is a W*-algebra on $\mathcal{H} \otimes \mathcal{K}$ obtained as the σ-weak closure of the *-subalgebra generated by $a \otimes b$ ($a \in M$, $b \in N$). Note that $M \otimes \mathbb{C}1_{\mathcal{K}} = \{a \otimes 1; a \in M\}$ and similarly for $\mathbb{C}1_{\mathcal{H}} \otimes N$.

Appendix G. Infinite Tensor Products

J. von Neumann, On infinite direct products, Composi. Math., 6(1939), 1–77.

A. Guichardet, Produits tensoriels infinis et représentations des relations d'anticommutation, Annales scientifiques de l'É.N.S., 83(1966), 1–52.

An algebraically sophisticated way to introduce finite tensor products $\bigotimes V_i$ is to define them as subspaces of multilinear functionals on $\prod V_i^*$: Given a finite family $\{v_i\}_{i\in I}$ of vectors, let $\otimes_{i\in I} v_i$ be a multilinear functional defined by

$$\otimes_{i\in I} v_i : (v_i^*) \mapsto \prod_{i\in I} \langle v_i, v_i^* \rangle.$$

Then $\bigotimes_{i\in I} V_i$ is the linear span of $\{\otimes_{i\in I} v_i\}$. From the associativity of direct products, the associativity for tensor products follows: If $I = \bigsqcup_{j\in J} I_j$, there is a natural isomorphism

$$\bigotimes_{j\in J} \left(\bigotimes_{i\in I_j} V_i \right) \cong \bigotimes_{i\in I} V_i.$$

In particular, given a decomposition $I = I' \sqcup I''$ and a family $\{v_{i''}\}_{i''\in I''}$,

$$\bigotimes_{i'\in I'} V_{i'} \ni \bigotimes_{i'\in I'} v_{i'} \mapsto \bigotimes_{i\in I} v_i \in \bigotimes_{i\in I} V_i \quad \text{with } \{v_i\} = \{v_{i'}\} \cup \{v_{i''}\}$$

is extended to a linear map.

When the index set is linearly ordered, it is customary to notationally reflect it in a geometric position. For example, if $I = \{1, 2, \dots, n\}$, we shall write $\otimes_{i\in I} v_i = v_1 \otimes \cdots \otimes v_n$.

Now consider a general family $\{\mathcal{A}_i\}$ of unital *-algebras. Then the family of unital *-algebras $\{\bigotimes_{i\in F} \mathcal{A}_i\}$, where F runs through finite subsets of I, is directed for inclusions of F by the map

$$\bigotimes_{i\in F} a_i \mapsto \bigotimes_{i\in F'} a_i',$$

where

$$a_i' = \begin{cases} a_i & \text{if } i \in F, \\ 1_i & \text{otherwise.} \end{cases}$$

The inductive limit $\varinjlim_{F\to I} \bigotimes_{i\in F} \mathcal{A}_i$ is denoted by $\bigotimes_{i\in I} \mathcal{A}_i$ and called the algebraic tensor product of $\{\mathcal{A}_i\}_{i\in I}$. Note that $\bigotimes \mathcal{A}_i$ is generated by $\otimes a_i$ ($a_i = 1$ except for finitely many indices) and $\bigotimes \mathcal{A}_i$ is unitary if so is each $\mathcal{A}_i$. Given a family $\{\varphi_i\}$ of states, a state $\varphi = \otimes\varphi_i$ of $\bigotimes \mathcal{A}_i$ is defined by $\varphi(\otimes a_i) = \prod \varphi_i(a_i)$. The positivity of φ is a consequence of

$$\sum_{j,k} \overline{z_j} z_k \varphi(\otimes_{i\in I} a_{i,j}^* a_{i,k}) = \sum_{j,k} \overline{z_j} z_k \prod_i \varphi_i(a_{i,j}^* a_{i,k}) \geq 0.$$

For a family of positive functionals $\{\varphi_i\}$, the positive functional $\otimes\varphi_i$ is defined by $\prod \varphi_i(1) \otimes (\varphi_i/\varphi_i(1))$ if $\prod \varphi_i(1) < \infty$. Note that $\otimes varphi_i$ is bounded if so is each φ_i under the condition $\prod \varphi_i(1) < \infty$, which is therefore identified with a bounded positive functional on the C*-algebra associated to $\bigotimes \mathcal{A}_i$.

The following formula looks quite reasonable but is in the heart of the celebrated Kakutani dichotomy on infinite product measures.

Theorem G.1. Assume that φ_i and ψ_i are bounded states for each $i \in I$. The, for $x = \otimes x_i$ and $y = \otimes y_i$ in $\bigotimes \mathcal{A}_i$, we have

$$(x(\otimes\varphi_i)^{1/2} | (\otimes\psi_i)^{1/2} y) = \prod_{i\in I} (x_i \varphi_i^{1/2} | \psi_i^{1/2} y_i).$$

Corollary G.2. If $\prod_{i\in I\setminus F} (\varphi_i^{1/2} | \psi_i^{1/2}) = 0$ for any finite subset F of I, then $\otimes\varphi_i$ and $\otimes\psi_i$ are disjoint.

Given a family $\{z_\alpha\}_{\alpha\in I}$ of complex numbers, let $z_F = \prod_{\alpha\in F} z_\alpha$ for a finite subset $F \subset I$. If $\{z_F\}$ is a coonvergent net of complex numbers, its limit is denoted by $\prod_{\alpha\in I} z_\alpha$ and we see $\prod_{\alpha\in I} |z_\alpha| = \left|\prod_{\alpha\in I} z_\alpha\right|$.

If $z_\alpha = 0$ for some $\alpha \in I$, then $\prod_{\alpha\in I} z_\alpha = 0$. If not, $\{|z_G|\}$ is bounded for any finite $G \subset I$ satisfying $|z_\alpha| > 1$ ($\alpha \in G$). In fact, if it is not bounded, given any finite $F \subset I$, we can choose $G \subset I \setminus F$ so that $|z_G|$ is arbitrarily large while $|z_F| \neq 0$, i.e., $|z_{F\sqcup G}|$ can be arbitrarily large, which contradicts with the convergence of $\prod_{\alpha\in I} |z_\alpha|$.

Thus the condition $\prod z_\alpha = 0$ is equivalent to (i) $z_\alpha = 0$ for some $\alpha \in I$ or (ii) $\prod_{\alpha:|z_\alpha|>1} |z_\alpha| < \infty$ and $\prod_{\alpha:|z_\alpha|\leq 1} |z_\alpha| = 0$.

Now assume that $\prod z_\alpha \neq 0$. Then $(\prod z_\alpha)^{-1} = \prod z_\alpha^{-1} \neq 0$, implies that

$$\prod_{\alpha:|z_\alpha|>1} |z_\alpha| < \infty, \qquad \prod_{\alpha:|z_\alpha|\leq 1} |z_\alpha| > 0$$

and $\prod e^{i\theta_\alpha}$ is convergent to a complex number of modulus one, where $z_\alpha = |z_\alpha| e^{i\theta_\alpha}$ with $-\pi < \theta_\alpha \leq \pi$. From the convergence of $\prod e^{i\theta_\alpha}$, we see that, given $\epsilon > 0$, we can find a finite $F \subset I$ such that $|\theta_\alpha| \leq \epsilon$ for any $\alpha \in I \setminus F$. Thus the convergence of $\prod e^{i\theta_\alpha}$ is equivalent to the convergence of $\sum \theta_\alpha$, which is combined with the convergence of $\sum_{\alpha\in I_\pm} \log |z_\alpha|$ ($I_\pm = \{\alpha \in I; \pm \log |z_\alpha| > 0\}$) to get the convergence of

$\sum \log z_\alpha$, i.e., $\sum |z_\alpha - 1| < \infty$. Conversely, this condition implies the absolute convergence of $\sum \log z_\alpha$ and we see that

$$\prod_{\alpha \in I} z_\alpha = e^{\sum \log z_\alpha}$$

is convergent.

Let $\{\mathcal{H}_i\}_{i\in I}$ be an infinite family of Hilbert spaces and $\{\iota_i \in \mathcal{H}_i\}$ be a family of unit vectors. Given a finite subset F of I, let

$$\mathcal{H}_F = \left(\bigoplus_\phi \mathcal{H}_{\phi(1)} \otimes \mathcal{H}_{\phi(2)} \otimes \cdots \otimes \mathcal{H}_{\phi(n)} \right)^{S_n}$$

be the symmetrized tensor product of $\{\mathcal{H}_i\}_{i\in F}$, where $\phi : \{1, \dots, n\} \to F$ $(n = |F|)$ runs through bijections and $(\ )^{S_n}$ denotes the fixed-point subspace under the obvious action of the symmetric group S_n. Thus, for each ϕ, $\mathcal{H}_F$ can be identified with the Hilbert space $\mathcal{H}_{\phi(1)} \otimes \cdots \otimes \mathcal{H}_{\phi(n)}$.

Define an embedding of $\mathcal{H}_F$ into $\mathcal{H}_{F'}$ for $F \subset F'$ by

$$\xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)} \mapsto \xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)} \otimes \iota_{\phi'(n+1)} \otimes \cdots \otimes \iota_{\phi'(n')},$$

where $n' = |F'|$ and $\phi' : \{1, \dots, n'\} \to F'$ is any extension of ϕ.

The inductive limit Hilbert space $\lim_{F\nearrow I} \mathcal{H}_F$ is called the infinite tensor product of $\{\mathcal{H}_i\}_{i\in I}$ with respect to the reference vector $\{\iota_i\}$ and denoted by $\bigotimes_{i\in I} \mathcal{H}_i$. The image of $\xi_F \in \mathcal{H}_F$ in $\bigotimes_{i\in I} \mathcal{H}_i$ is denoted by

$$\xi_F \otimes \bigotimes_{i \in I\setminus F} \iota_i.$$

Lemma G.3. Given a family of vectors $\{0 \neq \xi_i \in \mathcal{H}_i\}_{i\in I}$, let $\xi_F \in \mathcal{H}_F$ be defined by $\xi_{\phi(1)} \otimes \cdots \otimes \xi_{\phi(n)}$. Then the limit

$$\otimes \xi_i = \lim_{F\nearrow I} \xi_F \otimes \bigotimes_{i\in I\setminus F} \iota_i$$

exists in $\bigotimes_{i\in I} \mathcal{H}_i$ and $\otimes\xi_i \neq 0$ if and only if

$$\sum_{i\in I} |\log(\xi_i|\xi_i)| < \infty \quad \text{and} \quad \sum_{i\in I} |1 - (\iota_i|\xi_i)| < \infty.$$

Moreover, given another such family $\{\eta_i\}$, we have

$$(\otimes\xi_i|\otimes\eta_i) = \prod (\xi_i|\eta_i),$$

where the infinite product converges absolutely;

$$\sum_{i\in I} |1 - (\xi_i|\eta_i)| < \infty.$$

Appendix H. Tensor Products of Closed Operators

Let A and B be closable operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with their adjoints A^* and B^* densely defined. The algebraic tensor product $A \odot B$ is then an operator with its domain $D(A \odot B)$ equal to the algebraic tensor product $D(A) \odot D(B) \subset \mathcal{H} \otimes \mathcal{K}$. Then $A \odot B$ is closable in view of $(A \odot B)^* \supset A^* \odot B^*$.

Definition H.1. For closed operators A and B with A^* and B^* densely defined, we write $A \otimes B = (A \odot B)^{**}$.

Proposition H.2. We have $(A \odot B)^{**} = (A^* \odot B^*)^*$.

Proof. Since the graph of A^{**} (B^{**}) is approximated by that of A (B), we see $A \odot B \subset A^{**} \odot B^{**} \subset (A \odot B)^{**}$, i.e., the closure of $A^{**} \odot B^{**}$ is $(A \odot B)^{**}$. Thus the problem is reduce to the case $A = A^{**}$ and $B = B^{**}$, which is assumed in what follows.

Let $A = u|A|$ and $B = v|B|$ be the polar decompositions. Then as a closure of $A \odot B = (u \otimes v)(|A| \odot |B|)$, we have

$$(A \odot B)^{**} = (u \otimes v)(|A| \odot |B|)^{**},$$

whereas $A^* \odot B^* = (|A| \odot |B|)(u^* \otimes v^*)$ with $|A| \odot |B|$ supported by a partial isometry $u^* \otimes v^*$ implies

$$(A^* \odot B^*)^* = (u \otimes v)(|A| \odot |B|)^*.$$

In this way, the problem is further reduced to showing $(|A| \odot |B|)^{**} = (|A| \odot |B|)^*$, i.e., the essential self-adjointness of $|A| \odot |B|$. To see this, consider spectral decompositions

$$|A| = \int \alpha\, e(d\alpha), \quad |B| = \int \beta\, f(d\beta)$$

and let $\mathcal{H}_\infty$, $\mathcal{K}_\infty$ be spectrally bounded subspaces. Then vectors in $\mathcal{H}_\infty \odot \mathcal{K}_\infty$ are entirely analytic with respect to $|A| \odot |B|$, whence $|A| \odot |B|$ is essentially self-adjoint by Nelson's analytic vector theorem. $\square$

Corollary H.3. For closed operators A, B with $A = u|A|$, $B = v|B|$ their polar decompositions, $|A| \otimes |B|$ is positively self-adjoint and $(u \otimes v)(|A| \otimes |B|)$ gives the polar decomposition of $A \otimes B$.

Proof. Since the partial isometry $u \otimes v$ clearly supports $|A| \otimes |B|$, we just need to chek the positivity of $|A| \otimes |B|$. We first remark the positivity of $|A| \odot |B|$ in the sense that, for $\sum_j \xi_j \otimes \eta_j \in D(|A|) \odot D(|B|)$ with $\xi_j \in D(|A|)$ and $\eta_j \in D(|B|)$,

$$\sum_{j,k} (\xi_j \otimes \eta_j | |A| \xi_k \otimes |B| \eta_k) = \sum_{j,k} (\xi_j | |A| \xi_k)\, (\eta_j | |B| \eta_k) \geq 0$$

because $(\xi_j||A|\xi_k)$ and $(\eta_j||A|\eta_k)$ together with their complex conjugates are positive matrices and the last expression is the form of the trace of the product of two positive matrices.

The positivity of $|A|\otimes|B|$ follows from this: Given $\zeta \in D(|A|\otimes|B|)$, we can find a sequence $\zeta_n \in D(|A|\odot|B|)$ so that $\zeta_n \oplus (|A|\odot|B|)\zeta_n \to \zeta \oplus (|A|\otimes|B|)\zeta$, whence

$$(\zeta|(|A|\otimes|B|)\zeta) = \lim_n (\zeta_n|(|A|\odot|B|)\zeta_n) \geq 0.$$

$\square$

Appendix I. Polarity in Banach Spaces

Let X, Y be complex vector spaces and suppose that they are coupled by a non-degenerate bilinear form $\langle x, y\rangle$. If they are furnished with weak topologies, then continuous linear functionals are given by the coupling becuase the continuity of a linear functional $f : X \to \mathbb{C}$ relative to the seminorm $|\langle x, y_1\rangle| + \cdots + |\langle x, y_n\rangle|$ implies that f passes through the linear map $X \ni x \mapsto (\langle x, y_1\rangle, \ldots, \langle x, y_n\rangle) \in \mathbb{C}^n$. For a subspece E of X or Y, let $E^\perp$ be the polar of E with respect to the coupling $\langle\ ,\ \rangle$. Clearly $E \subset F$ implies $F^\perp \subset E^\perp$ and $E \subset E^{\perp\perp}$, which are combined to see that $E^{\perp\perp\perp} = E^\perp$. $E^{\perp\perp}$ is the weak closure of E by Hahn-Banach theorem. There is a one-to-one correspondence between weakly closed subspaces of X and weakly closed subspaces of Y by taking polars.

Now let X be a Banach space and $Y = X^*$ the dual Banach space of X.

Again, by Hahn-Banach theorem, norm closure and weak closure coincide for convex subsets of X. The weak topology on X^* via the natural coupling $\langle x, f\rangle = f(x)$ $(x \in X,\ f \in X^*)$ is referred to as the weak* topology to avoid confusion with the weak topology of the obvious coupling between X^* and X^{**}.

Proposition I.1. Let $F \subset X^*$ be a subspace. Then the weak* closure $F^{\perp\perp}$ of F is naturally identified with the dual $(X/F^\perp)^*$ of the quotient Banach space $X/F^\perp$ and we have

$$\sup\{|\langle x, f\rangle|; f \in F^{\perp\perp}, \|f\| \leq 1\} = \inf\{\|x + e\|; e \in F^\perp\}.$$

Proof. For $e \in F^\perp$ and $f \in f^{\perp\perp}$ with $\|f\| \leq 1$, we see

$$|\langle x, f\rangle| = |\langle x + e, f\rangle| \leq \|x + e\|$$

and then, by taking inf for e and sup for f,

$$\sup\{|\langle x, f\rangle|; f \in F^{\perp\perp}, \|f\| \leq 1\} \leq \inf\{\|x + e\|; e \in F^\perp\}.$$

By Hahn-Banach theorem, we can find $\varphi : X/F^\perp \to \mathbb{C}$ such that $\|\varphi\| = 1$ and $\varphi(x + F^\perp) = \|x + F^\perp\|$. Let $f \in F^{\perp\perp}$ be the composition $f(x) = \varphi(x + F^\perp)$. Then $\|f\| \leq 1$ and $|\langle x, f\rangle| = \|x + F^\perp\|$. □

Appendix J. Radon Measures

Riesz-Radon-Banach-Markov-Kakutani theorem.

Given a commutative C*-algebra $A = C_0(\Omega)$ with Ω a locally compact space, there is a one-to-one correspondence between elements of A^* and regular complex Borel measures on Ω by the relation

$$\varphi(a) = \int_\Omega a(\omega)\,\mu(d\omega)$$

so that $\|\varphi\| = |\mu|(\Omega)$. Under this correspondence, φ is positive if and only if μ is positive.

The essence in this correspondence can be summarized as follows:

Theorem J.1. Let $B(\Omega)$ be the *-algebra of bounded Baire functions. Then the natural embedding $C_0(\Omega) \to C_0(\Omega)^{**}$ is extended to a *-isomorphism of $B(\Omega)$ onto $C_0(\Omega)^{**}$ in such a way that, if a uniformly bounded sequence $f_n \in B(\Omega)$ converges to $f \in B(\Omega)$ point-wise, then $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ for any $\varphi \in C_0(\Omega)^*$.

Now it is immediate to get the spectral decomposition theorem. Let $\pi : C_0(\Omega) \to \mathcal{B}(\mathcal{H})$ be a *-representation on a Hilbert space $\mathcal{H}$. Since π is extended to a normal *-homomorphism $\pi^{**} : B(\Omega) = C_0(\Omega)^{**} \to \pi(C_0(\Omega))''$, if we define a projection-valued Baire measure E by $E(S) = \pi^{**}(1_S)$, then

$$\pi^{**}(f) = \int_\Omega f(\omega)\,E(d\omega) \quad \text{for } f \in B(\Omega).$$

Note that the class of Borel functions coincides with that of Baire functions when Ω is second countable. To avoid measure-theoretical complexities, we assume this condition from here on.

A group G is said to be locally compact if it is furnished with a locally compact topology so that the group operations, $G \times G \ni (a, b) \mapsto ab \in G$ and $G \ni g \mapsto g^{-1} \in G$ are continuous.

A positive Radon measure μ on a locally compact group G is called a left (resp. right) Haar measure if it is invariant under the left (resp. right) translations.

Theorem J.2. A left Haar measure exists and it is unique up to scalar multiplications.

Appendix K. Vector-valued Integration

Let X be a Banach space, μ be a finite (positive) measure on a measure space Ω, and f be a uniformly bounded X-valued function on Ω such that $\phi \circ f$ is μ-integrable for every $\phi \in X^*$. Then the integration

$$\int \langle f(\omega), \phi \rangle \, \mu(d\omega)$$

gives a linear functional $\int f(\omega)\, \mu(d\omega)$ on X^* with an obvious estimate

$$\left\| \int f(\omega)\, \mu(d\omega) \right\| \leq \mu(\Omega) \|\phi\|_{f(\Omega)} \leq \mu(\Omega)\|\phi\|_C,$$

where $\|\phi\|_S = \sup\{|\langle x, \phi \rangle|; x \in S\}$ for a subset S of X and C denotes the weak closure of the absolute convex hull of $f(\Omega)$. Thus, if C is weakly compact, $\int f\, d\mu$ is weak*-continuous by Mackey-Arens theorem and therefore realized by an element in X. This is especially the case when μ is a Radon measure on a compact space Ω and f is weakly continuous: The image $f(\Omega)$ is weakly compact and, by Krein-Smulian theorem, the weak closure of its convex hull is also weakly compact.

Note that we need somewhat longer arguments for the whole path of its proof. We shall here give a direct proof of $\int f d\mu \in X$ (in fact this is a preliminary argument in the proof of Krein-Smulian theorem) under the extra assumption of separability of X and uniform boundedness of $f(\Omega)$: By uniform boundedness, we first know that there is an $a \in X^{**}$ satisfying

$$\langle a, \phi \rangle = \int_\Omega \langle f(\omega), \phi \rangle \, \mu(d\omega)$$

for $\phi \in X^*$ and the problem is to show that a in fact belongs to $X \subset X^{**}$.

If not, $\|a + X\| > 0$ and Hahn-Banach theorem enables us to find a functional $\varphi : X^{**} \to \mathbb{C}$ of norm one satisfying $\varphi(X) = 0$ and $\varphi(a) = \|a + X\|$.

Take a countable set $\{a_j; j \geq 1\}$ which is norm-dense in X_1 and consider weak* neighborhoods of $\varphi \in X^{***}$ described by

$$|\varphi(a) - \varphi'(a)| \leq \frac{1}{n}, \quad |\varphi(a_j) - \varphi'(a_j)| \leq \frac{1}{n} \ (1 \leq j \leq n).$$

Since the unit ball X_1^* is weak*-dense in X_1^{***} (Goldstine's theorem), we can then find a sequence $\{\varphi_n\}$ in X_1^* so that

$$|\varphi(a) - \langle \varphi_n, a \rangle| \leq \frac{1}{n} \quad \text{and} \quad |\varphi(a_j) - \langle \varphi_n, a_j \rangle| \leq \frac{1}{n} \quad \text{for } 1 \leq j \leq n.$$

From the former $|\langle\varphi_n, a\rangle| \geq \|a+X\|-1/n$, whereas the latter inequalities $|\langle\varphi_n, a_j\rangle| \leq 1/n$ $(1 \leq j \leq n)$ mean that $\varphi_n \to 0$ in X_1^* with respect to the weak* topology.

Now, by dominated convergence theorem, $\lim_{n\to\infty} \langle\varphi_n, a\rangle = 0$, which contradicts with $|\langle\varphi_n, a\rangle| \geq \|a+X\| - 1/n > 0$.

Theorem K.1 (Helly)**.** Let X be a Banach space. Given a finite $\phi_1, \cdots, \phi_n \in X^*$, $x^{**} \in X_1^{**}$ and $r > 1$, we can find $x \in X_r$ so that $\phi_j(x) = x^{**}(\phi_j)$ for $1 \leq j \leq n$. Here $X_r = \{x \in X; \|x\| \leq r\}$.

Proof. We may assume that $\{\phi_1, \cdots, \phi_n\}$ is linearly independent. Then $\Phi : X \ni x \mapsto (\phi_1(x), \cdots, \phi_n(x)) \in \mathbb{C}^n$ is surjective and, given any $x^{**} \in X_1^{**}$, there exists an $a \in X$ such that $\Phi(a) = (x^{**}(\phi_1), \cdots, x^{**}(\phi_n))$. We need to show that $(a + \ker\Phi) \cap X_r \neq \emptyset$ for every $r \geq 1$.

If not, $\|a + \ker\Phi\| \geq r$ and, by Hahn-Banach, there is $\varphi \in X_1^*$ satisfying $\varphi(\ker\Phi) = 0$ and $|\varphi(a)| = \|a+\ker\Phi\|$. In view of $(\ker\Phi)^\perp = \mathbb{C}\phi_1 + \cdots + \mathbb{C}\phi_n$, we have an expression $\varphi = \lambda_1\phi_1 + \cdots + \lambda_n\phi_n$ with $\lambda_j \in \mathbb{C}$ and $\varphi(a) = \sum \lambda_j\phi_j(a) = \sum \lambda_j x^{**}(\phi_j) = x^{**}(\varphi)$. Now

$$r \leq |\varphi(a)| = |x^{**}(\varphi)| \leq \|x^{**}\|\,\|\varphi\| \leq 1,$$

a contradiction. □

Corollary K.2 (Goldstine)**.** X_1 is weak*-dense in X_1^{**}.

Theorem K.3. Let a locally compact group G act on a Banach space X by isometries and assume that $G \ni g \mapsto \phi(gx)$ is continuous for every $x \in X$ and $\phi \in X^*$. Then $G \ni g \mapsto gx \in X$ is norm-continuous.

Proof. Since the vector-valued function gx of $g \in G$ is weakly continuous, the integration $fx = \int_G f(g)gx\,dg \in X$ is well-defined for $f \in C_c(G)$ so that

$$\phi\left(\int_G f(g)gx\,dg\right) = \int_G f(g)\phi(gx)\,dg,$$

which gives an estimate

$$\left\|\int_G f(g)gx\,dg\right\| \leq \|x\| \int_G |f(g)|\,dg = \|x\|\,\|f\|_1.$$

Note that $f_1(f_2x) = (f_1 * f_2)x$ for $f_j \in C_c(G)$.

Since $g(fx) = (gf)x$ with $(gf)(g') = f(g^{-1}g')$, we see that

$$\|g(fx) - fx\| \leq \|x\|\|gf - f\|_1 \to 0$$

as $g \to e$; $G \ni g \mapsto g(fx) \in X$ is norm-continuous. Thus, if we set

$$Y = \{y \in X; gy \in X \text{ is norm-continuous in } g \in G\},$$

it includes $C_c(G)X$. We claim that Y is norm-dense in X. By Hahn-Banach theorem, this is equivalent to the weak density of Y. So, let $\phi \in X^*$ vanish on Y and we shall check that $\phi = 0$.

$$0 = \phi(fx) = \int_G f(g)\phi(gx)\, dg$$

for any $f \in C_c(G)$ and, by choosing approximate delta functions as an f, we have $\phi(x) = 0$ for any $x \in X$. □

Remark 9.

(i) If $\delta_n \in C_c(G)$ is an approximate unit in the convolution algebra $L^1(G)$, then $\|\delta_n x - x\| \to 0$ $(n \to \infty)$ by a three-epsilon argument.

(ii) The dualized action of G on X^* is not necessarily norm-continuous.

Appendix L. Sesquilinear Forms

A sesquilinear form θ on a complex vector space D is said to be hermitian (resp. positive) if $\theta(x,x) \in \mathbb{R}$ (resp. $\theta(x,x) \geq 0$) for $x \in D$. By the polarization identity, a hermitian form satisfies $\theta(y,x) = \overline{\theta(x.y)}$.

A positive form θ defined on a dense linear subspace D of a Hilbert space $\mathcal{H}$ is said to be closed if D is complete with respect to the inner product $(\xi|\eta)_\theta = (\xi|\eta) + \theta(\xi,\eta)$.

Example L.1. Let Θ be a densely defined positive operator in $\mathcal{H}$. Then $\theta(\xi,\eta) = (\xi|\Theta\eta)$ is a positive form on $D(\Theta)$.

Example L.2. A positive form associated to a positive operator Θ is closable in the following sense:

Let D be the completion of $D(\Theta)$ relative to the inner product $(\ |\ )_\theta$. Since the embedding $D(\Theta) \subset \mathcal{H}$ is norm-decreasing, it gives rise to a contractive linear map $\phi : D \to \mathcal{H}$, which turns out to be injective: Suppose that $\xi \in D$ and $\phi(\xi) = 0$. Then we can find a sequence $\xi_n \in D(\Theta)$ such that $\|\xi_n - \xi\|_\theta \to 0$ and $\|\xi_n\| \to 0$.

$$\|\xi\|_\theta^2 = \lim_{m,n\to\infty} (\xi_m|\xi_n)_\theta = \lim_{n\to\infty}\lim_{m\to\infty} \Big((\xi_m|\xi_n) + (\xi_m|\Theta\xi_n)\Big) = 0.$$

Clearly $D(\Theta) \subset \phi(D)$ and $(\phi^{-1}\xi|\phi^{-1}\eta)_\theta - (\xi|\eta)$ is a closed positive form which extends θ.

Given a closed positive form θ on $D \subset \mathcal{H}$, we want to get a positive self-adjoint operator Θ such that $\theta(\xi,\eta) = (\xi|\Theta\eta)$. To see this, let $\phi : D \to \mathcal{H}$ denote the embedding, which is norm-decreasing, i.e., $\|\phi(\xi)\| \leq \|\xi\|_\theta$ for $\xi \in D$, and set $R = \phi\phi^* : \mathcal{H} \to \mathcal{H}$, which is a positive contraction and satisfies the relation

$$(\xi|R\eta)_\theta = (\xi|\phi^*\eta)_\theta = (\xi|\eta), \quad \xi,\eta \in \mathcal{H}.$$

Since D is dense in $\mathcal{H}$, ϕ^* is injective and so is R. Thus R^{-1} with $D(R^{-1}) = R\mathcal{H} = \phi^*\mathcal{H} \subset D$ is a positive self-adjoint operator satisfying $R^{-1} \geq 1$. We notice here that $\phi^*\mathcal{H}$ is dense in D. In fact, if $\eta \in D$ is orthogonal to $\phi^*\mathcal{H}$, $0 = (\eta|\phi^*\xi)_\theta = (\eta|\xi)$ for any $\xi \in \mathcal{H}$ implies $\eta = 0$.

Let $\Theta = R^{-1} - 1$ with $D(H) = D(R^{-1}) = \phi^*\mathcal{H}$ be a positive self-adjoint operator. Then

$$\begin{aligned}(\phi^*\xi|\Theta\phi^*\eta) &= (\phi\phi^*\xi|R^{-1}\phi\phi^*\eta) - (\phi^*\xi|\phi^*\eta) = (\phi\phi^*\xi|\eta) - (\phi^*\xi|\phi^*\eta) \\ &= (\phi^*\xi|\phi^*\eta)_\theta - (\phi^*\xi|\phi^*\eta) = \theta(\phi^*\xi, \phi^*\eta).\end{aligned}$$

Finally observe that $D = D(\Theta^{1/2})$ in view of the density of $D(\Theta)$ in D and θ coincides with the closure of the positive form $(\xi|\Theta\eta)$ on $D(\Theta)$.

We shall review basics on the Pusz-Woronowicz theory of functional calculus on sesquilinear forms. Let α, β be positive (sesquilinear) forms on a complex vector space H. By a **representation** of the unordered pair $\{\alpha, \beta\}$, we shall mean a linear map $i : H \to K$ of H into a Hilbert space K together with positive (self-adjoint) operators A, B in K such that A commutes with B in the spectral sense, $i(H)$ is a core for the self-adjoint operator $A + B$ and

$$\alpha(x, y) = (i(x)|Ai(y)), \quad \beta(x, y) = (i(x)|Bi(y))$$

for $x, y \in H$. Note that $i(H)$ is included in the domains of $A = \frac{A}{A+B+I}(A + B + I)$ and $B = \frac{B}{A+B+I}(A + B + I)$. When A and B are bounded, we say that the representation is **bounded**. Note that, the core condition is reduced to the density of $i(H)$ in K for a bounded representation.

If A and B are commuting self-adjoint operators with spectral measures $e_A(ds)$, $e_B(dt)$ respectively and $f(s, t)$ be a complex-valued Borel function on $\sigma(A) \times \sigma(B) \subset \mathbb{R}^2$, then the normal operator $f(A, B)$ is defined by

$$f(A, B)\xi = \int_{\sigma(A)\times\sigma(B)} f(s, t)e_A(ds)e_B(dt)\xi.$$

Lemma L.3. Any pair of positive forms α, β on H admits a bounded representation.

Proof. Let K be the Hilbert space associated to the positive form $\alpha+\beta$ and $i : H \to K$ be the natural map. By the Riesz lemma, we have bounded operators A and B on K representing α and β respectively, which commute becuase of $A + B = 1_K$. □

Lemma L.4 (cf. Reed-Simon §VIII.6)**.**

(i) Let D be a core of a positive self-adjoint operator C on a Hilbert space $\mathcal{H}$. Then D is a core for $C^{1/2}$. In particular, we have the domain inclusion $D(C) \subset D(C^{1/2})$.

(ii) Let A and B be commuting positive (self-adjoint) operators on a Hilbert space $\mathcal{H}$ with $A + B$ denoting the closure of $A|_{D(A)\cap D(B)} + B|_{D(A)\cap D(B)}$. Let $D \subset \mathcal{H}$ be a core for the positive operator $A + B$, then D is a core for A and B as well.

Proof. (i) By a spectral representation of C, we may assume that $\mathcal{H} = \int_{t\geq 0} \mathcal{H}(t)\, \mu(dt)$ with C given by multiplication of t. Then any vector in $D(C^{1/2})$ is of the form $\int_{t\geq 0}^{\oplus} f(t)\, \mu(dt)$ with $\{f(t) \in \mathcal{H}(t)\}_{t\geq 0}$ a measurable field satisfying

$$\int_{t\geq 0} (f(t)|f(t))_t\, \mu(dt) < +\infty, \quad \int_{t>0} t(f(t)|f(t))_t\, \mu(dt) < +\infty.$$

Assume that $f \in \mathcal{H}$ satisfy $f(t) = 0$ for $t > M$ with $M \geq 1$. Then $f \in D(C)$ and, by assumption, we can find a sequence $\{f_n(t)\}_{n\geq 1}$ in D such that $\|f_n - f\| \to 0$ and $\|Cf_n - Cf\| \to 0$, i.e.,

$$\int_{t\leq M} \|f_n(t) - f(t)\|_t^2\, \mu(dt) + \int_{t>M} \|f_n(t)\|_t^2\, \mu(dt) \to 0$$

and

$$\int_{t\leq M} t^2\|f_n(t) - f(t)\|_t^2\, \mu(dt) + \int_{t>M} t^2\|f_n(t)\|_t^2\, \mu(dt) \to 0,$$

which imply

$$\begin{aligned}\|C^{1/2}f_n - C^{1/2}f\|^2 &= \int_{t\leq M} t\|f_n(t) - f(t)\|_t^2\, \mu(dt) + \int_{t>M} t\|f_n(t)\|_t^2\, \mu(dt)\\ &\leq M\int_{t\leq M} \|f_n(t) - f(t)\|_t^2\, \mu(dt) + \int_{t>M} t^2\|f_n(t)\|_t^2\, \mu(dt)\\ &\to 0\end{aligned}$$

Thus spectrally truncated vectors for $C^{1/2}$ are approximated by vectors in D relative to the graph norm, which in turn constitute a core for $C^{1/2}$.

(ii) By the trivial inclusion $\frac{A}{A+B+I}(A + B + I) \subset A$ of unbounded operators, $D \subset D(A+B) \subset D(A)$. Let $\xi \in D(A+B) = D(A+B+I)$ be a vector in the spectral subspace of condition $A + B + I \leq M$ with M a sufficiently large positive real number, then we can find a sequence $\{\xi_n\}$ in D such that $\|\xi_n-\xi\| \to 0$ and $\|(A+B+I)\xi_n-(A+B+I)\xi\| \to 0$.

Then

$$\|A\xi_n - A\xi\| = \left\| \frac{A}{A+B+I}(A+B+I)\xi_n - \frac{A}{A+B+I}(A+B+I)\xi \right\| \to 0$$

implies that the domain of the closure of $A|_D$ contains a dense set of entirely analytic vectors of A, whence $A|_D$ is essentially self-adjoint. □

A complex-valued Borel function f on the closed first quadrant $[0,\infty)^2$ is called a **form function** if it is locally bounded and homogeneous of degree one; f is bounded when restricted to a compact subset of $[0,\infty)^2$ and $f(rs, rt) = rf(s,t)$ for $r, s, t \geq 0$. Clearly $f(0,0) = 0$ and there is a one-to-one correspondence between form functions and bounded Borel functions on the unit interval $[0,1]$ by the restriction $f(t, 1-t)$ $(0 \leq t \leq 1)$. Let $\mathcal{F}$ be the the vector space of form functions.

Theorem L.5. For $f \in \mathcal{F}$, the sesquilinear form on H defined by

$$\gamma(x,y) = (i(x)|f(A,B)i(y)), \quad x, y \in H$$

does not depend on the choice of representations of $\{\alpha, \beta\}$, which will be reasonably denoted by $\gamma = f(\alpha, \beta)$.

Proof. Let $\mathcal{F}_0$ be the set of functions $f \in \mathcal{F}$ satisfying the property in the theorem. Clearly $\mathcal{F}_0$ is a linear subspace of $\mathcal{F}$, closed under taking pointwise limit in a locally uniformly bounded fashion and $\alpha, \beta \in \mathcal{F}_0$. By the lemma below, if $f \geq 0$ and $g \geq 0$ belong to $\mathcal{F}_0$, we have $fg/(f+g) \in \mathcal{F}_0$. Thus, for $\mu > 0$,

$$\frac{\mu st}{s+\mu t}, \quad \frac{((s+\mu t)/2)^2}{s+\mu t}$$

are functions in $\mathcal{F}_0$ and, as a linear combination of these,

$$\frac{(s+t)^2}{(1-\lambda)s+t} = \frac{s+t}{1-\lambda s/(s+t)}$$

belongs to $\mathcal{F}_0$ for $0 < \lambda < 1$. Thus, extracting asymptotics as $\lambda \to +0$, $\frac{s^{n+1}}{(s+t)^n} \in \mathcal{F}_0$ $(n = 0, 1, 2, \dots)$ and then by Weierstrass approximation theorem continuous functions in $\mathcal{F}$ are included in $\mathcal{F}_0$. Since $\mathcal{F}_0$ is closed under taking locally bounded sequential limits, we conclude that $\mathcal{F}_0 = \mathcal{F}$. □

Lemma L.6. Let $\mathcal{F}_0$ be the set of functions $f \in \mathcal{F}$ satisfying the property in the theorem and let $f, g \in \mathcal{F}_0$ take values in $[0,\infty)$. Then $\frac{fg}{f+g}$ belongs to $\mathcal{F}_0$.

Proof. Let (i, K, A, B) be a bounded representation. Then the harmonic mean $C = \frac{f(A,B)g(A,B)}{f(A,B)+g(A,B)}$ of positive operators $f(A, B)$ and $g(A, B)$ is characterized by

$$(\zeta|C\zeta) = \inf\{(\xi|f(A, B)\xi)+(\eta|g(A, B)\eta); \xi, \eta \in K, \xi+\eta = \zeta\}, \quad \zeta \in K.$$

Since $i(H)$ is dense in K, this implies the assertion □

For $z \in \mathbb{C}$ in the strip region $0 \leq \mathrm{Re}z \leq 1$, $\alpha^z\beta^{1-z}(x, y)$ is continuous and holomorphic in $0 < \mathrm{Re}z < 1$ for any $x, y \in H$, which is referred as a Uhlmann's interpolation between α and β. The boundary part $\alpha^{1-t}\beta^t$ $(0 \leq t \leq 1)$, which is a continuous family of positive forms and characterized by

$$\sqrt{(\alpha^{1-s}\beta^s)(\alpha^{1-t}\beta^t)} = \alpha^{1-(s+t)/2}\beta^{(s+t)/2}$$

for $0 \leq s, t \leq 1$.

Lemma L.7.

(i) If $\alpha \leq \alpha'$ and $\beta \leq \beta'$, then $\sqrt{\alpha\beta} \leq \sqrt{\alpha'\beta'}$.

(ii) If α', α'', β', β'' are positive forms and $0 \leq s \leq 1$,

$$s\sqrt{\alpha'\beta'} + (1 - s)\sqrt{\alpha''\beta''} \leq \sqrt{(s\alpha' + (1 - s)\alpha'')(s\beta' + (1 - s)\beta'')}.$$

Theorem L.8 (Uhlmann)**.** Uhlmann's boundary interpolations satisfy the following inequalities. Under the same situations as above, we have $\alpha^{1-t}\beta^t \leq (\alpha')^{1-t}(\beta')^t$ and

$$s(\alpha')^{1-t}(\beta')^t+(1-s)(\alpha'')^{1-t}(\beta'')^t \leq (s\alpha'+(1-s)\alpha'')^{1-t}(s\beta'+(1-s)\beta'')^t$$

for $0 \leq t \leq 1$.

Proof. Let I be the set of parameters $0 \leq t \leq 1$ satisfying the inequalities. Then $0, 1 \in I$ and $t, t' \in I$ implies $(t + t')/2 \in I$. Since I is a closed subset, this means $I = [0, 1]$. □

Definition L.9. Let α and β be positive forms on a vector space H. A hermitian form γ on H is said to be **dominated** by $\{\alpha, \beta\}$ if $|\gamma(x, y)|^2 \leq \alpha(x, x)\,\beta(y, y)$ for $x, y \in H$. Note that the order of α and β is irrelevant in the domination.

Theorem L.10 (Pusz-Woronowicz)**.** Let α, β be positive forms on a complex vector space H. Then, for $x \in H$, we have the following variational expression.

$$\sqrt{\alpha\beta}(x, x) = \sup\{\gamma(x, x); \gamma \text{ is a positive form dominated by } \{\alpha, \beta\}\}.$$

Proof. Let $(i : H \to K, A, B)$ be a representation of $\{\alpha, \beta\}$. We first prove the formula for bounded representations. Assume that a positive form γ is dominated by $\{\alpha, \beta\}$. Then the inequality

$$|\gamma(x,y)|^2 \leq \alpha(x,x)\,\beta(y,y) \leq \|A\|\,\|B\|\|\,i(x)\|^2\,\|i(y)\|^2$$

enables us to find a positive bounded operator C on K such that $\gamma(x,y) = (i(x)|Ci(y))$. Since $i(H)$ is dense in K, we have

$$|(\xi|C\eta)| \leq (\xi|A\xi)\,(\eta|B\eta) \leq (\xi|(A+\epsilon)\xi)\,(\eta|(B+\epsilon)\eta)$$

for any $\epsilon > 0$. Replacing ξ and η with $(A+\epsilon)^{-1/2}\xi$ and $(B+\epsilon)^{-1/2}\eta$ respectively, we have $\|(A+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2}\| \leq 1$ and hence

$$(A+\epsilon)^{-1/2}C(B+\epsilon)^{-1}C(A+\epsilon)^{-1/2} \leq 1_K.$$

Multiplying the positive operator

$$(B+\epsilon)^{-1/2}(A+\epsilon)^{1/2} = (A+\epsilon)^{1/2}(B+\epsilon)^{-1/2}$$

from the left and right sides, we get

$$\left((B+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2}\right)^2 \leq (A+\epsilon)(B+\epsilon)^{-1}$$

and then by taking square roots (taking square roots is operator-monotone)

$$(B+\epsilon)^{-1/2}C(B+\epsilon)^{-1/2} \leq (A+\epsilon)^{1/2}(B+\epsilon)^{-1/2}$$

and therefore $C \leq (A+\epsilon)^{1/2}(B+\epsilon)^{1/2}$. Thus $C \leq A^{1/2}B^{1/2}$.

Now let us deal with the case of unbounded A and B. Since $i(H)$ is assumed to be a core for $A+B+I$, it is a core for $(A+B+I)^{1/2}$ as well and, if we set $j(x) = (A+B+I)^{1/2}i(x)$, the linear map $j : H \to K$ has a dense range. By the identity

$$\begin{aligned}
(j(x)|\frac{A}{A+B+I}j(x)) &= (\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)|\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)) \\
&= (A^{1/2}i(x)|A^{1/2}i(x)) = \alpha(x,x)
\end{aligned}$$

and a similar expression for $\beta(x,x)$, we obtain a bounded representation $(j, \frac{A}{A+B+I}, \frac{B}{A+B+I})$ and then

$$\begin{aligned}
\sqrt{\alpha\beta}(x,x) &= (j(x)|\left(\frac{A}{A+B+I}\right)^{1/2}\left(\frac{B}{A+B+I}\right)^{1/2}j(x)) \\
&= (\frac{A^{1/2}}{(A+B+I)^{1/2}}j(x)|\frac{B^{1/2}}{(A+B+I)^{1/2}}j(x)) \\
&= (A^{1/2}i(x)|B^{1/2}i(x)).
\end{aligned}$$

□

Corollary L.11. Given positive forms α and β on H, we can find a positive form $\sqrt{\alpha\beta}$, called the **geometric mean** of α and β, satisfying

$$\sqrt{\alpha\beta}(x,x) = \sup\{\gamma(x,x); \gamma \text{ is a positive form dominated by } \{\alpha,\beta\}\}$$

for $x \in H$.

Remark 10. From the proof, we also have

$$\sqrt{\alpha\beta}(x,x) = \sup\{\gamma(x,x); \gamma \text{ is a hermitian form dominated by } \{\alpha,\beta\}\}.$$

Appendix M. Transition Probabilities

Let ω be a positive functional of a C*-algebra A. According to [Pusz-Woronowicz], we introduce two positive forms ω_L and ω_R on A defined by

$$\omega_L(x,y) = \omega(x^*y), \qquad \omega_R(x,y) = \omega(yx^*), \quad x,y \in A.$$

Lemma M.1. Let M be a W*-algebra and Let φ, ψ be positive normal functionals of a W*-algebra M. Then

$$\sqrt{\varphi_L\psi_R}(x,y) = \langle \varphi^{1/2}x^*\psi^{1/2}y\rangle \quad \text{for } x,y \in M.$$

Proof. By the positivity $\langle\varphi^{1/2}x^*\psi^{1/2}x\rangle = (x\varphi^{1/2}x^*|\psi^{1/2}) \geq 0$ and the Schwarz inequality $|\langle\varphi^{1/2}x^*\psi^{1/2}y\rangle|^2 \leq \varphi(x^*x)\psi(yy^*)$, the positive form $(x,y) \mapsto \langle\varphi^{1/2}x^*\psi^{1/2}y\rangle$ is dominated by $\{\varphi_L, \psi_R\}$.

Assume for the moment that φ and ψ are faithful and consider the embedding $i : M \ni x \mapsto x\varphi^{1/2} \in L^2(M)$. Then φ_L is represented by the identity operator, whereas

$$\psi(xx^*) = \|\psi^{1/2}x\|^2 = \|\psi^{1/2}(x\varphi^{1/2})\varphi^{-1/2}\|^2$$

shows that ψ_R is represented by the relative modular operator Δ ($\Delta(\xi) = \psi\xi\varphi^{-1}$). Note here that $M\varphi^{1/2}$ is a core for $\Delta^{1/2}$. Thus

$$\sqrt{\varphi_L\psi_R}(x,y) = (x\varphi^{1/2}|\Delta^{1/2}(y\varphi^{1/2})) = (x\varphi^{1/2}|\psi^{1/2}y) = \langle\varphi^{1/2}x^*\psi^{1/2}y\rangle.$$

Now we relax φ and ψ to have no-trivial supports. Let e be the support projection of $\varphi + \psi$. Then it is the supprot for $\varphi_n = \varphi + \frac{1}{n}\psi$ and $\psi_n = \frac{1}{n}\varphi + \psi$ as well. In particular, φ_n and ψ_n are faithful on the reduced algebra eMe.

Let γ be a positive form on M dominated by $\{(\varphi_n)_L, (\psi_n)_R\}$. Then $\varphi_n(1-e) = 0 = \psi_n(1-e)$ shows that

$$|\gamma(x(1-e),(1-e)y)|^2 \leq \varphi_n((1-e)x^*x(1-e))\psi_n((1-e)yy^*(1-e)) = 0,$$

i.e., $\gamma(x,y) = \gamma(xe,ey)$ for $x,y \in M$, whence we have

$$\gamma(x,y) = \gamma(xe,ey) = \overline{\gamma(ey,xe)} = \overline{\gamma(eye,exe)} = \gamma(exe,eye).$$

Since the restriction $\gamma|_{eMe}$ is dominated by $(\varphi_n|_{eMe})_L$ and $(\psi_n|_{eMe})_R$ with φ_n and ψ_n faithful on eMe, we have

$$\gamma(x,x) = \gamma(exe, exe) \leq \langle \varphi_n^{1/2} ex^* e \psi_n^{1/2} exe \rangle = \langle \varphi_n^{1/2} x^* \psi_n^{1/2} x \rangle.$$

Taking the limit $n \to \infty$, we obtain $\gamma(x,x) \leq \langle \varphi^{1/2} x^* \psi^{1/2} x \rangle$ in view of the Powers-Størmer inequality. □

Remark 11.

(i) The case $\varphi = \psi$ is implicitly considered in [PW].
(ii) In the notation of [U,relative entropy], we have

$$QF_t(\varphi_L, \psi_R)(x,y) = \langle \varphi^{1-t} x^* \psi^t y \rangle$$

for $0 \leq t \leq 1$ and $x, y \in M$.

Given a positive functional φ of a C*-algebra A, let $\widetilde{\varphi}$ be the associated normal functional on the W*-envelope A^{**} through the canonical duality pairing.

Lemma M.2. Let φ and ψ be positive functionals on a C*-algebra A with $\widetilde{\varphi}$ and $\widetilde{\psi}$ the corresponding normal functionals on A^{**}. Then

$$\sqrt{\varphi_L \psi_R}(x,y) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle \quad \text{for } x, y \in A \subset A^{**}.$$

Proof. The positive form $A \times A \ni (x,y) \mapsto \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle$ (recall that $x^* \widetilde{\psi}^{1/2} x$ is in the positive cone to see the positivity) is dominated by $\widetilde{\varphi}_L$ and $\widetilde{\psi}_R$ because of

$$|\langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle|^2 \leq \widetilde{\varphi}(x^*x) \widetilde{\psi}(yy^*) = \varphi(x^*x)\psi(yy^*).$$

Consequently,

$$\langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \leq \sqrt{\varphi_L \psi_R}(x,x) \quad \text{for } x \in A.$$

To get the reverse inequality, let γ be a positive form on $A \times A$ dominated by φ_L and ψ_R. Then we have the domination inequality

$$|\gamma(x,y)|^2 \leq \varphi(x^*x)\psi(yy^*) = \|x\widetilde{\varphi}^{1/2}\|^2 \, \|\widetilde{\psi}^{1/2} y\|^2.$$

Since A is dense in A^{**} relative to the σ^*-topology, we see that γ is extended to a positive form $\widetilde{\gamma}$ on $A^{**} \times A^{**}$ so that

$$|\widetilde{\gamma}(x,y)|^2 \leq \|x\widetilde{\varphi}^{1/2}\|^2 \, \|\widetilde{\psi}^{1/2} y\|^2 \quad \text{for } x, y \in A^{**},$$

whence

$$\gamma(x,x) = \widetilde{\gamma}(x,x) \leq \sqrt{\widetilde{\varphi}_L \widetilde{\psi}_R}(x,x) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \quad \text{for } x \in A.$$

Maximization on γ then yields the inequality

$$\sqrt{\varphi_L \psi_R}(x,x) \leq \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} x \rangle \quad \text{for } x \in A$$

and we are done. □

Corollary M.3. Given a normal state φ of a W*-algebra M, let $\widetilde{\varphi}$ be the associated normal state of the second dual W*-algebra M^{**}. Then

$$L^2(M) \ni \varphi^{1/2} \mapsto \widetilde{\varphi}^{1/2} \in L^2(M^{**})$$

defines an isometry of M-M bimodules.

Proof. Combining two lemmas just proved, we have

$$\langle \varphi^{1/2} x^* \psi^{1/2} y \rangle = \sqrt{\varphi_L \psi_R}(x, y) = \langle \widetilde{\varphi}^{1/2} x^* \widetilde{\psi}^{1/2} y \rangle$$

for $x, y \in M$. □

In what follows, $\varphi^{1/2}$ is identified with $\widetilde{\varphi}^{1/2}$ via the above isometry: Given a positive normal functional φ of a W*-algebra M, $\varphi^{1/2}$ is used to stand for a vector commonly contained in the increasing sequence of Hilbert spaces

$$L^2(M) \subset L^2(M^{**}) \subset L^2(M^{****}) \subset \dots .$$

In accordance with this convention, the formula in the previous lemma then takes the form

$$(x\varphi^{1/2} | \psi^{1/2} y) = \sqrt{\varphi_L \psi_R}(x, y) \quad \text{for } x, y \in A.$$

Here the left hand side is the inner product in $L^2(A^{**})$, whereas the right hand side is the geometric mean of positive forms on the C*-algebra A. Note that, the formula is compatible with the invariance of geometric means:

$$\sqrt{\varphi_L \psi_R}(x, y) = \sqrt{\psi_L \varphi_R}(y^*, x^*) = \sqrt{\varphi_R \psi_L}(y^*, x^*).$$

Remark 12. A W*-algebra M satisfies $M_* = M^*$ if and only if $\dim M < +\infty$. In fact, if $\dim M = \infty$, we can find a sequence of non-zero projections $\{p_n\}_{n \geq 1}$ in M such that $\sum_n p_n = 1$. In other words, M contains $\ell^\infty(\mathbb{N})$ as a W*-subalgebra. Let f be a singular state of $\ell^\infty(\mathbb{N})$ and extend it to a state φ of M. If $M_* = M^*$ in addition, φ is normal, which contradicts with

$$1 = \varphi(1_M) = \sum_n \varphi(p_n) = \sum_n f(p_n) = 0.$$

Remark 13.

(i) When φ and ψ are vector states of a full operator algebra $\mathcal{L}(\mathcal{H})$ associated to normalized vectors ξ, η in $\mathcal{H}$, our transition amplitude $(\varphi^{1/2} | \psi^{1/2})$ is reduced to the transition probability $|(\xi | \eta)|^2$.

(ii) Let $P(\varphi,\psi)$ be the transition probability between states in the sense of Kakutani-Bures-Uhlmann. Then we have $P(\varphi,\psi) = \langle|\varphi^{1/2}\psi^{1/2}|\rangle^2$ (cf. [Ragio]) and

$$(\varphi^{1/2}|\psi^{1/2})^2 \leq P(\varphi,\psi) \leq (\varphi^{1/2}|\psi^{1/2})$$

for states φ and ψ on a C*-algebra.

In the text, the transition probability described above is utilized to analyse the universal representations of C*-algebras. We shall here show that the main construction remains valid under some positivity assumption on geometric means of positive forms.

Given a finite family $\{\omega_j\}_{1\leq j\leq n}$ of positive functionals of a *-algebra $\mathcal{A}$, let ω be a positive functional of $M_n(\mathcal{A})$ defined by

$$X = (x_{jk}) \mapsto \omega(X) = \sum_{j=1}^n \omega_j(x_{jj}).$$

Lemma M.4. Let $E_{jk} \in M_n(\mathbb{C})$ be the matrix unit.

(i) The decomposition $M_n(\mathcal{A}) = \sum_{j,k} \mathcal{A}E_{jk}$ is orthogonal with respect to the positive form $\sqrt{\omega_L\omega_R}$.

(ii) For $x, y \in \mathcal{A}$, $\sqrt{\omega_L\omega_R}(xE_{jk}, yE_{jk}) = \sqrt{(\omega_k)_L(\omega_j)_R}(x,y)$.

Proof. (i) is a consequence of the fact that $\sum \mathcal{A}E_{jk}$ is orthogonal relative to both of ω_L and ω_R.

(ii) If a hermitian form Γ on $M_n(\mathcal{A})$ is dominated by $\{\omega_L, \omega_R\}$, then

$$|\Gamma(xR_{jk}, yE_{jk})|^2 \leq \omega(R_{kj}x^*yE_{jk})\omega(yE_{jk}E_{kj}x^*) = \omega_k(x^*y)\omega_j(yx^*)$$

shows that the hermitian form $\gamma(x,y) = \Gamma(xR_{jk}, yE_{jk})$ on $\mathcal{A}$ is dominated by $\{(\omega_k)L, (\omega_j)_R\}$, whence

$$\sqrt{\omega_L\omega_R}(xE_{jk}, xE_{jk}) = \sup \Gamma(xE_{jk}, xE_{jk}) \leq \sqrt{(\omega_k)_L(\omega_j)_R}(x,x)$$

for $x \in \mathcal{A}$.

Conversely, given a hermitian form γ dominated by $\{(\omega_k)_L, (\omega_j)_R\}$, the hermitian form $\Gamma(X,Y) = \gamma(x_{jk}, y_{jk})$ on $M_n(\mathcal{A})$ is dominated by $\{\omega_L, \omega_R\}$:

$$|\Gamma(X,Y)|^2 \leq \omega_k(x_{jk}^*x_{jk})\,\omega_j(y_{jk}y_{jk}^*) \leq \omega(X^*X)\,\omega(YY^*).$$

Thus,

$$\sqrt{(\omega_k)_L(\omega_j)_R}(x,x) = \sup \gamma(x,x) \leq \sqrt{\omega_L\omega_R}(xE_{jk}, xE_{jk}).$$

□

Definition M.5. A set $\mathcal{P}$ of positive functional on a *-algebra $\mathcal{A}$ is said to be **positive** if ω is a positive functional on $M_n(\mathcal{A})$ associated to a finite family $\{\omega_j\}$ in $\mathcal{P}$, then

$$\sqrt{\omega_L \omega_R}(XX^*, YY^*) \geq 0$$

for any $X, Y \in M_n(\mathcal{A})$.

Example M.6. The set of positive functionals on a C*-algebra is positive.

The positivity of $\sqrt{\varphi_L \varphi_R}(a^*a, bb^*)$ is highly non-trivial. Any counter example?

We now imitate the construction of standard Hilber spaces. Let $\mathcal{P}$ be a positive set of positive functionals. On the free algebraic sum

$$\sum_{\varphi \in \mathcal{P}} \mathcal{A} \otimes \varphi^{1/2} \otimes \mathcal{A},$$

introduce a sesquilinear for by

$$\left(\sum_j x_j \otimes \omega_j^{1/2} \otimes y_j \middle| \sum_k x'_k \otimes \omega_k^{1/2} \otimes y'_k \right) = \sum_{j,k} \sqrt{(\omega_j)_L (\omega_k)_R}((x'_k)^* x_j, y'_k y_j^*),$$

which is positive because

$$\begin{aligned} \sum_{j,k} \sqrt{(\omega_j)_L (\omega_k)_R}(x_k^* x_j, y_k y_j^*) &= \sum_{j,k} \sqrt{\omega_L \omega_R}(x_k^* x_j E_{kj}, y_k y_j^* E_{kj}) \\ &= \sqrt{\omega_L \omega_R}(X^* X, YY^*) \geq 0, \end{aligned}$$

where

$$X = \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \dots & 0 \end{pmatrix}.$$

The quotient inner product space is denoted by $\mathcal{L}^2(\mathcal{A}, \mathcal{P})$ with the quotient vector of $\sum_j x_j \otimes \omega_j^{1/2} \otimes y_j$ with respect to this positive form denoted by

$$\sum_j x_j \omega_j^{1/2} y_j.$$

From

$$\|\sum_j y_j^* \omega_j^{1/2} x_j^*\|^2 = \sum_{j,k} \sqrt{(\omega_j)_L(\omega_k)_R}(y_k y_j^*, x_k^* x_j)$$
$$= \sum_{j,k} \overline{\sqrt{(\omega_j)_L(\omega_k)_R}(y_k y_j^*, x_k^* x_j)}$$
$$= \sum_{j,k} \sqrt{(\omega_j)_L(\omega_k)_R}(x_k^* x_j, y_k y_j^*)$$
$$= \|\sum_j x_j \omega_j^{1/2} y_j\|^2,$$

the conjugation is well-defined by

$$\left(\sum_j x_j \omega_j^{1/2} y_j\right)^* = \sum_j y_j^* \omega_j^{1/2} x_j^*.$$

From the formula in the definition of pre-iiner product, we have

$$\left(\sum_j (a x_j) \otimes \omega_j^{1/2} \otimes y_j \middle| \sum_k x'_k \otimes \omega_k^{1/2} \otimes y'_k\right)$$
$$= \left(\sum_j x_j \otimes \omega_j^{1/2} \otimes y_j \middle| \sum_k (a^* x'_k) \otimes \omega_k^{1/2} \otimes y'_k\right)$$

for $a \in \mathcal{A}$, whence the left multiplication of $a \in \mathcal{A}$ on the quotient inner product space is well-defined. Similarly for the right multiplication.

In this way, we have constructed a *-bimodule $\mathcal{L}^2(\mathcal{A}, \mathcal{P})$ of $\mathcal{A}$.

A linear map $\Phi : A \to B$ between C*-algebras is said to be a Schwartz map if it satisfies the operator inequality $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ for $a \in A$.

Theorem M.7 (Uhlmann, relative entropy, Proposition 17)**.** Let $\Phi : A \to B$ be a unital Schwarz map between unital C*-algebras. Then, for $\varphi, \psi \in B_+^*$,

$$(\varphi^{1/2} | \psi^{1/2}) \leq ((\varphi \circ \Phi)^{1/2} | (\psi \circ \Phi)^{1/2}).$$

Proof. Let $\gamma : B \times B \to \mathbb{C}$ be a positive form dominated by $\{\varphi_L, \psi_R\}$. Then

$$|\gamma(\Phi(x), \Phi(y))|^2 \leq \varphi(\Phi(x)^*\Phi(x))\psi(\Phi(y)\Phi(y)^*) \leq \varphi(\Phi(x^*x))\psi(\Phi(yy^*))$$

shows that the positive form $A \times A \ni (x, y) \mapsto \gamma(\Phi(x), \Phi(y))$ is dominated by $\{(\varphi \circ \Phi)_L, (\psi \circ \Phi)_R\}$. Thus

$$\gamma(1,1) = \gamma(\Phi(1), \Phi(1)) \leq \sqrt{(\varphi \circ \Phi)_L (\psi \circ \Phi)_R}(1,1) = ((\varphi \circ \Phi)^{1/2} | (\psi \circ \Phi)^{1/2}).$$

Maximizing $\gamma(1,1)$ with respect to γ, we obtain the inequality. $\square$

Example M.8. Consider an inclusion of matrix algebras $\pi : M_n(\mathbb{C}) \ni x \mapsto x \otimes 1 \in M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$. By the isomorphism $M_n(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_{2n}(\mathbb{C})$, π takes the form

$$\pi(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Let a_j, b_j $(j = 1, 2)$ be hermitian matrices in $M_n(\mathbb{C})$ and set

$$a = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}.$$

Consider positive linear functionals on $M_{2n}(\mathbb{C})$ defined by $\varphi(y) = \mathrm{trace}(a^2 y)$ and $\psi(y) = \mathrm{trace}(b^2 y)$ for $y \in M_{2n}(\mathbb{C})$. Then we see

$$(\varphi \circ \pi)(x) = 2\mathrm{trace}((a_1^2 + a_2^2)x), \quad (\psi \circ \pi)(x) = 2\mathrm{trace}((b_1^2 + b_2^2)x)$$

for $x \in M_n(\mathbb{C})$ and the inequality $(\varphi^{1/2}|\psi^{1/2}) \leq ((\varphi \circ \pi)^{1/2}|(\psi \circ \pi)^{1/2})$ takes the form

$$\mathrm{trace}(|a|\,|b|) \leq \mathrm{trace}((a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}).$$

In view of the Jordan decompositions $a = a_+ - a_-$, $b = b_+ - b_-$ with $|a| = a_+ + a_-$, $|b| = b_+ + b_-$, we see $\mathrm{trace}(ab) \leq \mathrm{trace}(|a|\,|b|)$, which is combined with above inequality to get $\mathrm{trace}(a_1 b_1 + a_2 b_2) \leq \mathrm{trace}((a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2})$.

Now, by an obvious induction on m, we conclude the following: Given hermitian matrices $a_1, \dots, a_m$ and $b_1, \dots, b_m$ in $M_n(\mathbb{C})$, we have the inequality

$$\mathrm{trace}(a_1 b_1 + \cdots + a_m b_m) \leq \mathrm{trace}((a_1^2 + \cdots + a_m^2)^{1/2}(b_1^2 + \cdots + b_m^2)^{1/2}).$$

Theorem M.9. Let φ and ψ be positive functional on a C*-algebra A with unit 1_A. Let $\{A_n\}_{n \in \mathcal{N}}$ be an increasing net of C*-subalgebras of A containing 1_A in common and assume that, given any $a \in A$, we can find a net $\{a_n \in A_n\}_{n \in \mathcal{N}}$ in A satisfying

$$\lim_{n \to \infty} a_n \varphi^{1/2} = a\varphi^{1/2}, \quad \lim_{n \to \infty} \psi^{1/2} a_n = \psi^{1/2} a$$

in the norm topology of $L^2(A)$. Set $\varphi_n = \varphi|_{A_n}, \psi_n = \psi|_{A_n} \in A_n^*$. Then the net $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}_{n \in \mathcal{N}}$ is decreasing and converges to $(\varphi^{1/2}|\psi^{1/2})$.

Proof. The net $\{(\varphi_n^{1/2}|\psi_n^{1/2})\}$ is decreasing with $(\varphi^{1/2}|\psi^{1/2})$ a lower bound by the coarse-graining inequality.

Let e_n and f_n be projections in $\mathcal{B}(L^2(A))$ defined by

$$e_n L^2(A) = \overline{A_n \varphi^{1/2}}, \quad f_n L^2(A) = \overline{\psi^{1/2} A_n}.$$

By the variational expression of geometric mean, we can find positive forms $\gamma_n : A_n \times A_n \to \mathbb{C}$ for $n \in \mathcal{N}$ so that each γ_n is dominated by $\{(\varphi_n)_L, (\psi_n)_R\}$ and satisfies

$$\gamma_n(1,1) \geq (\varphi_n^{1/2}|\psi_n^{1/2}) - \epsilon_n,$$

where $\{\epsilon_n\}$ is a net of positive reals converging to 0. From the domination inequality of γ_n, we can find a linear map $C'_n : \overline{\psi^{1/2}A_n} \to \overline{A_n\varphi^{1/2}}$ satisfying

$$\gamma_n(x,y) = (x\varphi^{1/2}|C'_n(\psi^{1/2}y)) \quad \text{for } x, y \in A_n$$

and $\|C'_n\| \leq 1$. Let $C_n = e_n C'_n f_n : \overline{\psi^{1/2}A} \to \overline{A\varphi^{1/2}}$. Since $\|C_n\| \leq 1$, we may assume that $C_n \to C$ in weak operator topology by passing to a subnet if necessary. Now set

$$\gamma(x,y) = (x\varphi^{1/2}|C(\psi^{1/2}y)),$$

which is a sesquilinear form on A satisfying $|\gamma(x,y)| \leq \|x\varphi^{1/2}\| \, \|\psi^{1/2}y\|$. Moreover, if $x \in A_m$ for some $m \in \mathcal{N}$,

$$\gamma(x,x) = \lim_{n\to\infty} (x\varphi^{1/2}|C_n(\psi^{1/2}x)) = \lim_{n\to\infty} \gamma_n(x,x) \geq 0,$$

which shows that γ is positive on $\bigcup_{m\in\mathcal{N}} A_m$ and then on A by the approximation assumption. Thus, γ is a positive form dominated by $\{\varphi_L, \psi_R\}$ and the variational estimate is used again to get

$$\begin{aligned}(\varphi^{1/2}|\psi^{1/2}) \geq \gamma(1,1) &= \lim_{n\to\infty} (\varphi^{1/2}|C_n\psi^{1/2}) = \lim_{n\to\infty} \gamma_n(1,1) \\ &\geq \lim_{n\to\infty} \big((\varphi_n^{1/2}|\psi_n^{1/2}) - \epsilon_n\big) = \lim_{n\to\infty} (\varphi_n^{1/2}|\psi_n^{1/2}).\end{aligned}$$

□

Appendix N. Random Operators

Random linear operators / A.V. Skorohod

A random operator is a family of operators parametrized by elements in a Borel space in such a way that its dependence is considered to be measurable in some sense.

When a measure is not specified, the measurability means that for Borel structures.

N.1. **Polar Decomposition.** Let $\{\mathcal{H}_\omega\}$ be a measurable field of separable Hilbert spaces and $\{T_\omega : D_\omega \to \mathcal{H}_\omega\}$ be a family of densely defined closed operators which is measurable in the sense that we can find a sequence of measurable sections $\{\xi_n\}_{n\geq 1}$ so that $\sum_{n\geq 1} \mathbb{C}\xi_n(\omega)$ is a core for T_ω at almost every $\omega \in \Omega$ and sections $\{T_\omega \xi_n(\omega)\}_{\omega\in\Omega}$ $(n \geq 1)$ are measurable. Let

$$\mathcal{H}_\omega \oplus \mathcal{H}_\omega = \{\xi \oplus T_\omega \xi; \xi \in D(T_\omega)\} + \{T_\omega^* \eta \oplus -\eta; \eta \in D(T_\omega^*)\}$$

be an orthogonal decomposition associated with the graphs of T_ω and T_ω^*. Let $E_\omega \in \mathcal{B}(\mathcal{H}_\omega \oplus \mathcal{H}_\omega)$ be the projection to the graph of T_ω. By the measurability assumption on $\{T_\omega\}$, $\{E_\omega\}$ is a measurable field of projections and hence so is $\{1_{\mathcal{H}_\omega} - E_\omega\}$. Thus $\{T_\omega^*\}$ is measurable because the second component of $\{(1 - E_\Omega)(\zeta_j(\omega) \oplus \zeta_k(\omega)); j, k \geq 1\}$ is a core for T_ω^*, where $\{\zeta_n\}_{n\geq 1}$ is any sequence of measurable sections such that $\{\zeta_n(\omega); n \geq 1\}$ is dense in $\mathcal{H}_\omega$ at almost every $\omega \in \Omega$.

Ginve a measurable section $\zeta(\omega)$ of $\{\mathcal{H}_\omega\}$, the orthogonal decomposition

$$\zeta(\omega) \oplus 0 = (\xi(\omega) \oplus T_\omega \xi(\omega)) + (T_\omega^* \eta(\omega) \oplus -\eta(\omega))$$

with $\xi(\omega)$ and $\{\eta(\omega)\}$ measurable sections of $\{\mathcal{H}_\omega\}$ and belonging to $D(T_\omega)$ and $D(T_\omega^*)$ respectively. The relation $\xi(\omega) = (1 + T_\omega^* T_\omega)^{-1}\zeta(\omega)$ reveals that $\{(1 + T_\omega^* T_\omega)^{-1}\}$ and then

$$\frac{T_\omega^* T_\omega}{1 + T_\omega^* T_\omega} = 1 - \frac{1}{1 + T_\omega^* T_\omega}$$

are measurable. Since the square roots of a bounded positive operator is realized as a uniform limit of polynomials,

$$\sqrt{\frac{T_\omega^* T_\omega}{1 + T_\omega^* T_\omega}}$$

is measurable as well. Now replace T_ω with tT_Ω $(t > 0)$ and then divide the result by t to get a measurable family of positive operators

$$\sqrt{\frac{T_\omega^* T_\omega}{1 + t^2 T_\omega^* T_\omega}}.$$

Thanks to the spectral calculus, we then see that

$$|T_\omega|\xi_n(\omega) = \lim_{t\to+0} \sqrt{\frac{T_\omega^* T_\omega}{1 + t^2 T_\omega^* T_\omega}}\xi_n(\omega)$$

is a measurable section for $n \geq 1$. Since the partial isometry part V_ω in the polar decomposition of T_ω, is given by $|T_\omega|\xi \mapsto T_\omega \xi$ $(\xi \in D(T_\omega) =$

$D(|T_\omega|))$, $\{V_\omega\}$ maps measurable sections $|T_\omega|\xi_n(\omega)$ into measurable sections $T_\omega\xi_n(\omega)$ for $n \geq 1$. Thus $\{V_\omega\}$ is measurable.

Now let T be a densely defined operator in $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega\, \mu(d\omega)$ defined by

$$T\xi = \int_\Omega^\oplus T_\omega\xi(\omega)\, \mu(d\omega)$$

for $\xi = \int_\Omega^\oplus \xi(\omega)\, \mu(d\omega)$ satisfying

$$\int_\Omega \|T_\omega\xi(\omega)\|^2\, \mu(d\omega) < \infty.$$

Since the graph of T is equal to $E(\mathcal{H}\oplus\mathcal{H})$ with the projectio E defined by

$$E = \int_\Omega^\oplus E_\omega\, \mu(d\omega)$$

T is a closed operator. Furthermore,

$$|T| = \int_\Omega^\oplus |T_\omega|\, \mu(d\omega)$$

and

$$V = \int_\Omega^\oplus V_\omega\, \mu(d\omega)$$

is the partial isometry part in the polar decompositive of T.

N.2. **Sesquilinear Forms.** Let Ω be a Borel space with $\mathfrak{B}(\Omega)$ denoting the complex vector space of Borel functions on Ω. Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of complex vector spaces parametrized by elements in Ω and let $\mathfrak{H}$ be a vector space consisting of sections of $\{H_\omega\}$ fulfilling the conditions:

(i) $\xi = \{\xi(\omega)\} \in \mathfrak{H}$ and $f \in \mathfrak{B}(\Omega)$ imply $f\xi = \{f(\omega)\xi(\omega)\} \in \mathfrak{H}$.

(ii) $\mathfrak{H}$ is closed under taking point-wise sequential limits.

A family $\{\alpha_\omega\}$ of sesquilinear forms is said to be measurable if $\alpha(\xi, \eta) = \{\alpha_\omega(\xi(\omega), \eta(\omega))\} \in \mathfrak{B}(\Omega)$ for any $\xi, \eta \in \mathfrak{H}$.

Let $\phi_\omega(s, t)$ be a family of form functions which is measurable as a function of $(\omega, s, t) \in \Omega \times [0, \infty)^2$. Then, for mearurable families $\{\alpha_\omega\}$, $\{\beta_\omega\}$ of positive sesquilinear forms, the family $\{\phi_\omega(\alpha_\omega, \beta_\omega)\}$ of sesquilinear forms is measurable.

To this this, let $\mathcal{H}_\omega$ be the Hilbert space associated to the positive sesquilinear form $\alpha_\omega + \beta_\omega$ on H_ω and furnish $\{\mathcal{H}_\omega\}$ with the measurable field structure induced from $\mathfrak{H}$. Then the operator representation a_ω, b_ω of α_ω, β_ω gives measurable families $\{a_\omega\}$, $\{b_\omega\}$.

Let Φ be the set of measurable functions ϕ on $\Omega \times [0, 1]$ such that $r_\omega = \sup\{|\phi(\omega, s)|; 0 \leq s \leq 1\} < \infty$ for each $\omega \in \Omega$ and Φ_0 be the subset consisting of functions ϕ for which $\{\phi(\omega, a_\omega)\}$ is a measurable family of operators on $\mathcal{H}_\omega$. Note that

$$\phi_\omega(\alpha_\omega, \beta_\omega)(\xi(\omega), \eta(\omega)) = (i(\xi(\omega))|\phi(\omega, a_\omega)i(\eta(\omega))),$$

where $\phi(\omega, s) = \phi_\omega(s, 1-s)$, and the measurability of $\{\phi_\omega(\alpha_\omega, \beta_\omega)\}$ follows from that of the operator family $\{\phi(\omega, a_\omega)\}$.

Clearly $\mathfrak{B}_b(\Omega \times [0,1]) \subset \Phi$ [15] and each ϕ is pointwise limit of the sequence $\phi_n \in \mathfrak{B}_b(\Omega \times [0,1])$ defined by

$$\phi_n(\omega, s) = \begin{cases} \phi(\omega, s) & \text{if } r_\omega \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, to see $\Phi = \Phi_0$, it suffices to check $\mathfrak{B}_b(\Omega \times [0,1]) \subset \Phi_0$. In fact, Φ_0 contains $\mathfrak{B}_b(\Omega) \otimes \mathbb{C}[s]$ and is closed under taking uniformly bounded pointwise sequential limits. Thus it contains $\mathfrak{B}_b(\Omega) \otimes C[0,1]$ by the Weierstrass approximation theorem and then the whole $\mathfrak{B}_b(\Omega \times [0,1])$ and we are done.

N.3. **Normal Homomorphisms.** Let $\{\phi_\omega : M_\omega \to N_\omega\}$ be a family of normal *-homomorphisms and suppose that it is measurable: Given an adapted operator family $\{a(\omega) \in M_\omega\}$, the family $\{\phi_\omega(a(\omega))\}$ is adapted to $\{N_\omega\}$. Then we can find a measurable family $\{e_\omega \in \mathcal{B}(\mathcal{H}_\omega \otimes \ell^2)\}$ of projections belongin to the commutant of M_ω on $\mathcal{H}_\omega \otimes \ell^2$ and a measurable family of isometries $\{U_\omega^* : \mathcal{K}_\omega \to e_\omega(\mathcal{H}_\omega \otimes \ell^2)\}$ such that

$$\phi_\omega(a) = U_\omega(a \otimes 1)U_\omega^* \quad \text{for } \omega \in \Omega \text{ and } a \in M_\omega.$$

Appendix O. Periodic Entirely Holomorphic Functions

Let $f(z)$ be an entirely holomorphic function satisfying $f(z + 2\pi) = f(z)$. Apply the Cauchy theorem to a function $f(z)e^{-inz}$ ($n \in \mathbb{Z}$) and a rectangle $z = x + iy$ ($0 \leq x \leq 2\pi$, $0 \leq y/b \leq 1$) with $0 \neq b \in \mathbb{R}$ to get

$$2\pi f_n \equiv \int_0^{2\pi} f(x)e^{-inx}\, dx = \int_0^{2\pi} f(x+ib)e^{-in(x+ib)}\, dx.$$

For $n \neq 0$ and y satisfying $ny \leq 0$, we then have an estimate

$$2\pi|f_n| \leq e^{ny} \int_0^{2\pi} |f(x+iy)|\, dx \leq e^{-|y|} \int_0^{2\pi} |f(x+iy)|\, dx.$$

Consequently, the asymptotic condition

$$\lim_{y \to \pm\infty} e^{-|y|} \int_0^{2\pi} |f(x+iy)|\, dx = 0$$

implies $f_n = 0$ for $n \neq 0$, i.e., $f(z)$ is a constant function.

[15] $\mathfrak{B}_b$ indicates the bounded Borel functions.

The above equality also shows that the Fourier coefficients of the function $f(x+ib)$ are of the form

$$\frac{1}{2\pi}\int_0^{2\pi} f(x+ib)e^{-inx}\,dx = f_n e^{-nb}$$

and

$$f(z) = \sum_{n\in\mathbb{Z}} f_n e^{inz}.$$

In view of $|f_n e^{-nb}| \to 0$ as $|n| \to \infty$, $|f_n| = O(e^{-r|n|})$ for any $r > 0$ and the summation is absolutely convergent for any z.

Appendix P. Hilbert Algebras

Dixmier, Von Neumann Algebras, North-Holland, 1981.

A *-algebra H is called a **Hilbert algebra** if it is furnished with an inner product $(\ |\ )$ such that $(x|y) = (y^*|x^*)$ for $x, y \in H$ and the left multiplication gives a bounded non-degenerate *-representation of H; $(ax|y) = (x|a^*y)$ for $a, x, y \in H$ and H^2 is dense in H. The right multiplication is then bounded as well due to $(xa)^* = a^*x^*$ and compatible with the *-operation in view of

$$(xa|y) = (y^*|a^*x^*) = (ay^*|x^*) = (x|ya^*).$$

Example P.1. A typical example of Hilbert algebra is the algebra of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ with the inner product $(x|y) = \mathrm{trace}(x^*y)$. Another example is provided by a W*-algebra M with a faithful $\tau \in M_*^+$ satisfying $\tau(xy) = \tau(yx)$ for $x, y \in M$: $H = M$ with the inner product given by $(x|y) = \tau(x^*y)$. The general Hilbert algebra then turns out to be a kind of mixture of these.

Let $\mathcal{H}$ be the Hilbert space completion of H, which is *-bimodule of H in an obvious manner. An element $\xi \in \mathcal{H}$ is sadi to be left-bounded (resp. right-bounded) if $H \ni x \mapsto \xi x$ (resp. $x \mapsto x\xi$) is bounded. Denote the associated bounded operators on $\mathcal{H}$ by $l(\xi)$ and $r(\xi)$ respectively. Clearly $l(\xi) \in \mathrm{End}(\mathcal{H}_H)$ and $r(\xi) \in \mathrm{End}({}_H\mathcal{H})$. Elements in H are left and right bounded with l and r given by left and right multiplications.

If ξ is left-bounded,

$$(\xi^*x|y) = (\xi^*|yx^*) = (xy^*|\xi) = (x|\xi y) = (l(\xi)^*x|y)$$

for $x, y \in H$ shows that ξ^* is left-bounded and $l(\xi^*) = l(\xi)^*$, whereas

$$(x\xi^*|y) = (\xi^*|x^*y) = (y^*x|\xi) = (y^*|\xi x^*) = (y^*|l(\xi)x^*)$$

for $x, y \in H$ shows that ξ^* is right-bounded and $r(\xi^*) = *l(\xi)*$.

Consequently, left and right boundedness on elements in $\mathcal{H}$ are equivalent; we simply refer to them as being bounded. Let $B \supset H$ be the set of bounded vectors in $\mathcal{H}$. It is immediate to see that $HBH \subset B$ and

$$l(a\xi b) = l(a\xi b) = l(a)l(\xi)l(b), \quad r(a\xi b) = r(b)r(\xi)r(a)$$

for $a, b \in H$ and $\xi \in B$.

Lemma P.2.

(i) For $\xi, \eta \in B$, $l(\xi)$ and $r(\eta)$ commute.

(ii) If $L \in r(H)'$ and $x, y \in H$, then $xL(y) \in B$ with $(x(Ly))^* = y^*L^*(x^*)$ and $l(xL(y)) = l(x)Ll(y)$.

Proof. (i) Let ξ and η be bounded. Then, for $x, y \in H$,

$$(l(\xi)r(\eta)x|y) = (x\eta|\xi^*y) = (y^*\xi|\eta^*x^*) = (\eta y^*|x^*\xi^*) = (\xi x|y\eta^*) = (r(\eta)l(\xi)x|y)$$

shows $l(\xi)r(\eta) = r(\eta)l(\xi)$.

(ii) Since L and L^* leave the set of left-bounded elements invariant globally so that $l(L\xi) = Ll(\xi)$ and $l(L^*\xi) = L^*l(\xi)$ for a left-bounded $\xi \in \mathcal{H}$, $xL(y)$ belongs to B and the remaining follows from

$$l(x)Ll(y) = l(x(Ly)), \quad (l(x)Ll(y))^* = l(y^*)l(L^*x^*) = l(y^*(L^*x^*)).$$

□

Corollary P.3. We have $l(H)' = r(H)''$ and $l(H)'' = r(H)'$.

Proof. Since $l(H) \subset l(B) \subset r(B)' \subset r(H)'$ by (i), it suffices to show that $l(H)'' \supset r(B)'$ and $r(H)' \subset l(B)''$. By the symmetry of left and right, we shall only check the latter.

Let $L \in r(H)'$ and $R \in l(B)'$. We need to show that $LR = RL$. In view of $xL(y) \in B$ and $l(xL(y)) = l(x)Ll(y)$ for $x, y \in H$, we have

$$l(x)Ll(y)R = l(xL(y))R = Rl(xL(y)) = Rl(x)Ll(y).$$

Since the representation is non-degenerate, the identity is approximated by elements in $l(H)$ and we are done. □

For $\xi, \eta \in B$, we claim $l(\xi)\eta = \xi r(\eta)$. In fact, taking an approximating sequence $y_n \in H$ of $\eta \in \mathcal{H}$, we have for any $a \in H$

$$(l(\xi)\eta|a) = \lim(\xi y_n|a) = \lim(\xi|ay_n^*) = (\xi|a\eta^*) = (\xi|r(\eta^*)a) = (r(\eta)\xi|a).$$

We denote this common element by $\xi\eta$, which belongs to B in view of $l(\xi) \in r(H)'$ and $\xi\eta = l(\xi)\eta$. Clearly this extends the multiplication in H and makes B into a *-algebra in view of

$$(\xi\eta)^* = (l(\xi)\eta)^* = r(\xi^*)\eta^* = \eta^*\xi^*,$$
$$(\xi\eta)\zeta = r(\zeta)l(\xi)\eta = l(\xi)r(\zeta)\eta = \xi(\eta\zeta).$$

If we denote the von Neumann algebra $l(H)'' = r(H)'$ by N and extend the $*$-bimodule structure in $\mathcal{H}$ up to N by $\xi a = (a^*\xi^*)^*$ ($a \in N$, $\xi \in \mathcal{H}$), B is invariant under the biaction of N so that $l(a\xi b) = l(a)l(\xi)l(b)$ and $r(a\xi b) = r(b)r(\xi)r(a)$ for $a, b \in N$ and $\xi \in B$.

Lemma P.4 (Partition of Unity)**.** Let M be a W*-algebra.

(i) If $a, b \in M$ satisfy $b^*b \leq a^*a$, then there exists unique $c \in M$ satisfying $b = ca$ and $c[a^*] = c$, so that $\|c\| \leq 1$.

(ii) Let $\{a_j\}_{j\in I}$ be a family of elements in a W*-algebra M such that $\sum_j a_j^* a_j$ is weak*-convergent in M and set $a = (\sum_j a_j^* a_j)^{1/2}$. If $c_j \in M$ is defined by $a_j = c_j a$ and $[c_n] \leq [a]$, then $\sum_{j\in I} c_j^* c_j = [a]$ and $a = \sum_{j\in I} c_j^* a_j$.

Proof. (i) follows from $\|b\xi\|^2 \leq \|a\xi\|^2$ for $\xi \in L^2(M)$ and (ii) from $\sum_j a c_j^* c_j a = a^2 = a[a]a$ by noticing that $c_j^* c_j$'s are supported by $[a]$. □

Corollary P.5.

(i) If $b^*b \leq \sum_{j=1}^n a_j^* a_j$ with $b \in N$ and $a_j \in l(B)$, then $b \in l(B)$.

(ii) For a positive $a \in N_+$, $a \in l(B^2)$ if and only if $a^{1/2} \in l(B)$.

Proof. (i) From Lemma (ii), $a = \sum_{j=1}^n c_j a_j \in Nl(B) = l(NB) = l(B)$ and then $b = ca \in Nl(B) = l(B)$ by Lemma (i).

(ii) If $a = \sum x_j^* y_j \in l(B^2)$ with $x_j, y_j \in l(B)$

$$a = \sum_j \frac{1}{2}(x_j^* y_j + y_j^* x_j) \leq \sum_j (x_j^* x_j + y_j^* y_j)$$

and hence $a^{1/2} \in l(B)$ by part (i). The reverse implication is trivial. □

For $a \in N_+$, we set

$$\tau(a) = \begin{cases} (\xi|\xi) & \text{if } a = l(\xi)^* l(\xi) \text{ for some } \xi \in B, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly τ is faithful in the sense that $\tau(a) = 0$ implies $a = 0$ and semi-finite in the sense that $\{a \in N_+; \tau(a) < \infty\}$ is weak*-dense in N_+ (cf. Kaplansky's density theorem 4.23).

Additivity: $\tau(a+b) = \tau(a) + \tau(b)$ for $a, b \in N_+$. Since the condition $a + b \in l(B^2)$ is equivalent to $a, b \in l(B^2)$ by Corollary P.5, the claim is reduced to the additivity on $l(B^2) \cap N_+$, which follows from the fact that τ on $l(B^2) \cap N_+$ is extended to a linear functional (also denoted by τ) on $l(B^2)$ by

$$\tau(l(\xi^*\eta)) = (\xi|\eta) = \lim_{r(\zeta)\to 1} (\xi\zeta|\eta) = \lim_{r(\zeta)\to 1} (\zeta|\xi^*\eta).$$

Here $\zeta \in B$ and the limit is taken so that $r(\zeta)$ approaches to the identity operator in weak operator topology. Note that $l(B^2)$ is linearly spanned by $l(B^2) \cap N_+$ thanks to the polarization identity.

Trace property: $\tau(a^*a) = \tau(aa^*)$ for $a \in N$, $\tau(ab) = \tau(ba)$ for $a, b \in l(B)$ or for $a \in N$, $b \in l^2(B^2)$. Since B is invariant under uBu^* for a partial isometry in N, for $a \in N$, $a^*a \in l(B^2) \iff aa^* \in l(B^2)$ and, for $b = l(\beta)$, $c = l(\gamma)$ with $\beta, \gamma \in B$, $\tau(bc) = (\beta^*|\gamma) = (\gamma^*|\beta) = \tau(cb)$. Finally, if $a \in N$, $ab, ca \in l(B)$ and $\tau(abc) = \tau(cab) = \tau(bca)$.

Normality (continuity): Given an increasing family $\{a_i \in N_+\}$ indexed by a directed set I with a limit $a \in N_+$, we have $\sup \tau(a_i) = \tau(a)$.

Note that $\tau(a_i)$ is an increasing family in $[0, \infty]$ from the additivity of τ. Note also that $\{a_i^{1/2}\}$ is increasing[16] as well and converges[17] to $a^{1/2}$. It suffices to show $\sup \tau(a_i) \geq \tau(a)$ and, for this, we suppose that $\tau(a_i)$ is bounded. Then $a_i^{1/2} = l(\zeta_i) \in l(B)$ and, for $\xi, \eta \in B$,

$$(\xi|a^{1/2}\eta) = \lim(\xi|a_i^{1/2}\eta) = \lim(\xi|\zeta_i\eta) = \lim(\xi\eta^*|\zeta_i).$$

Since $(\zeta_i|\zeta_i) = \tau(a_i)$ is bounded by assumption, we may assume that ζ_i converges weakly to some $\zeta \in \mathcal{H}$ and we get $(\xi|a^{1/2}\eta) = (\xi\eta^*|\zeta) = (\xi|\zeta\eta)$ for any $\xi, \eta \in B$, which means that $\zeta \in B$ and $l(\zeta) = a^{1/2}$.

$$\tau(a) = (\zeta|\zeta) \leq \sup(\zeta_i|\zeta_i) = \sup \tau(a_i).$$

We now embed B^2 and B into N_* and $L^2(N)$ respectively in an algebraically compatible way. For this, we work with $l(B)$ instead of B. Recall that $l(B)$ is a weak*-dense *-ideal of N and the trace trace property $\tau(ax) = \tau(xa)$ of τ holds if $a \in l(B^2)$ and $x \in N$. For the choice $a = l(\xi)^* l(\eta)$,

$$\tau(ax) = \tau(l(\xi)^* l(\eta x)) = (\xi|\eta x) = (\xi x^*|\eta) = \tau(l(\xi x^*)^* l(\eta)) = \tau(xa).$$

The intermediate expression shows the weak* continuity of the functional $\tau(a(\cdot))$, which is denoted by $\tau a = a\tau$. The embedding $l(B^2) \ni a \mapsto a\tau \in N_*$ then preserves the N-biaction as well as the *-operation. It also preserves the positivity, i.e., $a\tau \geq 0 \iff a \geq 0$, in view of

$$\langle a\tau, l(\zeta\zeta^*)\rangle = \sum(\xi_j|\eta_j l(\zeta\zeta^*)) = \sum(\xi_j|\eta_j\zeta\zeta^*) = \sum(\zeta|\xi_j^*\eta_j\zeta) = (\zeta|a\zeta)$$

for $a = \sum l(\xi_j)^* l(\eta_j) \in l(B^2)$ and $\zeta \in B$. At the second equality, notice that $\beta l(\alpha) = (l(\alpha)^*\beta^*)^* = (\alpha^*\beta^*)^* = \beta\alpha$ for $\alpha, \beta \in B$. Since B is norm-dense in $\mathcal{H}$, so is $l(B^2)\tau = \tau l(B^2)$ in N_*.

We introduce a symbolical notation such as $b\tau^{1/2}$ to indicate the element $l^{-1}(b) \in B$ for $b \in l(B)$, i.e., $l(\beta)\tau^{1/2} = \beta$ for $\beta \in B$.

[16]Use $t^p = \dfrac{\sin \pi p}{\pi} \displaystyle\int_0^\infty \frac{t}{\lambda + t} \lambda^{p-1}\, d\lambda$ for $0 < p < 1$ and $t \geq 0$.

[17]The Taylor expansion of $\sqrt{1-t}$ converges uniformly on the interval $[-1, 1]$.

Note that $a^{1/2}\tau^{1/2}$ in $\mathcal{H}$ represents a left GNS-vector of $\varphi^{1/2}$ so that the correspondence $x\varphi^{1/2} \mapsto xa^{1/2}\tau^{1/2}$ is extended to a unitary map $L^2(N,\varphi) \to \overline{Na^{1/2}\tau^{1/2}} = \mathcal{H}[a]$ ($[a]$ being the support projection of a in N).

To get the whole identification $L^2(N) = \mathcal{H}$, we need therefore to check compatibility with (i) the inclusion $N\varphi^{1/2} \subset N\psi^{1/2}$ for $\varphi \leq \psi$ and (ii) the *-operation on $N\varphi^{1/2}$.

Lemma P.6. Let $\varphi = a\tau$ and $\psi = b\tau$ for $a, b \in l(B^2) \cap N_+$. Then $[\varphi] = [a]$, $[\psi] = [b]$ and $\varphi^{it}x\psi^{-it} = a^{it}xb^{-it}$ for $x \in [\varphi]N[\psi]$.

Proof. By Connes' trick, the problem is reduced to the case $a = b$.

We then prove the assertion by showing that the automorphism group $\sigma_t(x) = a^{it}xa^{-it}$ of $[a]N[a]$ satisfies the KMS-condition for φ. In fact, let $\beta = \beta^* \in B$ satisfy $a^{1/2} = l(\beta)$, $x \in l(B^2)$ and $y \in [a]N[a]$ be entirely analytic for σ_t. For the choice $x = l(\xi)^*l(\xi)$,

$$\varphi(x\sigma_t(y)) = (\beta|x\sigma_t(y)\beta) = (\xi\beta|\xi\sigma_t(y)\beta) = (\xi a^{1/2}|\xi\sigma_t(y)a^{1/2}) = (\xi|\xi\sigma_t(y)a).$$

Since $\sigma_t(y)a$ is analytically continued to $\sigma_{-i}(y)a = ay$ in norm, we have

$$\varphi(x\sigma_{-i}(y)) = (\xi|\xi ay) = (\xi|\xi\beta^2 y) = (\beta x|\beta y) = (y^*\beta|x^*\beta) = \varphi(yx).$$

$\square$

Exercise 52. Formulate matrix ampliations for Hilbert algebras.

We can now check the compatibility. (i) Since the condition $\varphi = a\tau \leq \psi = b\tau$ for $a, bl(B^2)$ is equivalent to $a \leq b$, we have $\varphi^{1/2} = a^{1/2}b^{-1/2}\psi^{1/2}$, whereas $l(a^{1/2}b^{-1/2}\beta) = a^{1/2}b^{-1/2}b^{1/2} = a^{1/2} = l(\alpha)$ shows that $(a^{1/2}b^{-1/2})(b^{1/2}\tau^{1/2}) = a^{1/2}\tau^{1/2}$.

(ii) Let $\alpha = \alpha^* \in B$ satisfy $a^{1/2} = l(\alpha)$. Since $x\varphi^{1/2} \in [\psi]N\psi^{1/2}$ for a sufficiently large $\psi \in l(B^2)\tau$, we may suppose that x is an entirely analytic element in $[\varphi]N[\varphi]$ for σ_t^φ to see that $(x\varphi^{1/2})^* = \sigma_{-i/2}^\varphi(x^*)\varphi^{1/2}$ corresponds to $\sigma_{-i/2}^\varphi(x^*)\alpha$ and $l(\sigma_{-i/2}^\varphi(x^*)\alpha) = \sigma_{-i/2}^\varphi(x^*)a^{1/2} = a^{1/2}x^* = l(\alpha x^*) = l((x\alpha)^*)$.

If we set $\tau^{1/2}l(\xi) = l(\xi)\tau^{1/2}$ for $\xi \in B$, it is also compatible with the *-operation: $(l(\xi)\tau^{1/2})^* = l(\xi)^*\tau^{1/2}$. In fact, letting $l(\xi) = ua^{1/2}$ be the polar decomposition and expressing $a^{1/2} = l(\alpha)$, $(l(\xi)\tau^{1/2})^* = (u\alpha)^* = \alpha u^* = l(\xi)^*\tau^{1/2}$.

So far, we have shown that the identification $L^2(N) = \mathcal{H}$ is compatible with the *-biactions of N so that $L^2(N)_+$ is the closure of B_+ in $\mathcal{H}$ ($l(B_+) = l(B) \cap N_+$). Finally we notice that, for $\xi, \eta \in B$, their product as elements in $L^2(N)$ is given by $l(\xi\eta)\tau \in N_*$, which is formally equal to $(\xi\eta)\tau^{1/2}$. In other words, the multiplication $\xi\eta$ in B is obtained from the natural product in N_* with the correcting factor $\tau^{-1/2}$ inserted.

Appendix Q. Measurable Operators

In this part, we present the basics in Segal-Nelson's according to the Haagerup's tyle described in [Terp].

Let us begin with recalling standard facts on unbounded operators. By an operator a on a Hilbert space $\mathcal{H}$, we shall mean a linear map $a : D(a) \to \mathcal{H}$ with $D(a)$ a linear subspace of $\mathcal{H}$. Associated to an operator a, we define its graph by $\mathcal{G}(a) = \{\xi \oplus a\xi; \xi \in D(a)\}$, which is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$. A linear subspace $\mathcal{G}$ of $\mathcal{H} \oplus \mathcal{H}$ is the graph of an operator if and only if $\mathcal{G} \cap (0 \oplus \mathcal{H}) = \{0 \oplus 0\}$. An operator a is said to be **closed** if $\mathcal{G}(a)$ is closed, **closable** if the closure of $\mathcal{G}(a)$ is the graph of an operator, and **densely defined** if $D(a)$ is dense in $\mathcal{H}$, respectively. For a closable operator a, its **closure** is an operator $\overline{a}$ specified by $\mathcal{G}(\overline{a}) = \overline{\mathcal{G}(a)}$. For a densely defined operator a, its **adjoint** is an operator a^* specified by $\mathcal{G}(a^*) = \mathcal{G}(a)^\dagger$, where $\mathcal{G}(a)^\dagger$ is the orthogonal complement of $\mathcal{G}(a)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A densely defined operator a is closable if and only if a^* is densely defined and, if this is the case, $\overline{a} = (a^*)^*$. For a closed operator a, its kernel $\ker a = \{\xi \in D(a); a\xi = 0\}$ is closed and the **support** $[a]$ of a is the projection to $(\ker a)^\perp$.

An operator b is called an **extension** of an operator a and denoted by $a \subset b$ if $\mathcal{G}(a) \subset \mathcal{G}(b)$. If $a \subset b$ with a densely defined, $b^* \subset a^*$.

An operator a is said to be **symmetric** (**self-adjoint**) if a is densely defined and satisfies $a \subset a^*$ ($a = a^*$). A symmetric operator is self-adjoint if and only if $\ker(a^* \pm i) = 0$ (von Neumann). There exists a one-to-one correspondence between self-adjoint operators and projection-valued measures on $\mathbb{R}$ by the relation of spectral decomposition

$$a = \int_{\mathbb{R}} \lambda E(d\lambda).$$

Note that $\ker a = E(\{0\})\mathcal{H}$ and $[a] = E(\mathbb{R} \setminus \{0\})$.

For operators a, b on $\mathcal{H}$, their sum is an operator $a + b$ defined by $D(a+b) = D(a) \cap D(b)$ and $(a+b)\xi = a\xi + b\xi$ for $\xi \in D(a+b)$, and their product is an operator ab defined by $D(ab) = \{\xi \in D(b); b\xi \in D(a)\}$ and $(ab)\xi = a(b\xi)$ for $\xi \in D(ab)$. In terms of graph, these are described by $\mathcal{G}(a+b) = \mathcal{G}(a) \cap \mathcal{G}(b)$ and $\mathcal{G}(ab) = \mathcal{G}(b) * \mathcal{G}(a)$, where

$$\mathcal{G}(b) * \mathcal{G}(a) = \{\xi \oplus \zeta; \xi \oplus \eta \in \mathcal{G}(b) \text{ and } \eta \oplus \zeta \in \mathcal{G}(a) \text{ for some } \eta \in \mathcal{H}\}.$$

An operator a is said to be **positive** if $(\xi | a\xi) \geq 0$ for $\xi \in D(a)$. For a positive self-adjoint operator a and a positive real $r \geq 0$, the positive self-adjoint operator a^r is defined by

$$a^r = \int_{\mathbb{R}} \lambda^r E(d\lambda).$$

For a densely defined closed operator a, obviously positive operator a^*a is self-adjoint and, if we denote $(a^*a)^{1/2}$ by $|a|$, there exists a partial isometry u satisfying $u^*u = [a^*a]$ and $a = u|a|$ (the **polar decomposition** of a). The polar decomposition is unique in the sense that if $u|a| = v|b|$ with partial isometries u and v satisfying $u^*u = [a^*a]$ and $v^*v = [b^*b]$, then $u = v$ and $|a| = |b|$.

We present E. Nelson's approach (cf. also Terp's thesis) to Segal's theory of non-commutative integrations. Let N be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with τ a faithful semifinite trace on N.

Recall that two projections p and q in a W*-algebra M are said to be **equivalent** and denoted by $p \sim q$ if we can find a partial isometry u in M satisfying $u^*u = p$ and $uu^* = q$. The following old lemma about an analogue of set-theoretical relations goes back to Murrey and von Neumann.

Lemma Q.1. Let p, q be projections in a W*-algebra M. Then we have $(p \vee q) - p \sim q - (p \wedge q)$.

Proof. In view of $\ker(pq^\perp) = q\mathcal{H} + (p^\perp \wedge q^\perp)\mathcal{H}$, we have

$$[pq^\perp] = q^\perp \wedge (p \vee q) = (p \vee q) - q$$

while, in view of $\ker(q^\perp p) = p^\perp\mathcal{H} + (p \wedge q)\mathcal{H}$,

$$[q^\perp p] = p \wedge (p \wedge q)^\perp = p - (p \wedge q).$$

These are then combined with $[pq^\perp] \sim [(pq^\perp)^*] = [p^\perp q]$ to get the assertion. □

Corollary Q.2. If $p \wedge q = 0$, p is equivalent to a subprojection of $1 - q$.

Proof.

$$p = 1 - p^\perp = (p \wedge q)^\perp - p^\perp = (p^\perp \vee q^\perp) - p^\perp \sim q^\perp - (p^\perp \wedge q^\perp) \leq q^\perp.$$

□

An operator a on $\mathcal{H}$ is said to be **affiliated** with N and denoted by $a \in' N$ if $u'a = au'$ for any unitary u' in N' or equivalently the graph of a is invariant under the diagonal action of u'.

A densely defined closed operator a with $a = u|a|$ its polar decomposition is affiliated with N if and only if u and all the spectral projections of $|a|$ belong to N.

A densely defined closable operator $a \in' N$ is said to be **τ-measurable** if for any $\epsilon > 0$ there exists a projection $p \in N$ satisfying $p\mathcal{H} \subset D(a)$

with $\|ap\| < \infty$ and $\tau(1-p) \leq \epsilon$. The set of τ-measurable operators is denoted by $\widetilde{N}$.

Lemma Q.3. Let a, b be densely defined closable operators affiliated with N and assume that there exist projections p, q in N such that $p\mathcal{H} \subset D(a)$, $q\mathcal{H} \subset D(b)$. Let $e = p \wedge q$ and $f = q \wedge (1 - [(1-p)bq])$ be projections in N. Then we have the following.

(i) $e\mathcal{H} \subset D(a+b)$, $\|(a+b)e\| \leq \|ap\| + \|bq\|$ and $\tau(1-e) \leq \tau(1-p) + \tau(1-q)$.
(ii) $f\mathcal{H} \subset D(ab)$, $\|abf\| \leq \|ap\| + \|bq\|$ and $\tau(1-f) \leq \tau(1-p) + \tau(1-q)$.

Proof. Since f is the projection to

$$\{\xi \in q\mathcal{H}; (1-p)b\xi = 0\} = (q\mathcal{H}) \cap \ker((1-p)bq),$$

we have $e\mathcal{H} \subset D(ab)$ in view of $be\mathcal{H} \subset p\mathcal{H} \subset D(a)$, $\|abe\| = \|apbe\| \leq \|ap\|\,\|be\| \leq \|ap\|\,\|bq\|$ and then

$$\begin{aligned}\tau(1-e) = \tau((1-q) \vee [(1-p)bq]) &\leq \tau(1-q) + \tau([(1-p)bq]) \\ &= \tau(1-q) + \tau([qb^*(1-p)]) \\ &\leq \tau(1-q) + \tau(1-p)\end{aligned}$$

□

Corollary Q.4. If $a, b \in \widetilde{N}$, then $a + b \in \widetilde{N}$ and $ab \in \widetilde{N}$.

Lemma Q.5. Let a be a densely defined closed operator affiliated with N. If a projection $p \in N$ satisfies $p\mathcal{H} \subset D(a)$ and $\delta = \|ap\| < \infty$, then the spectral projection e_δ of $|a|$ associated to the interval $[0, \delta]$ satisfies $\tau(1-e_\delta) \leq \tau(1-p)$.

Proof. Let $\delta = \|ap\| < \infty$. The projection e then satisfies $(1-e) \wedge p = 0$ because $\|a\xi\| \leq \delta\|\xi\|$ if $p\xi = \xi$ and $\|a\xi\| > \delta\|\xi\|$ if $0 \neq \xi \in (1-e)\mathcal{H}$. Thus $1-e$ is equivalent to a subprojection of $1-p$ and we have $\tau(1-e) \leq \tau(1-p)$. □

Corollary Q.6.

(i) A densely defined closed operator a affiliated with N is τ-measurable if and only if $\tau(1-e_\delta) < \infty$ for some $\delta > 0$.
(ii) If $a \in \widetilde{N}$, then $a^* \in \widetilde{N}$.

Proof. (i) If a is τ-measurable, the existence of $\delta > 0$ is a consequence of the lemma. Since $e_\delta\mathcal{H} \subset D(a)$ and $\|ae_\delta\| \leq \delta$ for any $\delta > 0$, $\tau(1-e_\delta) < \infty$ for some $\delta > 0$ implies

$$\lim_{\rho\to\infty} \tau(1-e_\rho) = \tau(1-e_\delta) - \lim_{\rho\to\infty} \tau(e_\rho - e_\delta) = \tau(1-e_\delta) - \tau(1-e_\delta) = 0$$

and the τ-measurablity of a follows.

(ii) Since $\overline{a}$ belongs to $\widetilde{N}$ as a closure of $a \in \widetilde{N}$ and since $a^* = \overline{a}^*$, we may assume that a is closed. Let $a = u|a|$ be the polar decomposition. Then $|a^*| = u|a|u^*$ and $ue_\delta u^*$ gives the spectral projection of $|a^*|$ corresponding to the interval $[0, \delta]$. The τ-measurability of a^* now follows from that of a in view of $\tau(1 - ue_\delta u^*) = \tau(1 - e_\delta)$. □

Theorem Q.7 (Identity)**.** Let $a, b \in \widetilde{N}$. If $a\xi = b\xi$ for $\xi \in D(a) \cap D(b)$, then $\overline{a} = \overline{b}$.

Proof. Let e and f be the projections to the closure of $\mathcal{G}(a)$ and $\mathcal{G}(b)$ respectively. Since a and b commute with N', e and f belong to the commutant $M_2(N)$ of the diagonal embedding $N' \subset M_2(\mathcal{B}(\mathcal{H}))$.

By τ-measurability and Lemma Q.3 (i), given any $\epsilon > 0$, we can find a projection $p \in N$ so that $p\mathcal{H} \subset D(a) \cap D(b)$, $\|ap\| = \|b\| < \infty$ and $\tau(1 - p) \leq \epsilon$.

Since $ap = bp$, we have $e \wedge (p \oplus p) = f \wedge (p \oplus p)$ and then $e \wedge f \wedge (p \oplus p) = e \wedge (p \oplus p)$. Thus $(e - e \wedge f) \wedge (p \oplus p) = 0$ and, thanks to Corollary Q.2, $\tau_2(e - e \wedge f) \leq \tau_2(p^\perp \oplus p^\perp) = 2\tau(1 - p) \leq 2\epsilon$ for any given $\epsilon > 0$. Consequently $\tau_2(e - e \wedge f) = 0$ and hence $e = e \wedge f$ by the faithfulness of τ_2. From the symmetry of arguments, we have $f = e \wedge f$ as well and conclude that $e = f$; $\mathcal{G}(\overline{a}) = \mathcal{G}(\overline{b})$. □

Corollary Q.8. If $a, b \in \widetilde{N}$ satisfy $a \subset b$, then $\overline{a} = \overline{b}$. For example, from $(ab)^* \subset b^*a^*$, $(ab)^* = b^*a^*$ and, from $a \subset a^*$, $a^{**} = a^*$.

Let $\tilde{N}$ be the set of closed τ-measurable operators on $\mathcal{H}$. Taking closure gives a surjection $\widetilde{N} \to \tilde{N}$ and $*$-algebraic operations in $\widetilde{N}$ are inherited to $\tilde{N}$: $\overline{a} + \overline{b} = \overline{a + b}$, $\overline{a}\overline{b} = \overline{ab}$ and $\overline{a}^* = \overline{a^*}$.

These are well-defined: Let $a_j, b_j \in \widetilde{N}$ $(j = 1, 2)$ satisfy $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Then a_1 and a_2 coincide on $D(a_1 \cap a_2)$ and, if we define $a \in \widetilde{N}$ so that $D(a) = D(a_1) \cap D(a_2)$ and $a = a_j$ on $D(a)$. Likewise we define $b \in \widetilde{N}$.

Now $a_j + b_j$ and $a_j b_j$ are extensions of $a + b$ and ab respectively, whence $\overline{a_1 + b_1} = \overline{a + b} = \overline{a_2 + b_2}$ and $\overline{a_1 b_1} = \overline{ab} = \overline{a_2 b_2}$ by the identity theorem. For taking adjoints, a^* extends a_j^* for $j = 1, 2$ and again the identity theorem is used to have $a_1^* = a^* = a_2^*$.

Since the associativity for sum and product holds in $\widetilde{N}$, the same holds in $\tilde{N}$. The distributive law which takes a form of inclusion $a(b + c) \supset ab + ac$ in $\widetilde{N}$ gives the form of strict identity in $\tilde{N}$, whereas the identity $(ab)^* = b^*a^*$ is always true in $\widetilde{N}^\tau$. Summarizing these, we conclude that

Theorem Q.9. We have a *-algebra $\overline{N}^\tau$ of densely defined closed τ-mearable operators on $\mathcal{H}$. Moreover, for a positive $a \in \overline{N}^\tau$, its positive powers a^r $(r > 0)$ also belong to $\overline{N}^\tau$.

In what follows, elements in $\widetilde{N}$ are implicitly identified with their images under the canonical map onto $\overline{N}^\tau$ so that $\widetilde{N}$ itself is identified with the *-algebra $\overline{N}^\tau$.

Given $\epsilon > 0$ and $\delta > 0$, consider the condition on $a \in \widetilde{N}$ that there exists a projection $p \in N$ satisfying $p\mathcal{H} \subset D(a)$, $\|ap\| \leq \delta$ and $\tau(1-p) \leq \epsilon$, which is equivalently described in terms of the spectral decomposition of $|a|$ by $\tau(1 - [\delta \wedge |a|]) \leq \epsilon$.

Let $N(\delta, \epsilon)$ be the totality of such elements. Here is a list of properties of $N(\delta, \epsilon)$ which can be checked immediately:

(i) $N(\delta, \epsilon)$ is increasing in (δ, ϵ) and satisfies

$$\bigcup_{\delta>0,\epsilon>0} N(\delta,\epsilon) = \widetilde{N}, \qquad \bigcap_{\delta>0,\epsilon>0} N(\delta,\epsilon) = \{0\}.$$

(ii) $N(\delta, \epsilon)^* = N(\delta, \epsilon)$ and $N(\delta, \epsilon) = \delta N(1, \epsilon)$.
(iii) $N(\delta_1, \epsilon_1) \cap N(\delta_2, \epsilon_2) \supset N(\delta_1 \wedge \delta_2, \epsilon_1 \wedge \epsilon_2)$.
(iv) $N(\delta_1, \epsilon_1) + N(\delta_2, \epsilon_2) \subset N(\delta_1 + \delta_2, \epsilon_1 + \epsilon_2)$.
(v) $N(\delta_1, \epsilon_1)N(\delta_2, \epsilon_2) \subset N(\delta_1\delta_2, \epsilon_1 + \epsilon_2)$.

Thus, the family $N(\delta, \epsilon)$ enables us to introduce a vector topology (the topology of convergence in measure) so that it gives a neighbourhood basis of 0 and makes $\widetilde{N} = \overline{N}^\tau$ a topological *-algebra. Now, we have the following, which as well as its proof is more or less an analogue of Riesz-Fisher theorem on classical L^p-spaces.

Theorem Q.10. The topological *-algebra $\widetilde{N}$ is complete and contains N as a dense *-subalgebra.

Proof. The density of N is immediate. So we focus on the completeness. Let $\{a_n\}$ be a Cauchy sequence in $\widetilde{N}$; given any (δ, ϵ), there eixts a natural number l such that $a_m - a_n \in N(\delta, \epsilon)$ for $m, n \geq l$.

It suffices to show that we can find a subsequence n' and $a \in \widetilde{N}$ so that $\lim_n a_{n'} = a$. To see this, by the denseity of N, we may assume that $a_n \in N$ from the outset. Choosing a subsequence k' so that $m, n \geq k' \Longrightarrow a_m - a_n \in N(1/2^k, 1/2^k)$ and replacing a_n with $a_{n'}$, we may further assume that $a_{n+1} - a_n \in N(1/2^n, 1/2^n)$. $\square$

Lemma Q.11. Let $h \in \widetilde{N}$ be a positive measurable operator. Then, $(0, \infty) + i\mathbb{R} \ni z \mapsto h^z \in \widetilde{N}$ is analytic with respect to the measure topology of $\widetilde{N}$.